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Existence of Monetary Equilibrium for Overlapping Generations Economies with Satiated Agents*

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Abstract

This paper treats a general equilibrium existence problem on overlapping generations (OLG) economies with *satiated agents* and *fiat money.* The setting is important because it allows the analysis to incorporate Pareto optimality problems of the type where the *value functions* do not necessarily converge. For example, this is the case for optimal allocations such as a *basic income* under a *budget deficit*, which can only be realized with policies that require non-negative wealth transfers for all agents. Because of the existence of satiated agents, our argument can be identified with a *dividend equilibrium* (Aumann and Dreze 1986) or an *equilibrium with slack* (Mas-Colell 1992). For OLG settings, however, some conditions (like arguments for strictly positive endowments, resource relatedness and minimum wealth conditions with negative prices, and limit arguments for dividends) must be reconsidered. We note that by considering the OLG framework together with satiated agents, not only can the existence of a monetary equilibrium be assumed in advance, but the existence of a monetary equilibrium based on an arbitrary money supply is guaranteed. Taking into account that our setting includes production (without discount factor), the model here provides a basis for a wide range of difficult problems, including the player's (equilibrium-dependent) *survival problem*, multi-sectoral *capital accumulation*, bequests, firm formation, and so on.

Keywords: Dividend Equilibrium, Monetary Equilibrium, Overlapping Generations Economy, Satiation, Basic Income.

JEL Classification: C62, D51, E40

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1 Introduction

In this paper, we treat the general equilibrium existence problem on overlapping generations (OLG) economies with possibly satiated agents and fiat money. The setting is important to incorporate monetary equilibrium allocations that are possible only after non-negative wealth transfers are introduced.

Because of the existence of satiated agents, the equilibrium concept that we are concerned with can be identified with a *dividend equilibrium* or an *equilibrium with slack* (see Aumann and Drèze 1986 or Mas-Colell 1992). Equilibrium existence theorems on dividend equilibria for such finite economies are treated in Mas-Colell (1992), Kajii (1996), Allouch and Florenzano (2013) and so forth. For the OLG framework, however, further considerations and arrangements for conditions are necessary.

To investigate the existence of equilibrium with slack or money, the approaches by Mas-Colell (1992) and Kajii (1996) were confined to finite time horizon cases. Their approaches also take the minimum wealth condition as given or as trivial under a certain initial endowment interiority condition. For the OLG framework, however, we must reconsider several conditions like strictly positive initial endowments and several limit arguments for equilibrium states including negative prices. In this paper, we use the condition of social non-satiatedness (SNS) of preferences and extend the method in Balasko and Shell (1980) and Balasko et al. (1980) to assure the existence of an equilibrium for the OLG economy.

In section 2, rigorous definitions for the economies treated in this paper will be given. The main theorem of the existence of equilibrium is given in Section 3, where several discussions and remarks are also provided. The proof of the theorem is given in Section 4.

2 Economies

Let N be the set of all positive integers and R be the set of real numbers. For finite set A, denote by $\sharp A$ the number of elements of A. We denote by R^A, the $\sharp A$ dimensional Euclidean space $R^{\sharp A}$ and by R^A_+ and R^A_{++}, the non-negative and strictly positive orthants of R^A, respectively. If A is a subset of B, R^A is canonically identified with the subset of R^B.

An *overlapping-generations economy*, or more simply, an *economy*, $\mathcal{E}$, is comprised of the following list:

(E.1) $\{I_t\}_{t=1}^{\infty}$: a countable family of mutually disjoint non-empty finite subsets of N such that $\bigcup_{t=1}^{\infty} I_t = I = N$. I_t is the index set of agents in *generation* t for each $t \in N$.

(E.2) $\{K_t\}_{t=1}^{\infty}$: a countable family of non-empty finite integer intervals, $K_t = \{k(t), k(t)+1, \cdots, k(t)+\ell(t)\}$ such that $\bigcup_{t=1}^{\infty} K_t = K = N$, $k(t) < k(t+1) \leqq k(t)+\ell(t)$ for all $t \in N$. K_t is the index set of commodities available to generation t for each $t \in N$. We suppose that $\ell = \max_{t \in N} \ell(t)$ exists.

(E.3) $\{Y_t\}_{t=1}^{\infty}$: a sequence of technologies. Y_t is a compact convex subset of $R^{K(t)}$ having 0 as its element, where $K(t)$ denotes the set $\bigcup_{s=1}^{t} K_s$. We suppose that for each t, there exists $\bar{t}$ such that for all $s \geqq \bar{t}$, $Y_s \cap R^{K(t)} = \{0\}$. Denote by $Y(t)$ the summation $\sum_{s=1}^{t} Y_s$. We suppose that technology Y_t is owned only by agents in $I(t+\ell)$ for a positive integer ℓ, where $I(t)$ denotes the set $\bigcup_{s=1}^{t} I_s$.

(E.4) $\{(\precsim_i, \omega_i)\}_{i\in I}$: countably many agents, where $\precsim_i$ is continuous and convex in the sense of Debreu (1959) on the commodity space of each generation, $R_+^{K_t}$, for each $i \in I_t$ and $t \in N$. The *initial endowment* of i, ω_i, is an element of $R_+^{K_t} \setminus \{0\}$ for each $i \in I_t$ and $t \in N$.

(E.5) $\{\{m_i^t\}_{i\in I}\}_{t=1}^{\infty}$: $m_i^t(y_t, p)$ is a non-negative continuous profit distribution function for agent i with respect to technology Y_t, where we suppose that $y_t \in Y_t$, $p \in R^N$, and $\sum_{i\in I(t+\ell)} m_i^t(y_t, p) = p \cdot y_t$.[1] We assume that for each $i \in I_t$, $m_i^s(y_s, p) = 0$ for all $s > t + \ell$.

We note that in (E.4), no monotonicity conditions are assumed, so that preferences are allowed to be satiated.

