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Socially Optimal Service hours with Special Offers*

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Abstract

It is very important for service industries to decentralize consumers at peak time, and thereby to increase sales not at peak time. This study discusses an optimal number of business hours for a service industry when the service provider offers a price discount immediately after the opening time and just before the closing time. For a specific ideal service time distribution of consumers, the optimal opening and closing time are explored. Clarified are the conditions under which an optimal number of business hours exists to maximize the social welfare. Numerical examples are also presented to illustrate the theoretical underpinnings of the proposed model.

KEY WORDS: Price discount, Service hours, Ideal service time distribution, Social welfare

JEL Classification: L80, M20

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1. Introduction

The regulations of business hours have traditionally generated a central issue in many European countries (see, e.g., De Meza (1984), Ferris (1990, 1991), Clemenz (1990, 1994), Inderst and Irmen (2005)). In the real world, however, it is very important for service providers to decentralize consumers at peak time and increase the sales not at peak time to increase the sales and/or the profit. Shy and Stenbacka (2006) have proposed a model to discuss an optimal number of business hours for a service provider against a specific consumers’ ideal service time distribution.

In the real circumstance, it can be observed that a special offer such as time discount has commonly implemented by service providers as an effective strategy, e.g., morning perm at a beauty salon, happy hour at a hotel, midnight discount of a telecommunications industry, special time discount in business logistics and so forth. From this point of view, Kim and Sandoh (2014) have introduced a special offer of discount of price immediately after the opening time and just before the closing time to discuss an optimal number of business hours, where the service provider is interested in maximizing his profit. This type of price promotion is an effective management tool since a service provider can attract extra consumers whose ideal or convenient service times are before the opening time or after the closing time.

In this study, we focus on the social welfare as an objective function to be maximized and clarify the conditions where an optimal number of business hours exists. Numerical examples are also presented to illustrate the insights of our analysis.

2. Model Formulation

2.1 Assumptions and notations

The assumptions along with their relevant notations in this study are as follows:

(1) Each individual consumer has her own ideal time to visit the provider to receive service from the service provider.

(2) Each consumer obtains utility, \( u_0 \), by purchasing a service product.

(3) The regular selling price of a service product is \( p \).

(4) During the price discount period, the provider sells his service product at price \( \alpha p \) as his special offer, where \( 0 < \alpha < 1 \).

(5) The length of the price discount period is denoted by \( \tau > 0 \).

(6) A consumer owes \( \omega \) per unit of time to shift her actual service time from her own ideal service time to purchase a service product.

(7) The opening and closing times are, respectively, denoted by \( t_o \) and \( t_c \), where we have \( 0 \leq t_o \leq t_c \leq 1 \).

(8) The raw price per service product is given by \( c_1 \), while the operation cost of the service provider per unit of time is \( c_2 \).

2.2 Ideal service time distribution

In this study, we assume that a customer distribution \( q_t \) in which the number of consumers with an ideal service time \( t \) (\( 0 \leq t \leq 1 \)) is

\[
q_t = \begin{cases} 
 n[\mu + 4(1-\mu)t], & 0 \leq t < \frac{1}{2} \\
 n[4-3\mu-4(1-\mu)t], & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
where \( n \) represents the population size and \( \mu \ (0 \leq \mu \leq 1) \) measures the degree of uniformity. Figure 1 shows the ideal time distribution given by Eq. (2.1) for \( n = 1 \) against various values of \( \mu \).

Shy and Stenbacka[6] have assumed the above ideal time distribution on the unit circle with the view to formalizing the idea that there are spillovers between time periods. In this study, however, we assume the same structure of the ideal time distribution on the unit time interval \([0, 1]\). This is because spillovers are an important factor only when the service provider sells his products for almost whole unit time period, and in such a situation the strategic determination of service hours might not be necessary.

We here introduce an additional assumption as follows:

(9) When the selling price is discounted to \( \alpha p \) at \( t \), demand quantity \( q_t \) increases \( \beta(\alpha)q_t \) to \( [1 + \beta(\alpha)]q_t \) for \( \beta(\alpha) > 0 \) and \( 0 < \alpha < 1 \).

\[
\eta = \frac{\beta(\alpha)q_t}{q_t} = \frac{\beta(\alpha)}{\alpha - 1} \frac{1}{1 - \alpha} \quad 0 < \alpha < 1, \tag{2.2}
\]

where \( \lim_{\alpha \to 0} \beta(\alpha) = 0 \).

