Poverty trap, dynamic gains from trade, and inferiority in consumption

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Summary. We extend the dynamic Heckscher-Ohlin model in Bond et al. (2009) and show that if a labor intensive good is inferior, then there may exist multiple steady states in autarky and poverty trap can arise. Also, it is shown that there is a possibility that under free trade, each country will reach a higher steady state level of welfare as a result of opening trade than in autarky. This contrasts sharply with the result in dynamic H-O models with normality in consumption: The country with a higher (lower) capital stock than the other will reach the steady state where the level of welfare is higher (lower) than that in autarky.

Key words: dynamic Heckscher-Ohlin model, poverty trap, inferior good, dynamic gains from trade

JEL Classification Numbers: E13, E21, F11, F43

1 Introduction

In dynamic general equilibrium models such as the Ramsey one, there is a negative relation between capital accumulation and the rental on capital: the rental rate decreases when capital stock (per unit of effective labor) increases. This guarantees the uniqueness and the saddle-point stability of the steady state, where the rental rate is equal to the sum of the discount factor and the depreciation rate.

Assuming identical and homothetic preferences with a constant intertemporal elasticity of substitution, which makes demands for goods independent of the international distribution of wealth, Chen [3] examined a dynamic two country Heckscher-Ohlin (H-O) model and showed that there is a continuum of steady state equilibria under free trade, with the world capital stock being constant across all of the steady states: The world capital stock that equalizes the rental rate to the sum of the discount factor and the depreciation rate is unique. Which steady state the world economy converges to is determined by the initial distribution of capital across countries. Since, in autarky, the rental rate in one country is lower than that in the other country with a lower capital stock, free trade among the countries yields an increase (decrease) in the rental on capital for the former (latter) country, and encourages (discourages) its capital accumulation. So, the initial ranking of factor endowment ratios among countries maintains along the dynamic equilibrium path,[1] and the country with initially higher (lower) capital stock than the other will have a higher (lower) capital stock and level of welfare at the free trade steady state than at the autarkic one.[2]

As shown in Bond et al. [2], the results in Chen [3] will substantially hold as long as goods are normal. Consider the case in which a labor intensive good is inferior, they also show that this can lead to multiple autarkic steady states and continua of free trade steady states, and that some steady states are saddle points, while the others are unstable (or indeterminacy can arise around them). Thus, dynamic H-O models may exhibit rich dynamic properties when the labor intensive good is inferior. This is because the inferiority implies that the more capital countries accumulate, the less the labor intensive good is demanded, and hence the more capital is needed for producing goods: It is possible that the rental on capital increases as capital stock does.

In this paper, utilizing the dynamic H-O model in Bond et al. [2], we will clarify the implication of inferiority in consumption in a closed economy as well as under free trade. Specifically, assuming that a labor intensive good is a necessity and it becomes an inferior good when households' income goes up, we will derive conditions on technologies and labor endowment under which there exist three steady state equilibria in autarky. It will be shown from a phase diagram that the middle steady state is unstable, while the others are saddle points, and poverty trap can arise due to inferiority in

[1] Assuming factor-generated externalities in a dynamic H-O model, Nishimura and Shimomura [7] show that the model exhibits indeterminacy, which implies that the international ranking of factor endowment ratios can differ from the initial ranking. Nishimura and Shimomura [8] and Doi et al. [4] show that indeterminacy can arise in dynamic trade models without externalities, if one good is an inferior good at the steady state.

[2] Atkeson and Kehoe [1] show substantially the same result in a small open economy.

consumption.[3] Under free trade, we will show that there is a possibility that both countries have "dynamic gains (losses) from trade" in the sense that each country will reach a higher (lower) steady state level of welfare as a result of opening trade than in autarky. This comes from the fact that the rental on capital is not monotonically decreasing in capital stocks under inferiority in consumption and contrasts sharply with the results in Chen [3] and Atkeson and Kehoe [1]: The country with a lower capital stock than the other necessarily has dynamic losses from trade.[4]

This paper is organized as follows. Section 2 presents the dynamic two country H-O model. Section 3 derives the steady state equilibria in autarky and under free trade. Section 4 proves that poverty trap can arise due to inferiority in consumption. Section 5 discusses the possibility of dynamic gains or losses from trade with and without international transfer of income. Section 6 provides the conclusion.

2 The Dynamic Two Country Heckscher-Ohlin Model

In this section we formulate the continuous-time version dynamic optimization problem for a representative country in a dynamic H-O model. By dynamic H-O model, we mean that each country has access to the same technology for producing two goods using a fixed factor (labor, L) and a reproducible factor (capital, K) under conditions of perfect competition and constant returns to scale. Good 1 is a pure consumption good, and the second good is a consumable capital good. Factors of production are assumed to be mobile between sectors within a country, but immobile internationally, and there are no markets for international borrowing and lending. We refer to the representative country as the home country: the corresponding behavioral relations for the other (foreign) country will be denoted by a "$*$."

We assume that the home and foreign countries are symmetric except for the initial capital endowment in each country. They have the same population normalized to be one, with each household having an endowment of labor, L, and a concave utility function u defined over consumption of goods 1 and 2, C_1 and C_2.

2.1 The Production Side

Letting F_i be the production function in sector i, we assume

Assumption 1: The production function in each sector is linearly homogeneous, twice differentiable, and strictly quasi-concave with $F_{iKK} \equiv \partial^2 F_i/\partial K^2 < 0$ and $F_{iLL} \equiv \partial^2 F_i/\partial L^2 < 0$. Both

[3] Notice that we assume away any externality nor strategic complementarity, which are commonly assumed in the literature on poverty traps. See Matsuyama [6] for a brief review of poverty trap in theoretical aspects.

[4] Bond et al. [2] consider only the case in which there are two autarkic steady states, one of which is a saddle point and the other is unstable. So, they did not discuss the possibilities of poverty trap and dynamic gains or losses from trade.

factors are indispensable for producing and the pure consumption good 1 is labor intensive.

Letting w denote the wage rate and r the rental on capital, the technology in sector i can be characterized by the unit cost function $\chi_i(w,r)$, $i=1,2$. The competitive profit conditions require that

$$p \leq \chi_1(w,r), \tag{1}$$
$$1 \leq \chi_2(w,r), \tag{2}$$

where good 2 is chosen as numeraire. The stock of capital is denoted by K. Factor market equilibrium requires that

$$1 = v_1 + v_2, \tag{3}$$
$$k = v_1\kappa_1(w/r) + v_2\kappa_2(w/r), \tag{4}$$

where v_i is the fraction of labor devoted to sector i, $k \equiv K/L$, and $\kappa_i(w/r) \equiv \chi_{ir}(w,r)/\chi_{iw}(w,r)$.

Solving for w and r when (1) and (2) hold with equality, we obtain the factor prices $(w(p), r(p))$ that are consistent with production of both goods. Notice that we have $pw'(p)/w(p) > 1$ and $r'(p) < 0$ due to the Stolper-Samuelson theorem, and that κ_i is increasing in p.

Defining p_0, p_∞, and k_i, respectively, as

$$p_0 \equiv \inf\{p|w(p) > 0\},$$
$$p_\infty \equiv \sup\{p|r(p) > 0\},$$
$$k_i(p) \equiv \kappa_i(w(p)/r(p)) \text{ for } p \in (p_0, p_\infty),$$

we formally state some properties of these functions.[5]

Lemma 1 *For any $p \in (p_0, p_\infty)$, we have* (i) $pw'(p)/w(p) > 1$ *and* $r'(p) < 0$; (ii) $k_1(p) < k_2(p)$; (iii) $k_i'(p) > 0$. *Also, we have* $\lim_{p\to p_0} w(p) = \lim_{p\to p_\infty} r(p) = 0$, $\lim_{p\to p_0} k_i(p) = 0$, *and* $\lim_{p\to p_\infty} k_i(p) = \infty$.

[5] Suppose that the production function in each sector has constant elasticity of substitution (CES) of σ_i:

$$F_i(K,L) = A_i(a_i L^{\frac{\sigma_i - 1}{\sigma_i}} + b_i K^{\frac{\sigma_i - 1}{\sigma_i}})^{\frac{\sigma_i}{\sigma_i - 1}},$$

where all the parameters are positive and $a_i + b_i = 1$. Notice that Assumption 1 implies that

$$\sigma_1 = \sigma_2 = \sigma \in (0,1] \text{ and } a_1 > a_2 \ (b_1 < b_2).$$

Then, we see that

$$p_0 = \begin{cases} \left(\frac{b_1}{b_2}\right)^{\frac{\sigma}{1-\sigma}} \frac{A_2}{A_1}, & \text{if } \sigma < 1, \\ 0, & \text{if } \sigma = 1, \end{cases} \quad \text{and} \quad p_\infty = \begin{cases} \left(\frac{a_1}{a_2}\right)^{\frac{\sigma}{1-\sigma}} \frac{A_2}{A_1}, & \text{if } \sigma < 1, \\ \infty, & \text{if } \sigma = 1. \end{cases}$$

These factor prices will satisfy full employment for $k \in [k_1(p), k_2(p)]$.