In the following, it would be convenient to identify the commodity space for each generation $R_+^{K_t}$ with a subset of $R^K = R^N$, which is the set of all functions from $K = N$ to R, by considering $x \in R_+^{K_t}$ a function that takes value 0 on $N \setminus K_t$. Then we can define the total commodity space for economy $\oplus_{t=1}^{\infty} R_+^{K_t}$ as the set of all finite sums of points in commodity spaces of generations. Clearly, $\oplus_{t=1}^{\infty} R_+^{K_t}$ can be identified with a subset of direct sum R_∞, the set of all finite real sequences, which is a subspace of the set of all real sequences, $R^\infty \approx R^N$ under the pointwise convergence topology.

Given an economy, $\boldsymbol{\mathcal{E}} = (\{I_t\}_{t=1}^{\infty}, \{K_t\}_{t=1}^{\infty}, \{Y_t\}_{t=1}^{\infty}, \{(\precsim_i, \omega_i)\}_{i\in I}, \{\{m_i^t\}_{i\in I}\}_{t=1}^{\infty})$, the *price space* for $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{P}}$, is defined as the set of all p in R^N. A sequence, $(x_i \in R_+^{K_t})_{i\in I}$, is called an *allocation* for $\boldsymbol{\mathcal{E}}$. Allocation $(x_i \in R_+^{K_t})_{i\in I}$ is said to be *feasible* if there exists a sequence $\{y_t \in Y_t\}_{t=1}^{\infty}$ such that

$$\sum_{t\in N} \sum_{i\in I_t} x_i = \sum_{t\in N} y_t + \sum_{t\in N} \sum_{i\in I_t} \omega_i, \tag{1}$$

where the summability in $R^K = R^N$ of both sides of the equality is assured by (E.2) and (E.3). In equation (1), the no free disposability is assumed since we do not use any monotonicity condition for each agent. Under the attainability condition (1) and the condition on Y_s and $R^{K(t)}$ in (E.3), we can obtain for each $i \in I_t$ a compact convex set $X_i \subset R_+^{K_t}$ such that every feasible allocation for i is a relative interior point of X_i, so that for individual maximization problem there is no loss of generality (under the convexity of preferences) to restrict i's consumption set to X_i instead of $R_+^{K_t}$.[2]

The list of price vector $p^* \in \boldsymbol{\mathcal{P}}$, non-negative slack variables $(M_i^* \in R_+)_{i\in I}$, and an allocation $(x_i^* \in R_+^{K_t})_{i\in I}$ feasible under $(y_t^* \in Y_t)_{t\in N}$, is called a *dividend equilibrium state* for $\boldsymbol{\mathcal{E}}$, if for each $t \in N$, $p^* \cdot y_t^* \geqq p^* \cdot y_t$ for all $y_t \in Y_t$, and for each $i \in I_t$, x_i^* is a $\precsim_i$-greatest

[1] In this paper, we use the inner product notation even for two infinite dimensional vectors in $R^N \approx R^\infty$ as long as one of which can be identified with a finite vector (through the identification like $R^{K_t} \subset R_\infty \subset R^\infty$ explained in the next following paragraphs), and the summability for the coordinate products is clearly warranted. For the continuity of m_i^s, we take the topology of $R^N \approx R^\infty$ as the product (pointwise convergence) topology.

[2] In this paper, for the sake of simplicity, we assume that each Y_t, $t = 1, 2, \ldots$, is compact. It follows that we can take such compact X_i for each i naturally from the feasibility condition (1). For more general cases, however, we have to ensure the compactness of attainable set by using conditions on asymptotic cones like Debreu (1959). The compactness of Y_t and X_i should be derived from such conditions.

element in the set

$$\{x_i \in X_i \subset R_+^{K_t} \mid p^* \cdot x_i \leqq p^* \cdot \omega_i + \sum_{s=1}^{t+\ell} m_i^s(y_s^*, p^*) + M_i^*\}. \tag{2}$$

Since the non-negative wealth transfer is possible to be identified with the money supply in perfect-foresight economies, a dividend equilibrium allocation can also be called a *monetary Wlaras allocation* for $\mathcal{E}$.

In order to assure the existence of equilibrium for OLG economies, a certain kind of resource relatedness condition is necessary. For this purpose, instead of the monotonicity of preferences, we assume the following condition.

(**SNS**: Socially Non-Satiated preference configuration)
For all generation t, for each commodity $k, s \in K_t$ and feasible allocation $x = (x_i)_{i \in I}$, there exists at least one agent $i \in I_t$ having a positive endowment of commodity s and non-empty open interval $(0, \epsilon) \subset R_+$ satisfying one of the following two conditions: (1) for every $\delta \in (0, \epsilon)$, $x_i + (0, \cdots, 0, \delta, 0, \cdots)$ is strictly preferred to x_i, or (2) for every $\delta \in (0, \epsilon)$, $x_i + (0, \cdots, 0, -\delta, 0, \cdots)$ is strictly preferred to x_i, where the non-zero entry of the additional vector, δ or $-\delta$, is the k-th coordinate.

The condition for a finite satiation economy with non-negative price settings was introduced in Murakami and Urai (2017) and Murakami and Urai (2019) as one of the simplest method to assure the *resource relatedness* for the entire economy. Condition SNS is automatically satisfied when for each generation t, there is at least one agent $i \in I_t$ whose preference is strictly monotonic and $\omega_i \in R_{++}^{K_t}$. On the other hand, the condition does not satisfied if there exists a commodity k such that under a certain feasible allocation $(x_i)_{i \in I}$, all agents possible to consume commodity k are satiated with k. We also note that SNS means that for all $t \in N$, $\sum_{i \in I_t} \omega_i \in R_{++}^{K_t}$.

3 Existence of Dividend Equilibrium

For each $T \in N$, denote by $\mathcal{E}(T)$ a finite truncated economy that consists of all agents in $I(T) = \bigcup_{s=1}^{T} I_s$. The proof of the existence theorem is essentially equivalent to the usual diagonal argument (see, Balasko-Call-Shell 1980, etc.), which derives the equilibrium of the total economy $\mathcal{E}$ as a limit of a sequence of equilibria for truncated economies. There are some difficulties, however, associated with the necessity to include preference saturation and the possibility of negative prices, together with the resource relatedness (SNS) condition specific in this paper. In particular, we use the Mas-Colell (1992) theorem to prove the existence of an equilibrium in economy $\mathcal{E}(T)$, but problems specific to OLG models (especially, in guaranteeing minimum wealth conditions for agents whose initial endowments cannot be strictly positive) prevent the theorem from a direct application (see, for example, equation (5) in section 4.2). We have the following theorem.