In the following, consumers involved in and represented by the demand quantity \( q_t \) are called type \( \mathcal{A} \), while those expressed by \( \beta(\alpha)q_t \) are referred to type \( \mathcal{B} \). Moreover, we concentrate upon the case where values of \( \alpha \) and \( \beta(\alpha) \) are both specified to specific values, and therefore \( \beta(\alpha) \) is written as \( \beta \) for simplicity.

3. Consumers’ Behavior

3.1 Best response

Since the ideal time distribution by Eq. (2.1) reveals a symmetrical shape, the opening time,
and the closing time, $t_c$, are also symmetrical with respect to $t = \frac{1}{2}$, accordingly we have

$$t_c = 1 - t_o.$$ 

Hence, we focus on the former half $[0, \frac{1}{2}]$ of period to discuss the opening time, $t_o$, hereafter.

(1) Type $A$ consumers’ response

When the provider offers early birds specials and/or closing time discount/sale, the best response of type $A$ consumers with ideal time $t$ becomes as follows:

i) If $t \in [0, t^{(1a)}_o]$, type $A$ consumers are reluctant to wait until $t_o$, and purchase no service product, where

$$t^{(1a)}_o = t_o - \frac{u_0 - \alpha p}{\omega}. \quad (3.1)$$

Consequently, their net utility, $U_i$, becomes

$$U_i = 0.$$

ii) If $t \in (t^{(1a)}_o, t_o]$, type $A$ consumers purchase a service product at the discounted price $\alpha p$ by waiting until $t_o$, and hence their net utility is given by

$$U_i = u_0 - \alpha p - \omega(t_o - t).$$

iii) Type $A$ consumers with their ideal time $t \in (t_o, t_o + \tau]$ purchase a service product at their own ideal time $t$, at the discounted price $\alpha p$. In this case, their net utility becomes

$$U_i = u_0 - \alpha p.$$

iv) In the case of $t \in (t_o + \tau, t^{(2)}_o]$, the consumers purchase a service product earlier than their own ideal time $t$, at the special price $\alpha p$, yielding

$$U_i = u_0 - \alpha p - \omega\{t - (t_o + \tau)\},$$

where

$$t^{(2)}_o = t_o + \tau + \frac{(1-\alpha)p}{\omega}. \quad (3.2)$$

It should be noted in Eq. (3.2) that $t^{(2)}_o \neq t_o + \tau + \frac{u_0 - \alpha p}{\omega}$ since consumers with their
own ideal time $t$, can obtain positive utility, $u_0 - p$, even at $t$, and $t_o^{(2)}$ should be derived from the condition in reference to $t$;

$$u_0 - \alpha p - \omega [t - (t_o + \tau)] \geq u_0 - p.$$  

v) When $t \in \left[ t_o^{(2)}, \frac{1}{2} \right]$, type $A$ consumers will purchase a service product at the regular price, $p$, at their ideal time $t$, and hence

$$U_t = u_0 - p.$$  

(2) Type $B$ consumers’ response.  

The best response of type $B$ consumers with their ideal time $t$ is described as follows:

i) If $t \in \left[ 0, t_o^{(b)} \right]$, type $B$ consumers would not wait until $t_o$ because they purchase no service, where

$$t_o^{(b)} = t_o - \frac{(1-\alpha)p}{\omega}.$$  

Consequently, their net utility becomes

$$U_t = 0.$$  

ii) If $t \in \left( t_o^{(b)}, t_o \right]$, type $B$ consumers purchases a service product at $\alpha p$, by shifting their actual service time from their own ideal time to $t_o$. In this case, the maximum value of their net utility can be represented by

$$U_t = (1-\alpha)p - \omega(t_o - t).$$  

iii) Consumers with $t \in \left( t_o, t_o + \tau \right]$ purchase a service product at the discounted price $\alpha p$, at their own ideal service time, and hence their maximum net utility can be expressed as

$$U_t = (1-\alpha)p.$$  

iv) In the case of $t \in \left( t_o + \tau, t_o^{(2)} \right]$, type $B$ consumers purchase a product earlier than their ideal time at $\alpha p$, and their maximum net utility becomes

$$U_t = (1-\alpha)p - \omega[t - (t_o + \tau)].$$
v) $t \in \left( t_o^{(2)}, \frac{1}{2} \right]$, type $B$ consumers would purchase no service product yielding

$$U_t = 0.$$ 

### 3.2 Domain of opening time

It is neither reasonable nor proper for a consumer with ideal time $t < 0$ to shift her actual service time to $t_o$, and thereby we assume

$$\min\left( t_o^{(1a)}, t_o^{(1b)} \right) = t_o^{(1a)} = t_o - \frac{u_o - \alpha p}{\omega} \geq 0,$$

which constrains the opening time to satisfy

$$t_o \geq \frac{u_o - (1-\alpha)p}{\omega}.$$ \hspace{1cm} \text{(3.4)}

The right-hand-side of Eq. (3.4) is denoted by $t_L$ in the following.