With incomplete specialization, we can express GDP as $[w(p) + r(p)k]L$. Applying the envelope theorem, we obtain the output of good i, Y_i, to be

$$Y_1(p,k) = \begin{cases} f_1(k)L, & \text{if } \; k < k_1(p) \Leftrightarrow p > p_1(k), \\ [w'(p) + r'(p)k]L, & \text{if } \; k \in [k_1(p), k_2(p)] \Leftrightarrow p \in [p_2(k), p_1(k)], \\ 0, & \text{if } \; k > k_2(p) \Leftrightarrow p < p_2(k), \end{cases} \tag{5}$$

$$Y_2(p,k) = \begin{cases} 0, & \text{if } \; k < k_1(p), \\ \{w(p) + r(p)k - p[w'(p) + r'(p)k]\}L, & \text{if } \; k \in [k_1(p), k_2(p)], \\ f_2(k)L, & \text{if } \; k > k_2(p), \end{cases} \tag{6}$$

where $f_i(k) \equiv F_i(K/L, 1)$ and p_i is the inverse function of k_i: $p_i(k_i(p)) = p$ for $p \in (p_0, p_\infty)$. The supply functions are linear in k with incomplete specialization.

2.2 The Consumption Side

We analyze the optimization problem for a representative household that owns L units of labor. We will impose the following restrictions on this utility function:

Assumption 2: The utility function is twice differentiable and strictly concave with $u_{11} < 0$ and $D \equiv u_{11}u_{22} - u_{12}u_{21} > 0$ for any $(C_1, C_2) \in \{(C_1, C_2) \in \mathbb{R}^2_+ | u_i(C_1, C_2) > 0,\ i = 1, 2\}$, and satisfies $\lim_{C_i \to 0} u_i(C_1, C_2) = \infty$ $(i = 1, 2)$ for any C_j $(j \neq i)$.

The representative household is assumed to maximize the discounted sum of its utilities

$$\max \int_0^\infty u(C_1, C_2) \exp(-\rho t) dt, \tag{7}$$

subject to its flow budget constraint

$$wL + rK = pC_1 + C_2 + \dot{K} + \delta K, \qquad K_0 \text{ given}, \tag{8}$$

where δ is the rate of depreciation on capital and ρ is the discount rate. The budget constraint reflects the assumed absence of an international capital market, since it requires that $pZ_1 + Z_2 = 0$, where $Z_1 = C_1 - Y_1$ $(Z_2 = C_2 + \dot{K} + \delta K - Y_2)$ is the excess demand for good 1 (2).

Solving the current value Hamiltonian for this problem yields the necessary conditions for the choice of consumption levels, the differential equation describing the evolution of the costate variable, λ, and the transversality conditions:

$$u_1(C_1, C_2) = \lambda p, \quad u_2(C_1, C_2) = \lambda, \tag{9}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r), \tag{10}$$

$$\lim_{t \to \infty} K(t)\lambda(t) \exp(-\rho t) = 0. \tag{11}$$

It will be useful for the subsequent analysis to invert the necessary conditions for choice of consumption levels to obtain consumption relations $C_i(p, \lambda)$ for $i = 1, 2$ and an expenditure relation $E(p, \lambda) \equiv pC_1(p, \lambda) + C_2(p, \lambda)$. The following Lemma, which is exactly the same as Lemma 1 in Bond et al. [2], establishes some properties of these functions.

Lemma 2 (i) $\lambda C_{1\lambda} = pC_{1p} + C_{2p}$. (ii) $E_\lambda = pC_{1\lambda} + C_{2\lambda} < 0$. (iii) $C_{1p} < 0$. (iv) $E_p = C_1 + \lambda C_{1\lambda}$.

Our expenditure relation differs from the standard expenditure function in that it holds constant the marginal utility of income, rather than the level of utility. Good i is normal if $C_{i\lambda} < 0$, so (ii) establishes that goods must be normal in total.

Using (8), (10), and the expenditure function, we have

$$\dot{k} = w + rk - e(p, \lambda) - \delta k, \tag{12}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r), \tag{13}$$

where $e(p, \lambda) \equiv E(p, \lambda)/L$. In the case of autarky, the system is closed by adding the market clearing condition for good 1 at home,

$$z_1(p, k, \lambda) \equiv c_1(p, \lambda) - y_1(p, k) = 0, \tag{14}$$

where $c_1(p, \lambda) \equiv C_1(p, \lambda)/L$ and $y_1(p, k) \equiv Y_1(p, k)/L$.

Notice that both goods are produced in any autarkic steady state equilibrium, where $\dot{k} = \dot{\lambda} = 0$ and (14) hold, because $y_1 = 0$ implies $z_1 > 0$ from (14) and $y_2 = 0$ does $\dot{k} = -c_2 - \delta k < 0$ from $py_1 = w + rk$, (12), and (14). So, the system can be expressed as

$$\dot{k} = w(p) + r(p)k - e(p, \lambda) - \delta k, \tag{15}$$

$$\dot{\lambda} = \lambda[\rho + \delta - r(p)], \tag{16}$$

$$0 = c_1(p, \lambda) - [w'(p) + r'(p)k], \tag{17}$$

around the autarkic steady state equilibrium, and these equations govern the evolution of (k, λ, p) under autarky.

2.3 The Foreign Country and World Market Equilibrium

The optimization problem for a foreign household is analogous to that for the home country and the solution of the foreign country's household optimization problem yields

$$\dot{k}^* = w^* + r^*k^* - e(p^*, \lambda^*) - \delta k^*, \tag{18}$$

$$\dot{\lambda}^* = \lambda^*(\rho + \delta - r^*). \tag{19}$$

In a free trade equilibrium, the price of good 1 will be equalized across countries and will be determined by the world market clearing condition for good 1,

$$z_1(p, k, \lambda) + z_1(p, k^*, \lambda^*) = 0. \tag{20}$$

From (13) and (19), we have $r = r^*$ when $\dot{\lambda} = \dot{\lambda}^* = 0$. Therefore, in any free trade steady state equilibrium, we have one of the following cases.

$$\begin{array}{rl} \text{case (i)} & k = k^* < k_1(p) \text{ and } r = r^* = pf_1'(k); \\ \text{case (ii)} & k, k^* \in [k_1(p), k_2(p)] \text{ and } r = r^* = r(p); \\ \text{case (iii)} & k = k^* > k_2(p) \text{ and } r = r^* = f_2'(k). \end{array}$$

As in autarky, case (i) and (iii) are inconsistent with the steady state equilibrium conditions: $\dot{k} + \dot{k}^* < 0$ in case (i) and $z_1 + z_1^* > 0$ in case (iii). So, both countries are incompletely specialized at any free trade steady state and around it, and hence the free trade equilibrium can be solved for the evolution of (k, k^*, λ, p) using (15), (16),

$$\dot{k}^* = w(p) + r(p)k^* - e(p, \lambda^*) - \delta k^*, \tag{21}$$

and

$$0 = c_1(p, \lambda) + c_1(p, \lambda^*) - [2w'(p) + r'(p)(k + k^*)], \tag{22}$$

where we have $\lambda^* = m\lambda$ for some $m > 0$. This is because any free trade equilibrium $\dot{\lambda}/\lambda = \dot{\lambda}^*/\lambda^*$ at each point in time as long as the conditions for factor price equalization are satisfied.

Suppose that $\theta \equiv \rho + \delta$ satisfies[6]

Assumption 3: $\theta < \sup\{r | \chi_2(w, r) = 1\}$.

Then, there is a price of good 1, $\tilde{p}$, such that $r(\tilde{p}) = \theta$ and $\tilde{p} \in (p_0, p_\infty)$. Notice that Assumption 3 is necessary and sufficient for the existence of a steady state price of good 1 at which $\dot{k} = \dot{k}^* = \dot{\lambda} = \dot{\lambda}^* = 0$.

Letting $\tilde{p} \equiv r^{-1}(\theta)$ and $\tilde{w} \equiv w(\tilde{p})$, we have the Lemma as follows.

Lemma 3 *The countries must be incompletely specialized in any autarkic or free trade steady state equilibrium. Under Assumption 3, the prices of good 1 and factors consistent with steady states exist and uniquely determined as $p = \tilde{p}$, $w = \tilde{w}$, and $r = \theta$.*

[6] In the case where the production function in each sector has CES of $\sigma \in (0, 1]$, the unit cost function in sector 2 is given by

$$\chi_2(w, r) = \frac{(a_2^\sigma w^{1-\sigma} + b_2^\sigma r^{1-\sigma})^{\frac{1}{1-\sigma}}}{A_2},$$

and hence, Assumption 3 will be satisfied if $\sigma = 1$, or $\sigma < 1$ and $\theta < A_2/b_2^{\frac{\sigma}{1-\sigma}}$.

3 Inferiority in Consumption and Multiple Equilibria

As Bond et al. [2] showed, multiple autarkic steady states can arise when the labor intensive good is inferior. We also consider the case. We assume that the labor intensive good 1 is a necessity (i.e. the income elasticity of good 1 is less than one) and it becomes an inferior good when households' income goes up. Since the slope of an income expansion path, $dC_2/dC_1|_{dp=0}$, is given by

$$\left.\frac{dC_2}{dC_1}\right|_{dp=0} = \frac{\partial C_2/\partial\lambda}{\partial C_1/\partial\lambda} = \frac{C_{2\lambda}}{C_{1\lambda}},$$

this assumption implies that for any pair (p, λ),

$$\frac{\partial}{\partial\lambda}\left[\left.\frac{dC_2}{dC_1}\right|_{dp=0}\right] = \frac{C_{2\lambda\lambda}C_{1\lambda} - C_{2\lambda}C_{1\lambda\lambda}}{C_{1\lambda}^2} < 0 \text{ and } C_{1\lambda} < 0,$$

or $C_{1\lambda} \geq 0$ holds. Formally, we assume

Assumption 4: For any price of good 1, there are two values of λ, denoted by $\lambda^1(p)$ and $\lambda^2(p)$, such that (i) $C_{2\lambda\lambda}C_{1\lambda} - C_{2\lambda}C_{1\lambda\lambda} \lesseqgtr 0$ if $\lambda \gtreqless \lambda^1(p)$; (ii) $C_{1\lambda} \lesseqgtr 0$ if $\lambda \gtreqless \lambda^2(p)$; (iii) $0 < \lambda^1(p) < \lambda^2(p) < \infty$.