Theorem 1 . *Under (E.1)-(E.5) and SNS conditions, an overlapping generations economy $\mathcal{E}$ has a dividend equilibrium.*

Proof: See Section 4. ■

Discussion 1 (Basic Income Policy)

Nowadays, government intervention for the purpose of redistribution to a certain entity has become more and more difficult and is not working practically. For example, resistances from corporations and rich people would be strong, and from the public-finance perspective, tax distortions are not desirable. This study focuses on government basic income-like spending policies as a possible new redistribution. Above all, this has a lump sum tax aspect and does not cause distortion (i.e., non-negative transfers could be accepted with lower resistances by all economic agents). By using the above theorem, we can say that there is an equilibrium and that the weak Pareto optimality in the sense of Balasko and Shell (1980) is always assured (the allocation cannot be improved by any finite agents). Of course, we have many Pareto optimal examples in OLG economy that are possible only after positive money supply (negative budgetary deficit of the government) is introduces. Under certain criteria, the equilibrium with BI is better. (This is where we need to extend the model to say that our new redistribution policy is properly supported by microeconomics and we need to expand the model.)

Discussion 2 (Capital Accumulation)

We also note that by considering the OLG framework together with the existence of satiated agents, we can easily introduce *bequests* and *capital accumulations* into the model. Indeed, if the existence of satiated agents and their consumptions are followed by appropriate non-negative wealth transfers to others, the situation cannot be distinguished from the existence of bequests as long as the determination problem of the heirs are exogenously given. It is also possible to understand that such bequests are bought by the government (so satiated agents are convinced in that their bequests are saved as government-bank deposits), lended to the next-generation technology, transformed to commodities in the future (including capital goods), and the capital accumulation (by using the fiat money or the fiscal deficits) takes place. For such circumstances, it is easy to construct Pareto optimal monetary equilibrium examples under which fiscal deficits (government bonds) do not necessarily be redeemed. (To obtain such Pareto optimal allocations, the fiscal deficits for eternal time periods are necessary.)

Discussion 3 (Survival Problem)

The framework of preference saturation and government dividends provides an interesting open question regarding the so-called general equilibrium with survival problem. Here, the survival problem is a general equilibrium existence problem in which, under some price conditions, the survival conditions of some of the consumers in the economy are not satisfied (i.e., the budget constraint set is empty under some price). Suppose that a group of people in the economy are able to saturate everyone under their initial holdings and technology, and that the surplus of output under this saturated situation is sufficient to survive all the rest of the people in the economy. Can the government's dividend policy then guarantee the existence of dividend equilibrium in the economy with a survival problem? We also note that

Discussion 4 (General Dynamic Game)

The first thing that characterizes our model from the viewpoint of infinite-horizon dynamic games is that it can handle problem setups and equilibrium paths in which the objective function and value function are not necessarily finite. Needless to say, this means that solution paths that cannot be captured by dynamic programming based on the Bellman equation are also subject to analysis. Second, while the above is well known in economic models with overlapping generations for consumption, in this paper it is treated in a general way (without the use of discount factors) by including production. Although individual production activities are directly depicted as closed within a finite period, they overlap generationally, so that the model introduces technology that spans an infinite number of periods as a whole. Thus, our model can serve as a basis for asking more general questions from a microeconomic standpoint toward capital accumulation, multi-sector growth issues, and in some cases, firm formation and industrial structure. Third, the government is introduced as the money-issuing entity, but its role is approximately minimal in an economic context, including the fact that the value of money may be zero (its value is determined by equilibrium). However, as a quid pro quo for introducing this special player, it is possible to deal with the survival problem as discussed in Discussion 3. This means that the setting can include situations where the players' choice set is empty, depending on how the equilibrium is (such as choices under prices), and allows for a more diverse treatment of hard-to-handle issues related to the game structure itself (such as bankruptcy and default).

4 Proof

4.1 Setting Truncated Economy till generation T

Let $\mathcal{E}(T) = (\{I_t\}_{t=1}^T, \{K_t\}_{t=1}^T, \{Y_t\}_{t=1}^T, \{\{(\precsim_i, \omega_i)\}_{i\in I(T)}, \{m_i^t\}\}_{i\in I(T)}\}_{t=1}^T)$ be an economy truncated at a generation T. Let h be an index for a good, and let p_h be a price for the good h. We denote by $H(T)$ the total number of goods till generation T, i.e., $H(T) = \sharp K(T) = k(T) + \ell(T)$. Price vectors for $\mathcal{E}(T)$ will be denoted like $p(T)$ or $p^*(T)$, and we define sets of prices, $\mathcal{P}(T)$, $\mathcal{P}^*(T)$ and $\mathcal{P}^\infty(T)$ as follows:

- $p(T) = (p_1, p_2, \cdots, p_h, \cdots, p_{H(T)})$,
- $p^*(T) = (p_1^*, p_2^*, \cdots, p_h^*, \cdots, p_{H(T)}^*)$,
- $\mathcal{P}(T) = \{p(T) \in R^{H(T)} | ||p(T)|| \leq 1\}$,
- $\mathcal{P}^*(T) = \{p^*(T) \in R^{H(T)} \backslash \{\mathbf{0}\} | ||p^*(T)|| \leq 1\}$,
- $\mathcal{P}^\infty(T) = \{(p_1^*, p_2^*, \cdots, p_{H(T)}^*, p_{H(T)+1}, \cdots) \in \mathcal{P}^*(T) \times R \times R \times R \cdots \mid ||p^*(T)|| \leq 1\}$.

Here, each production set is assumed to be compact.[3] So, naturally, each consumption set, X_i, in the economy can also be assumed to be compact. This is because the production plan is finite, and, therefore, consumers can consume at most finite amounts in total.