Likewise, it is reasonable to assume

$$t_o^{(2)} \leq \frac{1}{2},$$

which is equivalent to

$$t_o \leq \frac{1}{2} - \frac{(1-\alpha)p}{\omega}.$$ \hspace{1cm} \text{(3.5)}

The right-hand-side of Eq. (3.5) is denoted by $t_U$ hereafter.

It should be noted here that Eqs. (3.4) and (3.5) yield,

$$\frac{u_o - \alpha p}{\omega} \leq \frac{1}{2} - \frac{(1-\alpha)p}{\omega},$$

which signifies, at the same time, that $\omega$ should satisfy

$$\omega \geq \frac{2[u_o + (1-2\alpha)p]}{1-2\tau}.$$ \hspace{1cm} \text{(3.6)}

From Eqs. (3.4) and (3.5), the domain of $t_o$ is, as a result, given by

$$t_L \equiv \frac{u_o - \alpha p}{\omega} \leq t_o \leq \frac{1}{2} - \frac{(1-\alpha)p}{\omega} \equiv t_U.$$ \hspace{1cm} \text{(3.7)}
4. Social Welfare
4.1 Provider’s profit

Let \( Q_{A}(t_o) \) express the number of Type \( A \) consumers who purchase a service product at the discounted price \( \alpha p \), then we have
\[
Q_{1A}(t_o) = 2\int_{t_o}^{t_2} q_1 dt
= 2n\left[ \tau + \frac{u_0 + (1-2\alpha)p}{\omega} \right] \times \left[ \mu + 2(1 - \mu) \left( 2t_o + \tau - \frac{u_0 - p}{\omega} \right) \right]
\]  
(4.1)

By letting \( Q_{B}(t_o) \) signify the number of type \( B \) consumers who purchase a service product at \( \alpha p \), we have
\[
Q_{1B}(t_o) = 2\int_{t_o}^{t_2} q_2 dt
= 2n\beta\mu\left[ \tau + \frac{2(1-\alpha)p}{\omega} \right] + 4n\beta(1 - \mu)\left[ \tau + \frac{2(1-\alpha)p}{\omega} \right] (2t_o + \tau).
\]  
(4.2)

On the other hand, let us denote, by \( Q_{2}(t_o) \), the number of consumers who purchase a service product at the regular price \( p \), then we have
\[
Q_{2}(t_o) = 2\int_{t_o}^{t_2} q_2 dt
= 2n\left( \frac{1}{2} - t_o - \frac{1 - \alpha p}{\omega} \right) \times \left[ \mu + 2(1 - \mu) \left( \frac{1}{2} + t_o + \tau + \frac{1 - \alpha p}{\omega} \right) \right].
\]  
(4.3)

Hence, the provider’s profit is given by
\[
\Pi(t_o) = (\alpha p - c_i)\left[ Q_{1A}(t_o) + Q_{1B}(t_o) \right] + (p - c_i)Q_{2}(t_o) - c_2(1 - 2t_o).
\]  
(4.4)

We here introduce the following additional constraints so that the provider’s profit can take on a positive value at its demand peak and a negative value at its demand off-peak;
\[
n(2 - \mu)(p - c_i) > c_2, \quad \text{ (4.5)}
\]
\[
n\mu(p - c_i) < c_2. \quad \text{ (4.6)}
\]

Further, we also assume
\[
c_1 \leq \alpha p,
\]
not to lose profit by the special offer. This provides a lower bound for \( \alpha \) and consequently the domain of \( \alpha \) is given by
\[ \frac{c_1}{p} < \alpha < 1. \]  

4.2 Consumers’ surplus

Let us denote by \( C_i(t_o) \) the total surplus of Type \( i \) \((i = A, B)\) consumers, then we have

\[
C_a(t_o) = (u_0 - \alpha p)Q_{1a}(t_o) + (u_0 - p)Q_2(t_o) - 2\alpha \int_{t_o}^{t_o + \tau} (t_o - t) q dt \\
- 2\alpha \int_{t_o}^{t_o + \tau} [t - (t_o + \tau)] q dt, \\
C_B(t_o) = (1 - \alpha) p Q_{1b}(t_o) - 2\beta \int_{t_o}^{t_o + \tau} (t_o - t) \beta q dt - 2\beta \int_{t_o}^{t_o + \tau} [t - (t_o + \tau)] \beta q dt. 
\]