Then, as λ, which can be interpreted as the marginal utility of income, decreases from infinity to zero, the slope of any income expansion path becomes steeper, and the path bends backward and is asymptotic to the vertical line with $C_1 = \underline{C}_1(p) \equiv \lim_{\lambda\to 0} C_1(p, \lambda)$. Curve $\mathrm{Oe^3d^2e^2d^1e^1}$ in Figure 1 is the income expansion path with $p = \tilde{p}$. One example of the utility function that satisfies Assumptions 2 and 4 is as follows:[7]

$$u(C_1, C_2) = \alpha \ln C_1 + \beta \ln C_2 - \gamma C_1 C_2, \tag{23}$$

where all parameters are positive and $\alpha < \beta$.

3.1 Autarkic Steady State Equilibria

Letting $S(p)$ be the magnitude of the slope of the income expansion path at $(C_1, C_2) = (C_1(p, \lambda^1(p)), C_2(p, \lambda^1(p)))$, we will show that if

$$S(\tilde{p}) < \tilde{p} - \frac{\rho}{r'(\tilde{p})} \tag{24}$$

holds, then for some values of L, there exist three steady state equilibria in autarky.

From (17), we have

$$k = \frac{c_1(p, \lambda) - w'(p)}{r'(p)}. \tag{25}$$

Substituting (25) into (15) and rearranging it yields

[7] It is proven in the Appendix and Doi et al. [5].

$$\zeta_1(p)c_1(p,\lambda)+c_2(p,\lambda)=\zeta_2(p)-\dot{k}. \tag{26}$$

where $\zeta_1(p)\equiv p-[r(p)-\delta]/r'(p)$ and $\zeta_2(p)\equiv w(p)-[r(p)-\delta]w'(p)/r'(p)$.

Let $\tilde{\lambda}$ be the solution to

$$\zeta_1(\tilde{p})c_1(\tilde{p},\lambda)+c_2(\tilde{p},\lambda)=\zeta_2(\tilde{p})$$

and

$$k(\lambda)\equiv\frac{c_1(\tilde{p},\lambda)-w'(\tilde{p})}{r'(\tilde{p})}.$$

Then, the pair $(k(\tilde{\lambda}),\tilde{\lambda},\tilde{p})$ clearly satisfies (17) with $\dot{k}=\dot{\lambda}=0$: $(k(\tilde{\lambda}),\tilde{\lambda})$ is the pair of steady state values of k and λ.

Notice that $\tilde{\lambda}$ can be derived from the intersection of the line

$$\zeta_1(\tilde{p})C_1+C_2=\zeta_2(\tilde{p})L \tag{27}$$

and the income expansion path with $p=\tilde{p}$ (see Figure 1). So, we have the Lemma as follows.

Lemma 4 *Let (24) hold. Then, there is some range of values of L, $(\underline{L},\bar{L})$, such that there exist three steady state equilibria in autarky for $L\in(\underline{L},\bar{L})$, while there exists a unique one if $L<\underline{L}$ or $L>\bar{L}$.*

Remark 1 *Suppose that production technologies take the Cobb-Douglas form: $F_i(K,L)=A_iL^{a_i}K^{b_i}$. Then, $pr'(p)/r(p)$ is equal to $-a_2/(a_1-a_2)$, and hence we have $\zeta_1(\tilde{p})=(a_1\rho+a_2\delta)\tilde{p}/a_2\theta$. Also, suppose that households' preference is given by (23). Then, we have $S(\tilde{p})=\hat{s}(\alpha,\beta)\tilde{p}$, where $\hat{s}$ depends only on two parameters, α and β (see the Appendix). So, (24) holds, if and only if $\hat{s}(\alpha,\beta)<(a_1\rho+a_2\delta)/a_2\theta$. Notice that this condition is independent of the value of $\tilde{p}$.*

In the rest of paper, we assume

Assumption 5: $S(\tilde{p})<\zeta_1(\tilde{p})$ $(=\tilde{p}-\rho/r'(\tilde{p}))$ and $L\in(\underline{L},\bar{L})$ hold.

Then, we denote the value of λ at each of the three autarkic steady states by $\tilde{\lambda}^i(L)$ ($i=1,2,3$ and $\tilde{\lambda}^1<\tilde{\lambda}^2<\tilde{\lambda}^3$): Point e^i in Figure 1 is expressed as $(C_1(\tilde{p},\tilde{\lambda}^i),C_2(\tilde{p},\tilde{\lambda}^i))$.[8] As will be made clear in Section 4, the autarkic steady states with $\lambda=\tilde{\lambda}^1$ or $\tilde{\lambda}^3$ are saddle-point stable, while the other is unstable.[9]

[8] In order to simplify the presentation, we suppress the dependence of steady state values on L when there is no ambiguity.

[9] If both goods are normal, then the income expansion path with $p=\tilde{p}$ is upward-sloping for any λ. So, it is apparent from Figure 1 that the autarkic steady state equilibrium is unique (and one can verify that the equilibrium is saddle-point stable).

3.2 Free Trade Steady State Equilibria

Following Bond et al. [2], we derive free trade steady state equilibria.

First, from (15) and (16), the steady state capital stock per one unit of labor satisfy

$$\tilde{k}(\lambda) = \frac{e(\tilde{p}, \lambda) - \tilde{w}}{\rho}. \tag{28}$$

Notice that $\tilde{k}$ is strictly decreasing in λ from Lemma 2 and that $\tilde{k}(\lambda) \in (k_1(\tilde{p}), k_2(\tilde{p}))$ for $\lambda \in [\tilde{\lambda}^1(L), \tilde{\lambda}^3(L)]$ since $\tilde{k}^i(L) \equiv \tilde{k}(\tilde{\lambda}^i(L)) \in (k_1(\tilde{p}), k_2(\tilde{p}))$, $i = 1, 2, 3$. Let $\lambda_{\min}(L)$ and $\lambda_{\max}(L)$ denote, respectively, the solutions to $\tilde{k}(\lambda) = k_2(\tilde{p})$ and $\tilde{k}(\lambda) = k_1(\tilde{p})$, which necessarily exist and are decreasing in L because $e(\tilde{p}, \lambda) = E(\tilde{p}, \lambda)/L$, $\lim_{\lambda \to 0} E(\tilde{p}, \lambda) = \infty$, and $\lim_{\lambda \to \infty} E(\tilde{p}, \lambda) = 0$.

Substituting (28) into the excess demand for good 1, $z_1(p, k, \lambda)$, we obtain a steady state (per one unit of labor) excess demand function

$$\begin{aligned} \tilde{z}_1(\lambda) &= c_1(\tilde{p}, \lambda) - y_1(\tilde{p}, \tilde{k}(\lambda)) \\ &= -\frac{r'(\tilde{p})}{\rho}[\zeta_1(\tilde{p})c_1(\tilde{p}, \lambda) + c_2(\tilde{p}, \lambda) - \zeta_2(\tilde{p})] \end{aligned} \tag{29}$$

for $\lambda \in [\lambda_{\min}, \lambda_{\max}]$. Then, it is clear from Figure 1 that

$$\tilde{z}_1(\lambda) \begin{cases} > 0, & \text{if } \lambda \in [\lambda_{\min}, \tilde{\lambda}^1) \text{ or } \lambda \in (\tilde{\lambda}^2, \tilde{\lambda}^3), \\ < 0, & \text{if } \lambda \in (\tilde{\lambda}^1, \tilde{\lambda}^2) \text{ or } \lambda \in (\tilde{\lambda}^3, \lambda_{\max}]. \end{cases}$$

Notice that the slope of the income expansion path with $p = \tilde{p}$ is equal to $\zeta_1(\tilde{p})$ at points d_1 and d_2 in Figure 1. So, we see that there are exactly two values of λ, denoted by $\hat{\lambda}^i$ $(i = 1, 2)$, such that $\hat{\lambda}^1 \in (\tilde{\lambda}^1, \tilde{\lambda}^2)$, $\hat{\lambda}^2 \in (\tilde{\lambda}^2, \tilde{\lambda}^3)$, and $\left| C_{2\lambda}(\tilde{p}, \hat{\lambda}^i)/C_{1\lambda}(\tilde{p}, \hat{\lambda}^i) \right| = \zeta_1(\tilde{p})$, and hence we have

$$\tilde{z}_1'(\lambda) \begin{cases} > 0, & \text{if } \lambda \in (\hat{\lambda}^1, \hat{\lambda}^2), \\ < 0, & \text{if } \lambda \in [\lambda_{\min}, \hat{\lambda}^1) \text{ or } \lambda \in (\hat{\lambda}^2, \lambda_{\max}]. \end{cases}$$

Figure 2 is a graph of $\tilde{z}_1(\lambda)$ for some value of $L \in (\underline{L}, \bar{L})$.