In the following proof, we first restrict our argument to the truncated economy $\mathcal{E}(T)$.[4]

4.2 Budget constraint modification

To show the existence of an equilibrium for $\mathcal{E}(T)$, we have to modify the budget constraint in equation (2). Let us introduce a modified budget set for individual $i \in I_t \subset I(T)$ for price $p(T) \in \mathcal{P}(T)$, as follows:

- If $||p(T)|| \neq 0$, the modified budget set is described as

$$\{x_i \in X_i \subset R_+^{K_t} \mid p(T) \cdot x_i \leqq p(T) \cdot \omega_i + \sum_{s=1}^{T} m_i^s(y_s, p(T)) + \frac{1 - ||p(T)||}{||p(T)||}\}. \tag{3}$$

- Otherwise, it is

$$\{x_i \in X_i \subset R_+^{K_t} \mid p(T) \cdot x_i \leqq p(T) \cdot \omega_i + \sum_{s=1}^{T} m_i^s(y_s, p(T)) + \mathbb{A}\}, \tag{4}$$

where $\mathbb{A}$ is taken to be a large constant such that it covers the whole consumption set X_i for all $i \in I(T)$. Since X_i is compact and $I(T)$ is finite, an easy calculation shows that we can take $\mathbb{A}$ as a real number greater than the diameter of set X_i for all $i \in I(T)$.[5]

Adjustment term $\frac{1-||p(T)||}{||p(T)||}$ is a device to ensure the *minimum wealth* condition (Mas-Colell 1992, condition V) at every price whose norm is less than 1 (including the point, $||p(T)|| = 0$). For prices such that $||p(T)|| = 1$, however, we need a further elaboration since by the OLG framework, it is not appropriate to assume that an initial endowment for i is strongly positive.[6] So in order to ensure the minimum wealth condition for all $i \in I(T)$, some income redistribution is necessary. Since every ω_i is semi-positive, the minimum wealth condition is always satisfied as long as $p(T)$ is strictly positive. The minimum wealth condition is also satisfied for all agent as long as $p(T)$ has at least one negative coordinate. Therefore, the problem occurs only for points near to price $p(T)$ such that $\|p(T)\| = 1$ and $p(T)$ has at least one coordinate equal to 0, the relative boundary of $\{p(T) \in \mathcal{P}(T) | \, \|p(T)\| = 1\} \cap R_+^{K(T)}$ in the surface of $\mathcal{P}(T)$.

Here, we can utilize condition (SNS). As we shall see in subsection 4.6, for economy $\mathcal{E}(T)$, there is a positive lower bound b_l and upper bound b_u such that $|p_k|/|p_s| \in [b_l, b_u]$ for all $k, s \in K(T)$ as long as $p(T) = (p_1, p_2, \ldots, p_{H(T)}) \in \mathcal{P}(T)$ such that $\|p(T)\| = 1$ is an equilibrium price for $\mathcal{E}(T)$. This necessary condition for equilibrium prices also provides a minimum absolute value $\epsilon_s > 0$ for each coordinate, $s \in K(T)$. Indeed, if $|p_s| < \epsilon$, there

[3] We do not have to assume that the production set is compact. Assumptions on production sets based on their asymptotic cones like Debreu (1959) would be sufficient.

[4] We first focus on the truncated economy till time T. However, Balasko and Shell (1980) consider an economy with an infinite horizon setting directly. See, Balasko et al. (1980) for a more detailed treatment.

[5] In our setting, we have to consider the cases that $||p(T)|| = 0$ because we have to consider negative prices (the price space, $\mathcal{P}(T)$, is taken to be a ball shape containing $\{\mathbf{0}\}$).

[6] Only assumed in (E.4) is that for each $i \in I_t$, $\omega_i \in R_+^{K_t} \setminus \{0\}$ though we know by (SNS) that $\sum_{i \in I_t} \omega_i \in R_{++}^{K_t}$.

must exist at least one $k \in K(T) \setminus \{s\}$ such that $|p_k|$ is greater than $(1-\epsilon)/(H(T)-1)$.[7] Then, by taking $\epsilon > 0$ to be sufficiently small, we have $|p_k|/|p_s| > b_u$, a contradiction. Define $\epsilon > 0$ as the minimum of $\{\epsilon_s | \, s \in K(T)\}$. Denote by B the relative boundary of set $\{p(T) \in \mathcal{P}(T) | \, \|p(T)\| = 1\} \cap R_+^{K(T)}$ in the surface of the closed unit ball, $\mathcal{P}(T)$. Let U_ϵ be the open ϵ neighborhood of 0 and denote by $U(B)$ the open ϵ neighborhood, $U_\epsilon + \partial B$, of B. Note that every $p(T) \in U(B) \cap \mathcal{P}(T)$ cannot be an equilibrium price for $\mathcal{E}(T)$. Since $\mathcal{P}(T)$ is closed, set $\mathcal{P}(T) \setminus U(B)$ is closed, so for each $p(T) \in U(B)$, the distance between $p(T)$ and $\mathcal{P}(T) \setminus U(B)$, $d(p(T)) = dist(p(T), \mathcal{P}(T) \setminus U(B))$ is strictly positive (and less than or equal to ϵ). Let $\omega(T) \in R_{++}^{K(T)}$ (by SNS condition) be the vector $(\sum_{i \in I(T)} \omega_i)/\sharp I(T)$. Now, based on the modified budget set (3) and (4), we further modify budget set for $i \in I_t \subset I(T)$ under production $\{y_s\}_{s=1}^T$, by defining income function $m_i(\{y_s\}_{s=1}^T, p(T))$ as follows:

$$p(T) \cdot \left(\frac{\epsilon - d(p(T))}{\epsilon} \omega_i + \frac{d(p(T))}{\epsilon} \omega(T) \right) + \sum_{s=1}^{T} m_i^s(y_s, p(T)) = m_i(\{y_s\}_{s=1}^T, p(T)), \quad (5)$$

where m_i can be identified with the same notation in Mas-Colell (1992). So, we can directly apply the way employed in Mas-Colell (1992) for the existence of equilibrium prices for $\mathcal{E}(T)$. Note that if $p(T)$ belongs to $\mathcal{P}(T) \setminus U$, the left hand side of equation (5) is equivalent to $p(T) \cdot \omega_i + \sum_{s=1}^T m_i^s(y_s, p(T))$ in equations (3) and (4). For $p(T)$ in U, the summation $\sum_{i \in I(T)} p(T) \cdot \left(\frac{\epsilon - d(p(T))}{\epsilon} \omega_i + \frac{d(p(T))}{\epsilon} \omega(T) \right)$ is equal to $p(T) \cdot \sum_{i \in I(T)} \omega_i$, so m_i is nothing but a continuous income redistribution that is strictly positive for all agents (satisfying the minimum wealth condition V of Mas-Colell 1992).