Hence, the whole consumers’ surplus, which is denoted by \( C(t_o) \), is given by

\[
C(t_o) = C_a(t_o) + C_B(t_o). 
\]

4.3 Social welfare

The social welfare is defined by the sum of the total consumers’ surplus and the provider’s profit, i.e., let \( \Psi(t_o) \) denote the social welfare, then we have

\[
\Psi(t_o) = \Pi(t_o) + C(t_o) \\
= (u_0 - c_1)[Q_{1a}(t_o) + Q_2(t_o)] + (p - c_1)Q_{1b}(t_o) \\
- [D_1(t_o) + D_2(t_o)] - c_2(1 - 2t_o), 
\]

where

\[
D_1(t_o) = 2\alpha \int_{t_o}^{t_o + \tau} (t_o - t) q dt + \beta \int_{t_o}^{t_o + \tau} (t_o - t) \beta q dt, \\
D_2(t_o) = 2(1 + \beta)\alpha \int_{t_o}^{t_o + \tau} [t - (t_o + \tau)] q dt. 
\]

5. Optimal Strategy

This section seeks for the socially optimal opening time \( t_o^* \), which can provide an optimal closing time \( t_c^* \) by the symmetric structure of the ideal time distribution. Numerical examples are also presented to illustrate the proposed model formulation.

5.1 Analysis

From Eq. (4.11), we have
\[
\frac{d\Psi(t_o)}{dt_o} = (u_o - c_1) \left( \frac{dQ_{1,1}(t_o)}{dt_o} + \frac{dQ_2(t_o)}{dt_o} \right) + (p - c_1) \frac{dQ_{1,1}(t_o)}{dt_o} - \left[ \frac{dD_1(t_o)}{dt_o} + \frac{dD_2(t_o)}{dt_o} \right] + 2c_2. \quad (5.1)
\]

Let us denote, by \( \varphi(t_o) \), the right-hand-side of Eq. (5.1), and let \( a \) and \( b \) be defined by

\[
a = \frac{u_o - \alpha p}{\omega}, \quad b = \frac{(1 - \alpha) p}{\omega}.
\]

Then, we have

\[
\varphi(t_o) = (u_o - c_1) \left[ 8n(1 - \mu)(a - t_o) - 2n\mu \right] + (p - c_1)8n\beta(1 - \mu)(\tau + 2b) - 2n\omega \left[ 2(1 - \mu)(a^2 + b^2) + 4\beta(1 - \mu)b^2 \right] + 2c_2, \quad (5.2)
\]

which indicates \( \varphi(t_o) \) is strictly decreasing in \( t_o \).

In addition, we have

\[
\varphi \left( \frac{u_o - \alpha p}{\omega} \right) = -2n\mu(u_o - c_1) + (p - c_1)8n\beta(1 - \mu)(\tau + 2b) - 2n\omega \left[ 2(1 - \mu)(a^2 + b^2) + 4\beta(1 - \mu)b^2 \right] + 2c_2, \quad (5.3)
\]

\[
\varphi \left( \frac{1}{2} - \tau - \frac{(1 - \alpha) p}{\omega} \right) = 2n \left( u_o - c_1 \right) \left[ 2(1 - \mu)(2\tau + 2b + 2a - 1) - \mu \right] + (p - c_1)8n\beta(1 - \mu)(\tau + 2b) - 2n\omega \left[ 2(1 - \mu)(a^2 + b^2) + 4\beta(1 - \mu)b^2 \right] + 2c_2. \quad (5.4)
\]

Now, let \( A \) and \( B \) be defined by

\[
A \equiv -n\mu(u_o - c_1) + (p - c_1)4n\beta(1 - \mu)(\tau + 2b) - n\omega \left[ 2(1 - \mu)(a^2 + b^2) + 4\beta(1 - \mu)b^2 \right] + c_2, \quad (5.5)
\]

\[
B \equiv n \left( u_o - c_1 \right) \left[ 2(1 - \mu)(2\tau + 2b + 2a - 1) - \mu \right] + (p - c_1)4n\beta(1 - \mu)(\tau + 2b) - n\omega \left[ 2(1 - \mu)(a^2 + b^2) + 4\beta(1 - \mu)b^2 \right] + c_2, \quad (5.6)
\]
and then the optimal opening time, $t^o$, can be discussed under the following classification:

(a) If we have $A > 0$, further classification is necessary.

i) In the case of $B \geq 0$, $t^o$ is given by

$$t^o = \frac{1}{2} - \frac{(1-\alpha)p}{\omega} = t_U.$$  

ii) On the contrary, in case we have $B < 0$, $t^o$ is given by

$$t^o = a - \frac{\mu}{4(1-\mu)(u_0 - c_1)} + \frac{\beta(p - c_1)(\tau + 2b)}{u_0 - c_1} - \frac{\omega(a^2 + b^2)}{2(u_0 - c_1)} - \frac{\beta \omega b^2}{u_0 - c_1} + \frac{c_2}{4n(1-\mu)(u_0 - c_1)}.$$  

(b) If we have $A \leq 0$, then $\pi(t_o) \leq 0$ and hence

$$t^o = \frac{u_0 - \alpha p}{\omega} = t_L.$$  

As for the optimal opening time, $t^o$, we have the following proposition:

**Proposition 1** For the ideal time distribution with $\mu = 1$, if $u_0 - c_1 \geq \frac{c_2}{n}$, the opening time becomes

$$t^o = \frac{u_0 - \alpha p}{\omega} = t_L,$$

otherwise we have

$$t^o = \frac{1}{2} - \frac{(1-\alpha)p}{\omega} = t_U.$$  

**Proof.** In the case of $\mu = 1$, the relationship $u_0 - c_1 \geq \frac{c_2}{n}$ reveals $A \leq 0$ from Eq. (5.5) and with $\phi(t_o) \leq 0$. On the contrary, $u_0 - c_1 \leq \frac{c_2}{n}$ agrees with $B > 0$, accordingly we have $\phi(t_o) > 0$.

5.2 Numerical examples

This subsection presents numerical examples to illustrate the proposed model. Table 1 shows the optimal opening time, $t^o$, and its corresponding welfare, $\Psi(t^o)$, together with $t_L$ and $t_U$ against various values of $\mu$ and $\alpha$ when we set the parameters involved in the model as $(n, \tau, u_0, p, \omega, c_1, c_2, \beta) = (1, 0.05, 10, 9.49, 80, 4.99, 2.29, 0.35)$. It is observed in Table 1
that the optimal opening time, \( t^*_o \), satisfies \( t_L < t^*_o < t_U \) in the case of \( \mu = 0.25 \). In the other cases, we have \( t^*_o = t_L \). Table 1 indicates that as the distribution becomes closer to uniform, service hours maximizing the social welfare would increase towards “open 24 hours”.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.75</td>
<td>0.8</td>
<td>0.85</td>
</tr>
<tr>
<td>( t_L )</td>
<td>0.036</td>
<td>0.0301</td>
<td>0.0242</td>
</tr>
<tr>
<td>( t_U )</td>
<td>0.4203</td>
<td>0.4263</td>
<td>0.4322</td>
</tr>
<tr>
<td>( t^*_o )</td>
<td>0.1171</td>
<td>0.1149</td>
<td>0.1111</td>
</tr>
<tr>
<td>( \Psi(t^*_o) )</td>
<td>3.042</td>
<td>3.0358</td>
<td>3.0188</td>
</tr>
</tbody>
</table>

Figure 2 shows the shape of the social welfare, \( \Psi(t^*_o) \), for \( \mu = 0.25, 0.50 \) and 0.75 against \( \alpha = 0.8 \) with the other parameter values set to the same values in Table 1. It is also observed in Fig.1 that \( \Psi(t^*_o) \) has apparently its maximum when \( \mu = 0.25 \), while it is decreasing in \( t^*_o \) against \( \mu = 0.50, 0.75 \) taking its maximum at \( t^*_o = 0.0301 = t_L \).

6. Concluding Remarks
In this paper, we discussed an optimal number of business hours for a service provider, where he offers a special discounted price immediately after opening the store and just before closing it. Under a specific ideal service time distribution of consumers, derived
was the social welfare which is an objective function to be maximized. We clarified the conditions under which there exists an optimal opening time along with an optimal closing one. It was also shown that the optimal opening time decreases with increasing uniformity of the ideal time distribution. Numerical examples were also presented to illustrate the theoretical underpinnings of the proposed mathematical model.

The authors are to make a comparison between the results of this study and those in Kim and Sandoh(2015), which will appear in the forthcoming paper.

REFERENCES