Free trade steady state equilibria are obtained by solving $\tilde{z}_1(\lambda) + \tilde{z}_1(\lambda^*) = 0$. For example, each of the pairs $(\lambda, \lambda^*) = (\lambda_+^i, \lambda_-^j)$ $(i, j = 1, 2, 3)$ in Figure 2 corresponds to a free trade steady state equilibrium, where the home country imports good 1: $\tilde{z}_1(\lambda) > 0$. Since $\tilde{k}$ is decreasing in λ, we see that the capital abundant foreign exports the labor intensive good 1 (the static H-O theorem is violated) at the steady state equilibria with $(\lambda, \lambda^*) = (\lambda_+^i, \lambda_-^j)$ $(i = 2, 3$ and $i \geq j)$. As stated in Bond et al. [2], this occurs because the richer country demands less of the inferior labor intensive good, and this effect dominates its relatively lower supply of the labor intensive good at these equilibria.

Figure 3 illustrates the set of equilibrium pairs of λ and λ^* for some $L > (\underline{L} + \bar{L})/2$.[10] The slope of these loci are given by

$$\left.\frac{d\lambda^*}{d\lambda}\right|_{d\tilde{z}_1 + d\tilde{z}_1^* = 0} = -\frac{\tilde{z}_1'(\lambda)}{\tilde{z}_1'(\lambda^*)}.$$

Following Bond et al. [2], we have the following Lemma, which establishes the local stability of free trade equilibria.

Lemma 5 *Free trade steady state equilibria are locally saddle-point stable, if $\tilde{z}_1'(\lambda) + m\tilde{z}_1'(\lambda^*)$ is negative there, while they are unstable if it is positive.*

Proof. See Bond et al. [2]. ■

So, we can see that for m that sufficiently close to one, the ray from the origin $\lambda^* = m\lambda$ cuts the loci exactly three times (e.g. points E^i or e^i, $i = 1, 2, 3$ in Figure 3), and that $\tilde{z}_1'(\lambda) + m\tilde{z}_1'(\lambda^*)$ is positive at the middle point, while it is negative at the others. So, we have the following Proposition.

Proposition 1 *There exists an open interval $M(L)$ such that for any $m \in M(L)$, $\tilde{z}_1(\lambda) + \tilde{z}_1(m\lambda) = 0$ has exactly three solutions for λ. The highest and lowest values of λ correspond to the free trade steady state equilibria that is locally saddle-point stable, and the middle one does the unstable steady state equilibrium.*

4 Poverty Trap under Inferiority in Consumption

In this section, deriving the phase diagram of the closed economy, we examine the global stability of the autarkic steady state equilibria, and show that poverty trap can arise due to the inferiority in consumption of the labor intensive good.

4.1 The Autarkic Prices

First, notice that from Lemma 1 and (5) we have, for $p \in (p_0, p_\infty)$,

$$\max_k y_1(p, k) = f_1(k_1(p)), \quad \frac{df_1(k_1(p))}{dp} > 0, \text{ and } \lim_{p \to p_0} f_1(k_1(p)) = 0.$$

Let $\underline{p}(\lambda)$ be the solution to $c_1(p, \lambda) = f_1(k_1(p))$ and $\underline{k}(\lambda)$ denote $k_1(\underline{p}(\lambda))$. Then, $\lim_{\lambda \to \infty} c_1(p, \lambda) = 0$ for $^\forall p$ implies $\lim_{\lambda \to \infty} \underline{k}(\lambda) = 0$. Notice that for given λ, $\underline{p}(\lambda)$ uniquely exists between p_0 and p_∞ because $c_{1p} < 0$ and $\lim_{p \to p_\infty} c_1(p, \lambda) < \lim_{p \to p_\infty} f_1(k_1(p))$.[11]

[10] Notice that $L > (\underline{L} + \bar{L})/2$ implies $\tilde{z}_1(\hat{\lambda}^1) + \tilde{z}_1(\hat{\lambda}^2) < 0$, and hence there is only one value of λ^* that satisfies $\tilde{z}_1(\hat{\lambda}^1) + \tilde{z}_1(\lambda^*) = 0$, while there are three values of λ^* satisfying $\tilde{z}_1(\hat{\lambda}^2) + \tilde{z}_1(\lambda^*) = 0$, which yields the set of equilibrium pairs as in Figure 3. In Figure 3, we suppose that $\tilde{z}_1(\hat{\lambda}^1) + \tilde{z}_1(\lambda_{\min}) > 0$, $\tilde{z}_1(\hat{\lambda}^2) + \tilde{z}_1(\lambda_{\max}) < 0$, and $\tilde{z}_1(\lambda_{\min}) + \tilde{z}_1(\lambda_{\max}) > 0$, all of which does not matter in the following discussion.

[11] Suppose that $\lim_{p \to p_\infty} c_1(p, \lambda) \geq \lim_{p \to p_\infty} f_1(k_1(p))$ holds for some λ. Then, for $^\forall p \in (p_0, p_\infty)$, $c_1(p, \lambda) \geq w'(p) + r'(p)k_1(p)$ holds for such λ. However, it is impossible for $p = r^{-1}(\delta)$ (see Appendix 7.3).

We see from Figure 4 and (14) that for given $\lambda > 0$, the autarkic price of good 1, p^A, given by the intersection of $c_1(p, \lambda)$ and $y_1(p, k)$, decreases as k increases from zero to $\underline{k}(\lambda)$, and it increases as k increases from $\underline{k}(\lambda)$ to infinity. Also, we see that production is completely specialized to good 1 when $k \leq \underline{k}(\lambda)$, while it is incompletely specialized otherwise.

Based on the above, we have the Lemma as follows.

Lemma 6 *The autarkic price of good 1 is given by a continuous function of k and λ, and it has the properties below:*

$$\lim_{k\to 0} p^A(k,\lambda) = \infty, \quad \lim_{k\to\infty} p^A(k,\lambda) = p_\infty, \quad p^A(\underline{k}(\lambda),\lambda) = \underline{p}(\lambda), \quad \textit{and} \quad \frac{\partial p^A(k,\lambda)}{\partial k} \begin{cases} < 0, & \textit{if } k < \underline{k}(\lambda), \\ > 0, & \textit{if } k > \underline{k}(\lambda). \end{cases} \tag{30}$$

The autarkic factor prices are also given by functions of k and λ as follows:

$$r^A(k,\lambda) = \begin{cases} p^A(k,\lambda) f_1'(k), & \textit{if } k < \underline{k}(\lambda), \\ r(p^A(k,\lambda)), & \textit{otherwise}, \end{cases} \tag{31}$$

$$w^A(k,\lambda) = \begin{cases} p^A(k,\lambda)[f_1(k) - k f_1'(k)], & \textit{if } k < \underline{k}(\lambda), \\ w(p^A(k,\lambda)), & \textit{otherwise}. \end{cases} \tag{32}$$

(30) and (31) together imply that r^A is continuous in k and λ, and strictly decreasing in k with $\lim_{k\to 0} r^A(k,\lambda) = \infty$ *and* $\lim_{k\to\infty} r^A(k,\lambda) = 0$.

4.2 The Phase Diagram

Our first task is to determine the $\dot{\lambda} = 0$ locus in the (k, λ) plane. The $\dot{\lambda} = 0$ locus consists of two parts:

$$\lambda = 0 \quad \text{and} \quad r^A(k,\lambda) = \theta.$$

From Lemma 6, we see that for each $\lambda > 0$, there uniquely exists a value of k that satisfies $r^A(k, \lambda) = \theta$.

Under incomplete specialization, the locus along which the equilibrium price of good 1 is constant is determined as (25),

$$k = \frac{c_1(p,\lambda) - w'(p)}{r'(p)},$$

which is derived from the market clearing condition (17). So, the $\dot{\lambda} = 0$ locus partly consists of the set

$$\left\{ (k,\lambda) \,\middle|\, k = \frac{c_1(\tilde{p},\lambda) - w'(\tilde{p})}{r'(\tilde{p})} \text{ and } k \geq \underline{k}(\lambda) \right\}.$$

On the other hand, we see from Lemma 6 that for λ that violates[12]

$$\frac{c_1(\tilde{p},\lambda)-w'(\tilde{p})}{r'(\tilde{p})} \geq \underline{k}(\lambda), \tag{33}$$

the $\dot{\lambda}=0$ locus is given by the set $\{(k,\lambda)|p^A(k,\lambda)f_1'(k)=\theta\}$.

Based on the above, the $\dot{\lambda}=0$ locus can be drawn as in Figure 5 and $\dot{\lambda}$ is positive (negative) in the right (left) side of the locus.[13]

Next, we turn to the $\dot{k}=0$ locus. Since the locus must lie in the region where $k>\underline{k}(\lambda)$, we have

$$\begin{aligned}\dot{k} &= w\left(p^A(k,\lambda)\right)+r\left(p^A(k,\lambda)\right)k-e(p^A(k,\lambda),\lambda)-\delta k \\ &= 0.\end{aligned} \tag{34}$$

Then, totally differentiating of (34) yields

$$d\dot{k}=\left(r^A-\delta-\lambda c_{1\lambda}\frac{\partial p^A}{\partial k}\right)dk-\left(\lambda c_{1\lambda}\frac{\partial p^A}{\partial \lambda}+e_\lambda\right)d\lambda$$

where use has been made of Lemma 2 (iv) and the market clearing condition. It can be easily shown that the coefficient of $d\lambda$ is always positive,[14]

$$\lambda c_{1\lambda}(\partial p^A/\partial\lambda)+e_\lambda<0, \tag{35}$$

and that of dk is positive when $r^A>\delta$ and $c_{1\lambda}<0$.