4.3 Existence of Equilibrium for $\mathcal{E}(T)$

For economy $\mathcal{E}(T) = (\{I_t\}_{t=1}^T, \{K_t\}_{t=1}^T, \{Y_t\}_{t=1}^T, \{\{(\precsim_i, \omega_i)\}_{i \in I(T)}, \{m_i^t\}_{i \in I(T)}\}_{t=1}^T)$, a dividend equilibrium till generalization T is defined as a triple, $(p^*(T), \{\{x_i^*\}_{i \in I_t}\}_{t=1}^T, \{y_t^*\}_{t=1}^T)$ which satisfies the following conditions: (6) *feasibility*, (7) *profit maximization* and (8) *utility maximization*.

$$\sum_{t \in T} \sum_{i \in I_t} x_i^* = \sum_{t \in T} y_t^* + \sum_{t \in T} \sum_{i \in I_t} \omega_i, \quad (6)$$

$$y_t^* \in Y_t, \text{ and } p_t^* y_t \leq p_t^* y_t^* \text{ for all } y_t \in Y_t, \text{ for all } t = 1, \ldots, T. \quad (7)$$

For all $i \in I_t$ and $t = 1, \ldots, T$, x_i^* is a $\precsim_i$-greatest element in $X_i \subset R_+^{K_t}$ under the budget,

$$p^*(T) \cdot x_i \leq p^*(T) \cdot \omega_i + \sum_{s=1}^{T} m_i^s(y_s^*, p^*(T)) + \frac{1 - ||p^*(T)||}{||p^*(T)||}. \quad (8)$$

The proof is directly from Mas-Colell (1992) by using the modification (equation (5)) of consumer budgets. The last term $\frac{1-||p^*(T)||}{||p^*(T)||}$, can be identified with an equilibrium *monetary* (non-negative wealth) transfer to consumer $i \in I(T)$, $M_i^*(T)$, which is automatically equally to all consumers in $I(T)$ in our present setting. As will be discussed in subsection 4.8, the distribution ratio of money for each agent, $i \in I(T)$, can be set freely, by using parameter $Z_i \in R_{++}$ and considering term $Z_i \frac{1-||p^*(T)||}{||p^*(T)||}$ instead of $\frac{1-||p^*(T)||}{||p^*(T)||}$ for budget constraints in (3) and (8).[8]

[7] If not, the norm of $p(T)$ cannot be greater than or equal to 1.

[8] Since the term has the role to ensure the minimum wealth condition (so that the continuity of budget

4.4 Eliminate the case $\exists h$, $p_h^* = 0$ for economy $\mathcal{E}(T)$

If $p_h^*, h \leq H(T)$ is 0, from SNS condition, $\exists\ i \in I_t \subset I(T)$, $h \in K_t$ such that i has a positive initial endowment of $s \in K_t$ and $p_h = 0$ can never be a supporting price for i at every attainable allocation. It follows that till generation T, a competitive equilibrium (CE) price, $p^*(T)$ for $\boldsymbol{\mathcal{E}}(T)$ cannot contain a coordinate 0. Especially, CE price space till generation T, $\boldsymbol{\mathcal{P}}^*(T)$, does not contain 0.

$$\boldsymbol{\mathcal{P}}^*(T) = \{p^*(T) \in R^{H(T)} \backslash \{\mathbf{0}\} | ||p^*(T)|| \leq 1\}. \tag{9}$$

4.5 Bounds using price ratio

We need to derive an upper and lower bound of the competitive price for accommodating infinite price sequence.[9] Let me define $\overline{x_i}^+ = x_i^* + (\cdots, \delta_k, \cdots)$ or $\overline{x_i}^- = x_i^* + (\cdots, -\delta_k, \cdots)$, where $\delta_k, \in R_{++}$.

From SNS condition, $\exists i \in I_t$ in every generation t, for his consumption bundle $x_i \in R_+^{H(T)}$, we can take strictly preferred $\overline{x}_i^+$ or $\overline{x}_i^-$ by taking δ_k on any good k. Moreover, continuity of preference ensures that, on open ball $B^+_{r_k(x_i)}(\overline{x}_i^+) = \{x_i^{'} \in R_+^{H(T)} \mid d(x_i^{'}, \overline{x}_i^+) < r_{k(x_i)}\}$ or $B^-_{r_k(x_i)}(\overline{x}_i^-) = \{x_i^{'} \in R_+^{H(T)} \mid d(x_i^{'}, \overline{x}_i^-) < r_{k(x_i)}\}$. Note that $r_{k(x_i)} \in R_{++}$ is dependent on x_i and $\overline{x}_i^+$ or $\overline{x}_i^-$. When $r_{k(x_i)}$ is considerably small, within its open ball, we can pick up any $x_i^{'}$ that satisfies $x_i^{'} \succ_i x_i^{''}$. Note that $x_i^{''}$ is any arbitrary point within the open ball $B_q(x_i) = \{x_i^{''} \in R^{H(T)} \mid d(x_i^{''}, x_i^*) < q\}$.

The bounds of CE prices are decided so that the hyperplane through $B_q(x_i)$ made by the CE price vector does not cross $B^+_{r_k(x_i)}(\overline{x}_i^+)$ or $B^-_{r_k(x_i)}(\overline{x}_i^-)$. That is, for the individual, at $x_i^* \in R_+^{H(T)}$, he can pick up more desired consumption point. This is because from SNS condition under any $\delta_k \in R_{++}$, $\overline{x_i}^+ = x_i^* + (\cdots, \delta_k, \cdots)$ or $\overline{x_i}^- = x_i^* + (\cdots, -\delta_k, \cdots)$ would be also in his feasible set. It is contradiction to the property that x_i^* is a maximum point.