For each $p\in(p_0,p_\infty)$, there necessarily exists a intersection of the line $\zeta_1(p)C_1+C_2=\zeta_2(p)L$ and the income expansion path with p, say (C_1',C_2'). From (26), we see that the intersection corresponds to the point in (k,λ) plane where $\dot{k}=0$ as follows:

$$C_1(p,\lambda)=C_1' \;\text{ and }\; k=\frac{C_1'-w'(p)L}{r'(p)L}.$$

Since production is completely specialized to good i when $k=k_i(p)$, we see from (5) and (6) that for $p\in(p_0,p_\infty)$,

$$k_1(p)=\frac{pw'(p)-w(p)}{-pr'(p)+r(p)} \;\text{ and }\; k_2(p)=-\frac{w'(p)}{r'(p)}. \tag{36}$$

From Lemma 1 and (36), one can verify that ζ_2 and ζ_2/ζ_1 are both strictly increasing in p for $p<r^{-1}(\delta)$. So, as p increases from p_0 to $r^{-1}(\delta)$, the line $\zeta_1(p)C_1+C_2=\zeta_2(p)L$ sifts outward, while the income expansion path does inward.

[12] We will show in Appendix 7.3 that (33) holds for $^\forall\lambda>0$, if $C_1(\tilde{p},\lambda^2(\tilde{p}))$ is smaller than the C_1-intercept of the line $\zeta_1(\tilde{p})C_1+C_2=\zeta_2(\tilde{p})L$ as in Figure 1.

[13] For λ that violates (33), we have $\underline{p}(\lambda)>\tilde{p}$, and therefore $p^A(k_1(\tilde{p}),\lambda)f_1'(k_1(\tilde{p}))>\tilde{p}f_1'(k_1(\tilde{p}))=r(\tilde{p})>r(\underline{p}(\lambda))$, which implies that the locus with $r^A(k,\lambda)=\theta$ lies between $k_1(\tilde{p})$ and $\underline{k}(\lambda)$ for such λ.

[14] See the Appendix.

So, there are two values of p, p_m^- and p_m^+, such that (i) $p_m^- \in (p_0, \tilde{p})$ and $p_m^+ \in (\tilde{p}, r^{-1}(\delta))$; (ii) the $\dot{k} = 0$ locus cuts exactly three times the curve

$$k = \frac{c_1(p, \lambda) - w'(p)}{r'(p)} \text{ with } p \in (p_m^-, p_m^+);$$

(iii) the locus cuts exactly once the curve with $p \in (p_0, p_m^-)$ or $p \in (p_m^+, r^{-1}(\delta)]$.[15] Also, notice that $\lim_{p \to p_0} k_2(p) = 0$ implies the $\dot{k} = 0$ locus is asymptotic to the vertical axis. Thus, we can derive the $\dot{k} = 0$ locus, above (below) which $\dot{k}$ is positive (negative), as in Figure 5.

The phase diagram shows that two autarkic steady state equilibria, e^1 and e^3, are saddle-point stable, while the middle one is unstable, and we obtain the first main theorem as follows.

Theorem 2 *If an initial stock of capital is greater (smaller) than $\tilde{k}^2$, the economy converges to the highest (lowest) autarkic steady state equilibrium e^1 (e^3): Poverty trap arises due to inferiority in consumption.*

Remark 2 *Without the assumption of normality in consumption, the higher capital stock does not necessarily imply the lower rental on capital and there can be a non-monotonic relation between capital stock and its rental rate.*

Along the dynamic general equilibrium path where $\dot{k}$ is negative, the existing capital may be consumed as good 2. However, one may think that irreversible investment (or at least some costs of reversible investment) should be assumed, since we suppose that newly produced consumable capital is tradable but the existing one is internationally immobile. So, in the rest of paper, we assume

Assumption 6: $L > \hat{L} \equiv \zeta_2(\tilde{p})\bar{L}/[\zeta_2(\tilde{p}) + \delta k_1(\tilde{p})]$.

Then, we have

Lemma 7 *Consuming the existing capital and complete specialization to produce good 1 do not occur along the path from e^2 to e^3.*

Proof. See the Appendix. ■

5 Dynamic Gains or Losses from Trade

As stated above, we use the term "dynamic gains (losses) from trade" to denote the higher (lower) steady state level of welfare as a result of opening trade than in autarky. When both goods are normal, there are dynamic gains (losses) from trade for the country with a higher (lower) capital stock than the other, as shown in Chen [3], Atkeson and Kehoe [1], and Bond et al. [2]. This is

[15] Notice that p_m^+ must be smaller than $r^{-1}(\delta)$, because $\zeta_1(r^{-1}(\delta)) = p$ implies that the line $\zeta_1(r^{-1}(\delta))C_1 + C_2 = \zeta_2(r^{-1}(\delta))L$ cuts exactly once the income expansion path with $p = r^{-1}(\delta)$.

because without inferiority in consumption, the autarkic steady state equilibrium is unique and the rental on capital in the country with the higher capital stock is lower than the other's along their autarkic equilibrium paths, and hence free trade among the countries will yield an increase in the rental rate in the higher capital stock country and encourage its capital accumulation, and vice versa. Without the assumption of normality, however, it is possible that both the countries have dynamic gains or dynamic losses from trade.

5.1 Dynamic Equilibrium Paths with Free Trade

Figure 6 illustrates the set of equilibrium pairs of k and k^*, which can be derived by using (28). The slope of these loci are given by

$$\left.\frac{dk^*}{dk}\right|_{d\tilde{z}_1+d\tilde{z}_1^*=0} = -\frac{\tilde{z}_1'(\lambda)e_\lambda(\tilde{p},\lambda^*)}{\tilde{z}_1'(\lambda^*)e_\lambda(\tilde{p},\lambda)}$$

from $\tilde{k}'(\lambda) = e_\lambda(\tilde{p},\lambda)/\rho$. Points e^i, E^i, N in Figure 6 correspond to those in Figure 3, respectively. Points e^i $(i = 1, 2, 3)$ and N are all autarkic free trade steady state equilibria in the sense that the excess demand for good 1 is zero in both countries.

Notice that if the initial capital stock in each country is the same, $k_0 = k_0^*$, then for all $t \geq 0$, $k = k^*$ and $z_1 = z_1^* = 0$ along the dynamic general equilibrium path, which is substantially the same as in autarky. Therefore, if $k_0 = k_0^* > \tilde{k}^2$ $(< \tilde{k}^2)$, the economy converges to e^1 (e^3) as shown in Figure 6.

So, from Proposition 1, if m is sufficiently close to one, a dynamic general equilibrium path where $\lambda^* = m\lambda$ holds for all $t \geq 0$ has the following properties: (i) there are three steady states on the path; (ii) the middle one is unstable; (iii) the economy converges to the highest (lowest) steady state when the capital stock in each country is initially higher (lower) than in the middle steady state (e.g. see locus $\mathrm{E}^3\mathrm{E}^2\mathrm{IT}'\mathrm{E}^1$ in Figure 6). Suppose that $k_0 > \tilde{k}^2 > k_0^*$ and the pair (k_0, k_0^*) is given by point I in Figure 6. Then, it is apparent that both the countries have dynamic gains from trade: $k > \tilde{k}^1$ and $k^* > \tilde{k}^3$ (equivalently, $\lambda < \tilde{\lambda}^1$ and $\lambda^* < \tilde{\lambda}^3$) hold at point E^1. Thus, we have the second main result as follows.

Theorem 3 *There are two nonempty subsets of $\{(k_0, k_0^*)|k_0 > \tilde{k}^2 > k_0^*\}$ such that for each pair of one subset, both countries have dynamic gains from trade, while they have dynamic losses from trade for that of the other.*

Notice that the autarkic rental rate is greater (smaller) than the steady state rental on capital θ in the capital abundant home country (in the capital scarce foreign country), and that both countries will have dynamic gains from trade if the equalized rental on capital is greater than θ, and vice versa.

5.2 International Transfer of Income

Let us consider the case where the home and foreign countries had reached the highest and lowest autarkic steady states, respectively, before opening trade: the foreign country is in poverty trap. Since point N is a autarkic free trade steady state equilibrium, opening trade among the countries has no effect on their production patterns and levels of utility. Suppose that adding free trade, there is a foreign aid to overcome poverty trap such that the richer home country transfers a part of his capital income to the foreign country for ever. Since good 2 is a luxury good, this may yield an increase in demand for the capital intensive good 2 in foreign, increase the rental on capital, and encourage capital accumulation in foreign. Moreover, we show that there is a possibility of dynamic gains from trade for both the donor and recipient countries.

Let the amount of transfer at time t be

$$\mu(\tilde{k}^1 - \tilde{k}^3)r,$$

where μ is constant over time and $0 \leq \mu \leq 1/2$. Notice that as long as production in each country is incompletely specialized along the dynamic equilibrium path with the scheme of transfer, it mimics the path that starts from

$$(k_0, k_0^*) = ((1-\mu)\tilde{k}^1 + \mu\tilde{k}^3, \mu\tilde{k}^1 + (1-\mu)\tilde{k}^3),$$

which is on the line NT in Figure 6.