As follows, we prove formally for any goods pair s and k, the CE price rate is bounded. NOTE THAT the proof is incomplete and we will check it later.

4.5.1 x_{is}^* is inner or corner of consumption set

NOTE THAT we need to check this subsection later.

This is the case where x_{is}^* is inner of consumption set

Under CE price till time T, $p^*(T) = (p_1^*, p_2^*, \cdots, p_T^*)$, for $i \in I(T)$ who falls into individual with SNS condition, $p^*(T)$, should be at least on the region where hyperplane $h_i^{x_i^*}$, $h_i^{x_i^*} = \{x_i \in R_+^N \mid p^*(T)(x_i^* - x_i) = 0\}$ satisfies $h_i^{x_i^*} \cap B_{r_k(x_i^*)}(\overline{x}_i^+) = \phi$ or $h_i^{x_i^*} \cap B_{r_k(x_i^*)}(\overline{x}_i^-) = \phi$. Note that this creates "biggest" upper bound. (Here we use property of SNS condition that when $x_{ik}^* = 0$ better set is located at right side for sure. Otherwise just applying SNS condition does not help restrict the CE price space.

correspondence) for each agent at $p(T) = 0$, we cannot set Z_i to be equal to 0, as long as we use the method of Mas-Colell (1992)

[9] Only eliminating the case $\exists h$ $p_h^* = 0$ is not enough because we need closeness. Intuitively, eliminating $\exists h$ $p_h^* = 0$ leads the space to be open.

If $x^*_{is} = 0$ in this case the same argument, that is, the hyperplane through x^*_i does not help because $x^*_{is} = 0$ (in this case better set is not ball but hemisphere). Then, we use the SNS condition saying initial endowment of goods s is strictly positive. So, instead of using the hyperplane through x^*_i, we use the hyperplane through individual i's initial endowment. Always individual i can consume his initial endowment. Therefore, at least under the CE price the hyperplane through initial endowment $h_i^{\omega_i} = \{x_i \in R_+^N \mid p^*(T)(\omega_i - x_i) = 0\}$ does not intersect with his better set. (NOTE THAT this is also "biggest" upper bound. This means there is term $\sum_s^{t+l} m_i^s(y^s, p(T)) + \frac{1-||p(T)||}{||p(T)||}$. So, possible hyperplane is steeper also.

Particularly, We mention how to get the bound of arbitrary price ratio. Focusing on arbitrary two goods, let them goods s and k without loss of generality. Imagine exchange of the two goods through market. From SNS condition, $\forall t \in T$ there is at least one non-satiated individual. He strictly prefer $\overline{x}_i^+$ or $\overline{x}_i^-$ to $x_i^* = (x^*_{i1}, x^*_{i2}, \cdots, x^*_{ik}, x^*_{is} \cdots, x^*_{iT})$. Of course He can move $-p^*_k/p^*_s$ slope, exchanging unites of goods k instead of unites of s through market. We can regard this price rate as a kind of hyperplane through x^*_i or ω_i (this is corner solution case). If x^*_i is CE allocation, that $|p^*_k/p^*_s|$ holding CE is restricted to where the hyperplane passing through x^*_i (or ω_i), $h_i^{x^*_i}$ (or $h_i^{\omega_i}$), must not interact with $B^+_{r_k(x^*_i)}(\overline{x}_i^+)$ or $B^-_{r_k(x^*_i)}(\overline{x}_i^-)$. Otherwise that individual can achieve strictly preferred consumption, which is contradict to the condition of CE allocation.

4.5.2 Details

NOTE THAT we need to check this subsection later.

There are three cases (1). x^*_i is located on the inner of his consumption set, (2). x^*_i is located corner, while $x^*_{is} > 0$ and (3). x^*_i is located corner, while $x^*_{is} = 0$.

- (1). x^*_i is located on the inner and (2). x^*_i is located corner, while $x^*_{is} > 0$

We can use the discussion we mentioned above, open ball argument. So, we omit detailed explanation. See the above discussion.

- (3). x^*_i is located corner, while $x^*_{is} = 0$.

In this case, we need careful attention. As follows, we describe in detail. Suppose x^*_i is located corner while $x^*_{is} = 0$ (case (3)), the possible GE prices are the 4 cases. Moreover, we have to take into account the better set (left-sided or right-sided). In sum, suppose $p^*(T)$ and (x^*_i, y^*_t) are CE price and consumption for individual i with SNS condition, there are 8 possible cases.

- a. $p^*_k > 0$ $p^*_s > 0$ and better set is right-sided
- a'. $p^*_k > 0$ $p^*_s > 0$ and better set is left-sided
- b. $p^*_k > 0$ $p^*_s < 0$ and better set is right-sided
- b'. $p^*_k > 0$ $p^*_s < 0$ and better set is left-sided

- c. $p_k^* < 0$ $p_s^* > 0$ and better set is right-sided
- c'. $p_k^* < 0$ $p_s^* > 0$ and better set is left-sided
- d. $p_k^* < 0$ $p_s^* < 0$ and better set is right-sided
- d'. $p_k^* < 0$ $p_s^* < 0$ and better set is left-sided

We check whether we can get upper bound one by one in this subsection. Note that in every case it is enough to check the normal vector of $-p_k/p_s$ slope passing through $\omega_i = (\cdots, \omega_{is} > 0, \cdots) \in R_+^T$ and $(\cdots, x_{ki}^* + \epsilon - r, 0, \cdots) \in R_+^T$. This is because, even if there is additional term $\sum_s^{t+l} m_i^s(y^s, p(T)) + \frac{1-||p(T)||}{||p(T)||}$ in his budget constraint, the normal vector through $\omega_i = (\cdots, \omega_{is} > 0, \cdots) \in R_+^T$ must be steeper.

That is why in the following we only focus on the normal vector through ω_i.

Case a.: From strictly positiveness of ω_{is}, the $|p_k^*/p_s^*|$ is restricted as the a normal vector of ω_i does not intersect with the better set $B^+_{r_k(x_i^*)}(\overline{x}_i^+)$. Otherwise, individual i with SNS condition have incentive to change his GE consumption x_i^*. This is contradiction.