Since the steady states close to point N are locally saddle-point stable (see Figure 3 and Lemma 5), if μ (the ratio of transfer to home households' capital income) is small, then

$$k - \mu(\tilde{k}^1 - \tilde{k}^3) < \tilde{k}^1 \quad \text{and} \quad k^* + \mu(\tilde{k}^1 - \tilde{k}^3) > \tilde{k}^3$$

will hold at the steady state: the level of welfare in home will be lower at the steady state than at the autarkic one, and vice versa.

The following Lemma, which is proven in the Appendix, establishes conditions under which incomplete specialization will hold in both countries along the dynamic general equilibrium path with $\mu = 1/2$.

Lemma 8 *Suppose that $w''(p) > 0$ or $r''(p) < 0$ for $^\forall p \in (p_0, p_\infty)$, and that $\tilde{k}^1 + \tilde{k}^3 > 2\tilde{k}^2$ and $2C_1(\tilde{p}, \tilde{\lambda}^1) > C_1(\tilde{p}, \tilde{\lambda}^3)$ hold. Then, along the dynamic general equilibrium path with $\mu = 1/2$, the economy converges to the steady state with $k + k^* = 2\tilde{k}^1$ and both countries are incompletely specialized on the path.*

Notice that $w''(p) > 0$ holds for $^\forall p \in (p_0, p_\infty)$ when the production function in each sector has CES of $\sigma \in (0, 1]$, and that $\tilde{k}^1 + \tilde{k}^3 > 2\tilde{k}^2$ necessarily holds for L that close to $\bar{L}$ since $\tilde{k}^1$ and $\tilde{k}^3$ are increasing in L while $\tilde{k}^2$ is decreasing, and $\lim_{L \to \bar{L}} \tilde{k}^1(L) > \lim_{L \to \bar{L}} \tilde{k}^2(L) = \lim_{L \to \bar{L}} \tilde{k}^3(L)$. The inequality $2C_1(\tilde{p}, \tilde{\lambda}^1) > C_1(\tilde{p}, \tilde{\lambda}^3)$ holds when the difference between $S(\tilde{p})$ and $\zeta_1(\tilde{p})$ is sufficiently small or $2\underline{C}_1(\tilde{p}) \geq C_1(\tilde{p}, \lambda^2(\tilde{p}))$ holds.

We see from Lemma 8 that there are dynamic gains from trade for both the donor and recipient countries when μ is sufficiently close to one half (see locus from T′ to E^1 in Figure 6).

Theorem 4 *Let the conditions in Lemma 8 hold and the economy initially stay in the steady state with $(k, k^*) = (\tilde{k}^1, \tilde{k}^3)$. Then, if the amount of transfer is sufficiently large (μ is sufficiently close to one half), then the foreign country will overcome poverty trap, and the welfare levels in both the donor and recipient countries will be higher at the steady state than at their autarkic steady states, while the steady state level of welfare in home will be worse off when the amount of transfer is sufficiently small.*

6 Concluding Remarks

Our analysis has shown that when the labor intensive good is inferior, there can be a non-monotonic relation between capital stock and its rental rate, and hence multiple autarkic steady states and poverty trap can arise without any externality nor strategic complementarity, which are commonly assumed in the literature on poverty traps. We have also shown that there is a possibility that free trade between two countries one of which had escaped from poverty trap and the other in it will lead both countries out of poverty trap or into it. In the former (latter) case, each country will reach the higher (lower) steady state level of welfare as a result of opening trade than in autarky. This contrasts sharply with the result in dynamic H-O models with normality in consumption: The country with a higher (lower) capital stock than the other will reach the steady state where the level of welfare is higher (lower) than that in autarky.

7 Appendix

7.1 *Properties of the Utility Function (23)*

It can be easily shown that (23) satisfies Assumption 2 where the set $\{(C_1, C_2) \in \mathbb{R}^2_+ | u_i(C_1, C_2) > 0, i = 1, 2\}$ is given by $\{(C_1, C_2) \in \mathbb{R}^2_+ | \gamma C_1 C_2 < \alpha\}$.

The first order conditions for the choice of consumption levels, (9), are

$$\frac{\alpha - \gamma C_1 C_2}{C_1} = \lambda p, \tag{37}$$

$$\frac{\beta - \gamma C_1 C_2}{C_2} = \lambda, \tag{38}$$

which yield

$$(\gamma C_1 C_2)^2 - \left(\alpha + \beta + \frac{\lambda^2 p}{\gamma}\right) \gamma C_1 C_2 + \alpha\beta = 0.$$

Therefore, we have

$$\gamma C_1 C_2 = \frac{\alpha + \beta + \frac{\lambda^2 p}{\gamma} - \left[\left(\alpha + \beta + \frac{\lambda^2 p}{\gamma} \right)^2 - 4\alpha\beta \right]^{\frac{1}{2}}}{2}, \tag{39}$$

since consumption bundles (C_1, C_2) satisfy $\gamma C_1 C_2 < \alpha$. Let us denote the right-hand side of (39) by $q(\lambda^2 p)$. Then, we get

$$\lim_{\lambda^2 p \to 0} q(\lambda^2 p) = \alpha, \quad \lim_{\lambda^2 p \to \infty} q(\lambda^2 p) = 0, \text{ and } q'(\lambda^2 p) = -\frac{q(\lambda^2 p)^2}{\gamma[\alpha\beta - q(\lambda^2 p)^2]}.$$

Thus, for any positive p and λ, $q(\lambda^2 p)$ is between zero and α and strictly decreasing in $\lambda^2 p$.

Utilizing this function, we obtain consumption relations $C_i(p, \lambda)$ $(i = 1, 2)$ from (37) and (38) as follows:

$$C_1(p, \lambda) = \frac{\alpha - q(\lambda^2 p)}{\lambda p} \text{ and } C_2(p, \lambda) = \frac{\beta - q(\lambda^2 p)}{\lambda},$$

where

$$\lim_{\lambda \to 0} C_1(p, \lambda) = 0, \quad \lim_{\lambda \to 0} C_2(p, \lambda) = \infty, \text{ and } \lim_{\lambda \to \infty} C_i(p, \lambda) = 0 \ (i = 1, 2).$$

As a result of straightforward calculations, we get

$$C_{1\lambda}(p, \lambda) = -\frac{q(q^2 - 2\beta q + \alpha\beta)}{\gamma(\beta - q)(\alpha\beta - q^2)}, \tag{40}$$

$$C_{2\lambda}(p, \lambda) = -\frac{pq(q^2 - 2\alpha q + \alpha\beta)}{\gamma(\alpha - q)(\alpha\beta - q^2)}, \tag{41}$$

$$C_{1\lambda\lambda}(p, \lambda) = \frac{2\beta q(\alpha - q)\phi_1(q)}{\gamma(\beta - q)(\alpha\beta - q^2)^3 \lambda}, \tag{42}$$

$$C_{2\lambda\lambda}(p, \lambda) = \frac{2p\alpha q(\beta - q)\phi_2(q)}{\gamma(\alpha - q)(\alpha\beta - q^2)^3 \lambda}, \tag{43}$$

where

$$\phi_1(q) \equiv q^4 - 4\alpha q^3 + 6\alpha\beta q^2 - 4\alpha\beta^2 q + \alpha^2\beta^2,$$
$$\phi_2(q) \equiv q^4 - 4\beta q^3 + 6\alpha\beta q^2 - 4\alpha^2\beta q + \alpha^2\beta^2.$$

From (40)–(43), we have

$$C_{2\lambda\lambda} C_{1\lambda} - C_{2\lambda} C_{1\lambda\lambda} = \frac{2pq^3(\beta - \alpha)\psi(q)}{\gamma^2(\alpha - q)(\beta - q)(\alpha\beta - q^2)^3 \lambda},$$

where

$$\psi(q) \equiv q^4 - 6\alpha\beta q^2 + 4\alpha\beta(\alpha + \beta)q - 3\alpha^2\beta^2. \tag{44}$$

First, from (40) and (41), we see

$$C_{1\lambda} \lesseqgtr 0 \text{ if } q(\lambda^2 p) \lesseqgtr \beta - \sqrt{\beta(\beta - \alpha)} \text{ and } C_{2\lambda} < 0 \text{ for } {}^{\forall} q \in (0, \alpha),$$

where $\beta - \sqrt{\beta(\beta-\alpha)}$ is a solution to $q^2 - 2\beta q + \alpha\beta = 0$ and smaller than α.

Next, from (44), we have

$$\psi(0) = -3\alpha^2\beta^2 < 0,\ \psi(\alpha) = \alpha^2(\beta-\alpha)^2 > 0,\ \psi'(0) = 4\alpha\beta(\alpha+\beta) > 0,$$

and

$$\psi''(q) = 12(q^2 - \alpha\beta) < 0 \text{ for } q \in (0, \alpha).$$

So, we see that $\psi(q) = 0$ has a unique solution between zero and α, denoted by $\hat{q}(\alpha,\beta)$, and that $\psi(q)$ is negative (positive) if $q(\lambda^2 p)$ is smaller (greater) than $\hat{q}(\alpha,\beta)$, and hence

$$C_{2\lambda\lambda}C_{1\lambda} - C_{2\lambda}C_{1\lambda\lambda} \lesseqgtr 0 \text{ if } q(\lambda^2 p) \lesseqgtr \hat{q}(\alpha,\beta).$$

It can be easily shown that $\psi(\beta - \sqrt{\beta(\beta-\alpha)})$ is negative, which implies $\hat{q}(\alpha,\beta) > \beta - \sqrt{\beta(\beta-\alpha)}$. Therefore, we see that $0 < \lambda^1(p) < \lambda^2(p) < \infty$ holds because

$$\lambda^1(p) = \left[\frac{q^{-1}(\hat{q}(\alpha,\beta))}{p}\right]^{\frac{1}{2}} \quad \text{and } \lambda^2(p) = \left[\frac{q^{-1}(\beta - \sqrt{\beta(\beta-\alpha)})}{p}\right]^{\frac{1}{2}}.$$

Thus, (23) satisfies Assumption 4.

Notice that the slope of the income expansion path at $(C_1, C_2) = (C_1(p, \lambda^1(p)), C_2(p, \lambda^1(p)))$, is given by

$$\frac{(\beta-\hat{q})(\hat{q}^2 - 2\alpha\hat{q} + \alpha\beta)}{(\alpha-\hat{q})(\hat{q}^2 - 2\beta\hat{q} + \alpha\beta)}p.$$

So, we have

$$\hat{s}(\alpha,\beta) = -\frac{(\beta-\hat{q})(\hat{q}^2 - 2\alpha\hat{q} + \alpha\beta)}{(\alpha-\hat{q})(\hat{q}^2 - 2\beta\hat{q} + \alpha\beta)},$$

which is greater than one.

7.2 *Derivation of the inequality (35)*

Totally differentiating equations (9) with respect to C_1, C_2, p and λ, we derive

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} dC_1 \\ dC_2 \end{bmatrix} = \begin{bmatrix} p \\ 1 \end{bmatrix} d\lambda + \begin{bmatrix} \lambda \\ 0 \end{bmatrix} dp.$$

Since the determinant of the coefficient matrix, $D = u_{11}u_{22} - u_{12}^2$, is positive at any point where $u_i(C_1, C_2) > 0$ (Assumption 2) and therefore invertible, we obtain

$$C_{1\lambda}(p,\lambda) \equiv \frac{\partial C_1}{\partial \lambda} = \frac{1}{D}(u_{22}p - u_{12}), \tag{45}$$

$$C_{2\lambda}(p,\lambda) \equiv \frac{\partial C_2}{\partial \lambda} = \frac{1}{D}(u_{11} - u_{12}p), \tag{46}$$

$$C_{1p}(p,\lambda) \equiv \frac{\partial C_1}{\partial p} = \frac{1}{D}\lambda u_{22} < 0, \tag{47}$$

$$C_{2p}(p,\lambda) \equiv \frac{\partial C_2}{\partial p} = -\frac{1}{D}\lambda u_{12}.$$

The results of Lemma 2 follow immediately from these comparative statics results.

Totally differentiating (14) with respect to p and λ yields

$$\frac{\partial p^A}{\partial \lambda} = -\frac{c_{1\lambda}}{c_{1p} - y_{1p}}.$$

So, from (45)–(47) we have

$$\begin{aligned}
\lambda c_{1\lambda}\frac{\partial p^A}{\partial \lambda} + e_\lambda &= -\frac{\lambda c_{1\lambda}^2}{c_{1p} - y_{1p}} + pc_{1\lambda} + c_{2\lambda} \\
&\leq -\frac{\lambda c_{1\lambda}^2}{c_{1p}} + pc_{1\lambda} + c_{2\lambda} \\
&= c_{1\lambda}\left(\frac{u_{12}}{u_{22}} - p\right) + pc_{1\lambda} + c_{2\lambda} \\
&= \frac{1}{u_{22}} \\
&< 0.||
\end{aligned}$$

7.3 *Proof of Lemma 7*

From (25) and (36), production is incompletely specialized iff

$$k_1(p) < \frac{c_1(p,\lambda) - w'(p)}{r'(p)} < k_2(p) \tag{48}$$

$$\Leftrightarrow 0 < c_1(p,\lambda) < \frac{\zeta_2(p) + \delta k_1(p)}{\zeta_1(p)}. \tag{49}$$

Therefore, if $C_1(p, \lambda^2(p))$ is smaller than the C_1-intercept of the line $\zeta_1(p)C_1 + C_2 = [\zeta_2(p) + \delta k_1(p)]L$, which must be true at least for $p = r^{-1}(\delta)$, then

$$k_1(p) < \frac{c_1(p,\lambda) - w'(p)}{r'(p)} < k_2(p) \quad \text{for } {}^\forall\lambda > 0. \tag{50}$$

So, if $C_1(\tilde{p}, \lambda^2(\tilde{p})) < [\zeta_2(\tilde{p}) + \delta k_1(\tilde{p})]L/\zeta_1(\tilde{p})$, then (50) holds with $p = \tilde{p}$, and hence

$$k = \frac{c_1(\tilde{p},\lambda) - w'(\tilde{p})}{r'(\tilde{p})} > \underline{k}(\lambda)$$

for ${}^\forall\lambda > 0$: Complete specialization to produce good 1 do not occur along the path from e^2 to e^3.

For $L > \hat{L}$, we have $[\zeta_2(\tilde{p}) + \delta k_1(\tilde{p})]L/\zeta_1(\tilde{p}) > \zeta_2(\tilde{p})\bar{L}/\zeta_1(\tilde{p})$, the right-hand side of which must be greater than $C_1(\tilde{p}, \lambda^2(\tilde{p}))$ (see Figure 1). Also, we see that if the line $\zeta_1(\tilde{p})C_1 + C_2 = [\zeta_2(\tilde{p}) - \dot{k}]L$ cuts three times the income expansion path with $p = \tilde{p}$, then $\dot{k} > -\delta k_1(\tilde{p})$ holds. Therefore, $\dot{k}$ is greater than $-\delta k_1(\tilde{p})$ at the points in (k, λ) plane that satisfies $\lambda \geq \tilde{\lambda}^1$ and $k = [c_1(\tilde{p},\lambda) - w'(\tilde{p})]/r'(\tilde{p})$, and hence we have $\dot{k} > -\delta k$ (consuming the existing capital do not occur) along the path from e^2 to e^3.||

7.4 *Proof of Lemma 8*

Suppose $\mu = 1/2$ and $\tilde{k}^1 + \tilde{k}^3 > 2\tilde{k}^2$. Then, as long as both home and foreign are incompletely specialized, households' income including transfer is the same across the countries and it is initially greater than the income at the unstable autarkic steady state e^2, which implies that each of the country will accumulate capital along the autarkic dynamic general equilibrium path to the steady state e^1. So, we show that

$$k_1(p^A(k,\lambda)) < k + \frac{\tilde{k}^1 - \tilde{k}^3}{2} < k_2(p^A(k,\lambda)),$$
$$k_1(p^A(k,\lambda)) < k - \frac{\tilde{k}^1 - \tilde{k}^3}{2} < k_2(p^A(k,\lambda)),$$

hold along the path to e^1.

First, notice that $k_1(\tilde{p}) < \tilde{k}^3$ holds, and that along the path, $p^A(k,\lambda) \leq \tilde{p}$, and hence $k_i(p^A(k,\lambda)) \leq k_i(\tilde{p})$ $(i = 1,2)$ holds, where $k \in [(\tilde{k}^1 + \tilde{k}^3)/2, \tilde{k}_1]$. Therefore, it suffices to prove that

$$k + \frac{\tilde{k}^1 - \tilde{k}^3}{2} < k_2(p^A(k,\lambda))$$

holds for $k \in [(\tilde{k}^1 + \tilde{k}^3)/2, \tilde{k}_1]$.

Second, we have $c_1(\tilde{p}, \tilde{\lambda}^1) \leq c_1(\tilde{p}, \lambda) \leq c_1(p^A(k,\lambda),\lambda)$, which implies

$$w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1 \leq w'(p^A) + r'(p^A)k \tag{51}$$
$$\Leftrightarrow r'(p^A)k - r'(\tilde{p})\tilde{k}^1 \geq w'(\tilde{p}) - w'(p^A). \tag{52}$$

Suppose that $w''(p) > 0$ for ${}^\forall p \in (p_0, p_\infty)$. Then, $w'(\tilde{p}) \geq w'(p^A)$, and from (52) we see

$$r'(p^A) \geq \frac{\tilde{k}^1}{k} r'(\tilde{p}) > 2r'(\tilde{p}). \tag{53}$$

This inequality also holds when $r''(p) < 0$ for ${}^\forall p \in (p_0, p_\infty)$.