Case a'.: In this case the better set $B^-_{r_k(x_i^*)}(\overline{x}_i^-)$ is located within his feasible set at first, because $p_k^* > 0$ $p_s^* > 0$ means feasible set is left side of the normal vector. So, it is impossible that x_i^* is maximum point at first.

Case b.: From strictly positiveness of ω_{is}, the $|p_k^*/p_s^*|$ is restricted as the a normal vector of ω_i does not intersect with the better set $B^+_{r_k(x_i^*)}(\overline{x}_i^+)$. Otherwise, individual i with SNS condition have incentive to change his GE consumption x_i^*. This is contradiction.

Case b'.: In this case the better set $B^-_{r_k(x_i^*)}(\overline{x}_i^-)$ is located within his feasible set at first, because $p_k^* > 0$ $p_s^* < 0$ means feasible set is left side of the normal vector. So, it is impossible that x_i^* is maximum point at first.

Case c.: In this case the better set $B^+_{r_k(x_i^*)}(\overline{x}_i^+)$ is located within his feasible set at first, because $p_k^* < 0$ $p_s^* > 0$ means feasible set is right side of the normal vector. So, it is impossible that x_i^* is maximum point at first.

Case c': From strictly positiveness of ω_{is}, the $|p_k^*/p_s^*|$ is restricted as the a normal vector of ω_i does not intersect with the better set $B^-_{r_k(x_i^*)}(\overline{x}_i^-)$. Otherwise, individual i with SNS condition have incentive to change his GE consumption x_i^*. This is contradiction.

Case d: In this case the better set $B^+_{r_k(x_i^*)}(\overline{x}_i^+)$ is located within his feasible set at first, because $p_k^* < 0$ $p_s^* < 0$ means feasible set is right side of the normal vector. So, it is impossible that x_i^* is maximum point at first.

Case d': From strictly positiveness of ω_{is}, the $|p_k^*/p_s^*|$ is restricted as the a normal vector of ω_i does not intersect with the better set $B^-_{r_k(x_i^*)}(\overline{x}_i^-)$. Otherwise, individual i with SNS condition have incentive to change his GE consumption x_i^*. This is contradiction.

So, suppose $p^*(T)$ and (x_i^*, y_t^*) are CE price and consumption for individual i with SNS condition, only possible cases are a, b, c' or d'. In each case we can derive upper bound, as we will show in the "examples of such bounds" section.

4.5.3 Examples of such bounds

NOTE THAT we need to check this subsection later.
(Just to be sure, here we give a few examples of such bounds)

- Case.1 : x_{is}^* is inner of consumption set and better set is right

Let $x_i^* = (\cdots, x_{ik}^*, x_{is}^*, \cdots) \in R_+^T$ without loss of generality. When we focusing on goods k and s, we can always, from SNS condition, take better set $B_{r_k(x_i^*)}^+(\overline{x}_i^+) = \{x_i^{'} \in R_+^T \mid d(x_i^{'}, \overline{x}_i^+) < r_{k(x_i^*)}\}$,here $\overline{x}_i^+ = x_i^* = (\cdots, x_{ik}^* + \epsilon, x_{is}^*, \cdots)$. So, at CE the bounds of price rate p_k^* and p_s^* is decided in the following way (from elementary geometry)

$$||(\cdots, 0, p_k^*, p_s^*, 0, \cdots) + x_i^* - \overline{x}_i^+|| = \epsilon^2 - r^2 \wedge ||(\cdots, 0, p_k^*, p_s^*, 0, \cdots)|| = r.$$

So, given x_i^* and ϵ, r,

$$|p_k^*/p_s^*| = \frac{r}{\epsilon\sqrt{1-\epsilon}}$$

- Case.2 : $x_{is}^* = 0$ and better set is right

Let $\omega_i = (\cdots, \omega_{is} > 0, \cdots)$. In this situation, the ratio of p_k^* and p_s^* is decided as follows,

$$(\cdots, 0, p_k^*, p_s^*, 0, \cdots) \bullet ((\cdots, x_{ik}^* + \epsilon - r, 0, \cdots) - \omega_i) = 0.$$

So, given x_i^* and ϵ, r,

$$|p_k^*/p_s^*| = \frac{x_{ik}^* + \epsilon - r - \omega_{ik}}{\omega_{is}}$$

Similarly, we can derive CE price rate of other cases like better set is left side.
So we omit the rest.

4.6 Relationship between generation T and $T+1$

Let $b_l^{s,k}(T)$ and $b_u^{s,k}(T)$ be lower and upper bounds of the price ratio between goods $s \in K(T)$ and $k \in K(T)$ within which every CE price ratio till generation T must belong.

Next, for $T+1$, if we obtain the same two bounds, $b_l^{s,k}(T+1)$ and $b_u^{s,k}(T+1)$, we can replace them by $b_l^{s,k}(T)$ and $b_u^{s,k}(T)$, since if "p_k/p_s is not a CE price ratio (supporting all agents' optimal consumption vectors) till generation T," then it is clear that "p_k/p_s is not a CE price ratio (supporting all agents' optimal consumption vectors) till generation $T+1$," so the upper and lower bounds, $b_l^{s,k}(T)$ and $b_u^{s,k}(T)$, also defines the interval that CE price ratio p_k/p_s till generation $T+1$ must belong.

Hence, for each two commodities s and k, by considering the first T such that $s \in K(T)$ and $k \in K(T)$, we obtain the lower and upper bounds of the price ratio between goods s and k, $b_l^{s,k} = b_l^{s,k}(T)$ and $b_u^{s,k} = b_u^{s,k}(T)$, within which every CE price ratio (p_k/p_s) must belong.

Let us summarize the above by using a set-theoretic form.

For each s and k,

$$\mathcal{R}_{s,k}(T+1) \subseteq \mathcal{R}_{s,k}(T),$$

where $\mathcal{R}_{s,k}(T)$ is defined as

$$\mathcal{R}_{s,k}(T) = \{|p_k^*/p_s^*| \, | \, p_k^*/p_s^* \text{ is a CE price ratio till generation } T\}, \tag{10}$$

and

$$\mathcal{R}_{s,k}(T+1) = \{|p_k^*/p_s^*| \, | \, p_k^*/p_s^* \text{ is a CE price ratio till generation } T+1\}. \tag{11}$$

As is mentioned in section 4.2, we can use such lower and upper bounds for each commodity-price ratio to define a compact range for each commodity where general equilibrium prices (if such exist) necessarily belong. One of the simplest ways is to normalize the norm of the first coordinate of a price, p_1^*, which cannot be 0 as stated in section 4.4, to be 1. Now, we can define for each commodity $k = 1, 2, ...,$ a compact interval,

$$[a_k, b_k] \subset R, \; 0 < a_k \leqq b_k, \; k = 1, 2, \ldots, \tag{12}$$

such that the norm (the absolute value) of the k-th coordinate $|p_k^*|$ must belong to $[a_k, b_k]$ as long as $p^* = (p_1^*, p_2^*, \ldots)$ is a candidate for an equilibrium price. (Of course we can define a_1 and b_1 as $a_1 = b_1 = 1$.)

4.7 Limits of $x_i^*(T)$, $y_t^*(T)$ and $M_i^*(T)$

About the limits of competitive prices when $T \to \infty$, from the above discussion and by Tychonoff's theorem, we can obtain a compact set for CE prices (normalized their first coordinates as norm 1) as

$$\mathcal{P}^* = \{p = (p_1, p_2, \ldots) \in R^\infty | \, |p_k| \in [a_k, b_k] \text{ for all } k = 1, 2, \ldots\}. \tag{13}$$

For each $T = 1, 2, ...,$ and T-generation economy $\mathcal{E}(T)$, we have an equilibrium price $p^*(T)$ in $\mathcal{P}^*(T)$ identified with a subset of R^∞ through $\mathcal{P}^\infty(T) \subset R^\infty$. So, we can take a convergent subsequence of $\{p^*(T)\}_{T=1}^\infty$ in its compact competitive price set $\mathcal{P}^*$ when $T \to \infty$.[10] Note that the topology of $\mathcal{P}^* = \prod_{k=1}^\infty ([-b_k, -a_k] \cup [a_k, b_k])$ assured by Tychonoff's theorem for compactness is the product topology that is equivalent to the relativized topology to $\mathcal{P}^*$ of R^∞. In other words, without loss of generality (since the choice of finite truncation economies is arbitrary), we may identify $\{p^*(T)\}_{T=1}^\infty$ with a pointwise (coordinatewise) convergent sequence in R^∞ to a certain $p^* = (p_1^*, p_2^*, \ldots) \in \mathcal{P}^*$.

With the convergent sequence, $\{p^*(T)\}_{T=1}^\infty$, is associated three kinds of sequences of T-period equilibrium states, $\{x_i^*(T)\}_{T=1}^\infty$ for each $i \in I$, $\{y_t^*(T)\}_{T=1}^\infty$ for each $t = 1, 2, \ldots,$

[10] The argument here might be better to use the ordinary coordinatewise convergence together with the diagonal method instead of using Tychonoff's theorem. See Murakami and Urai (2017) for a detailed price convergence argument in OLG economy under SNS condition.

and $\{M_i^*(T)\}_{T=1}^{\infty}$ for each $i \in I$. Since the convergence of $p^*(T)$ to p^* is coordinatewise, and since for each $t = 1, 2, \ldots$, $y_t^*(T)$ is a solution of the ordinary profit maximization problem in $R^{K(t)}$, we may conclude that (by using the Berge's maximum theorem and taking a subsequence if necessary) that $y_t^*(T)$ converges to a point y_t^* such that y_t^* is a profit maximization point in the compact Y_t under p^*. Of course, the convergence of $y_t^*(T)$ to y_t^* together with $p^*(T)$ to p^* means the convergence of $m_i^t(y_t^*(T), p^*(T))$ to the amount $m_i^t(y_t^*, p^*)$ for each i and t since every m_i^t is continuous. It is also clear (since every X_i is compact) that we may suppose that (by taking subsequences repeatedly) $\{x_i^*(T)\}_{T=1}^{\infty}$ converges to $x_i^* \in X_i$ for each $i \in I$. As limits satisfying condition (6), it is also clear that the feasibility condition (1) is satisfied. Therefore, all we have to check is to assure each consumer's utility maximization under (2), although it should be noted that the value M_i^* for each i is still not fixed. Here, note that for each $i \in I_t$, sequence $\{M_i^*(T)\}_{T=1}^{\infty}$ (after following the normalization of prices, $p^*(T)$, $T = 1, 2, \ldots$), has an obvious upper bound, $\mathbb{A}_t \sum_{k=1}^{H(t)} |b_k|$, where $\mathbb{A}_t$ is the budget modification constant in $\mathcal{E}(t)$, since for equilibrium prices of $\mathcal{E}(t)$ normalized to the Euclidean norm of 1, the norm of another normalization based on the first coordinate cannot exceed the sum, $\sum_{k=1}^{H(t)} |b_k|$ (the value of maximum variational norm). It follows that, again by taking subsequences repeatedly (if necessary), we can assume for all i, sequence $\{M_i^*(T)\}_{T=1}^{\infty}$ converges to $M_i^* \in R_+$. Now it would be a routine task for every i to check that x_i^* is a utility maximization point under (2) since the argument is nothing but to check (again by using Berge's maximum theorem) the upper semicontinuity of Marshallian demand correspondence.

4.8 Extension to the case where $\frac{1-||p(T)||}{||p(T)||}$ is different from person to person

Since we use the method in Mas-Collel (1982), we have to take $M_i^*(T)$ as $M_i^*(T) = \frac{1-||p(T)||}{||p(T)||}$, i.e., the monetary transfer is the same for everyone in the economy till generation T. We can extend this situation to where each person's dividend is different.

Let $Z_i \in R_{++}$ be a dividend operator, which allows difference between the amounts of dividend an individual gets. With Z_i, we modify the budget constraint as follows,

$$p(T)x_i \leq p(T)\omega_i + \sum_{s=1}^{T} m_i^s(y_s, p(T)) + Z_i \frac{1 - ||p(T)||}{||p(T)||}. \tag{14}$$

As long as $Z_i \in R_{++}$, the minimum wealth condition still holds. So, the the rest of argument will not change. Hence, we can prove there is CE in the same way as before.

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