From (36), (51), and (53), we have

$$\begin{aligned}
k_2(p^A) - \left(k + \frac{\tilde{k}^1 - \tilde{k}^3}{2}\right) &= -\frac{1}{r'(p^A)}\left[w'(p^A) + r'(p^A)\left(k + \frac{\tilde{k}^1 - \tilde{k}^3}{2}\right)\right] \\
&\geq -\frac{1}{r'(p^A)}\left[w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1 + r'(p^A)\frac{\tilde{k}^1 - \tilde{k}^3}{2}\right] \\
&> -\frac{1}{r'(p^A)}\left[w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1 + r'(\tilde{p})(\tilde{k}^1 - \tilde{k}^3)\right] \\
&= -\frac{1}{r'(p^A)}\left\{2[w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1] - [w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^3]\right\} \\
&= -\frac{2c_1(\tilde{p},\tilde{\lambda}^1) - c_1(\tilde{p},\tilde{\lambda}^3)}{r'(p^A)}. ||
\end{aligned}$$

References

[1] Atkeson, A., Kehoe, P.: Paths of development for early- and late-boomers in a dynastic Heckscher-Ohlin model, Research Staff Report 256, Federal Reserve Bank of Minneapolis, (2000)

[2] Bond, E., Iwasa, K., Nishimura, K.: A dynamic two country Heckscher-Ohlin model with non-homothetic preferences, KIER Discussion Paper Series 686, Kyoto University, (2009)

[3] Chen, Z.: Long-run equilibria in a dynamic Heckscher-Ohlin model, Canadian Journal of Economics 25, 923–943, (1992)

[4] Doi, J., Iwasa, K., Shimomura, K.: Indeterminacy in the free-trade world, Journal of Difference Equations and Applications 13, 135–149, (2007)

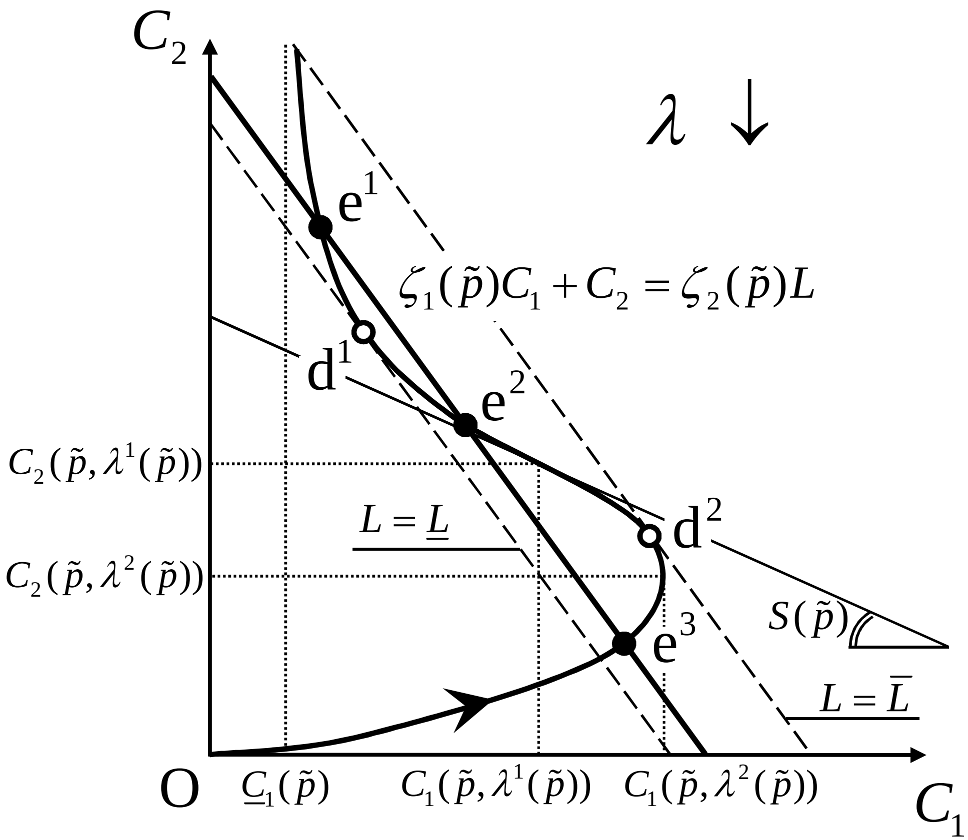
[5] Doi, J., Iwasa, K., Shimomura, K.: Giffen behavior independent of the wealth level, Economic Theory 41, 247–267, (2009)

[6] Matsuyama, K.: Poverty traps, in L. Blume and S. Durlauf, eds., The New Palgrave Dictionary of Economics, 2nd ed., New York: Palgrave Macmillan, Vol. 6, 561–565, (2008)

[7] Nishimura, K., Shimomura, K.: Trade and indeterminacy in a dynamic general equilibrium model, Journal of Economic Theory 105, 244–259, (2002)

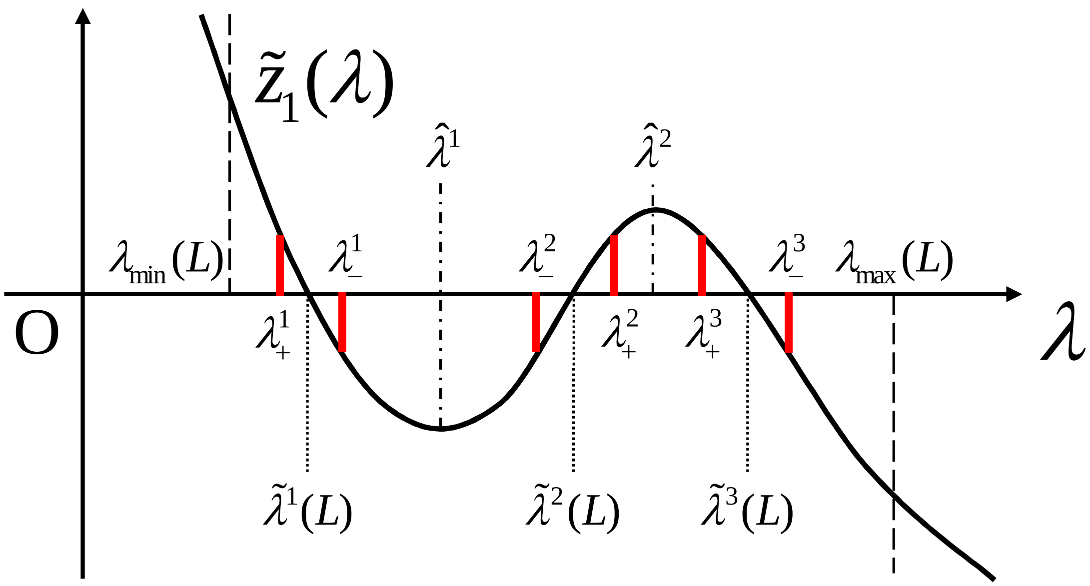
[8] Nishimura, K., Shimomura, K.: Indeterminacy in a dynamic two-country model, Economic Theory 29, 307–324, (2006)

Figure 1



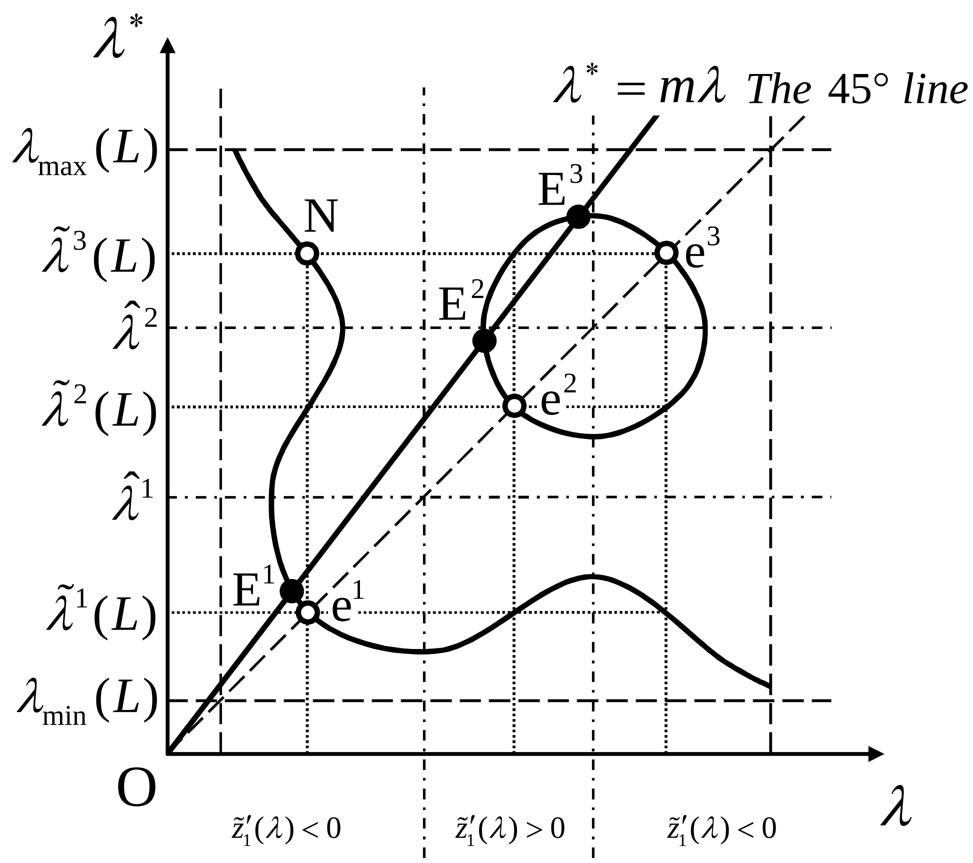
C_2
$\lambda \downarrow$
e^1
$\zeta_1(\tilde{p})C_1 + C_2 = \zeta_2(\tilde{p})L$
d^1
e^2
$C_2(\tilde{p}, \lambda^1(\tilde{p}))$
$L = \underline{L}$
d^2
$C_2(\tilde{p}, \lambda^2(\tilde{p}))$
$S(\tilde{p})$
e^3
$L = \bar{L}$
O
$\underline{C}_1(\tilde{p})$
$C_1(\tilde{p}, \lambda^1(\tilde{p}))$
$C_1(\tilde{p}, \lambda^2(\tilde{p}))$
C_1

Figure 2



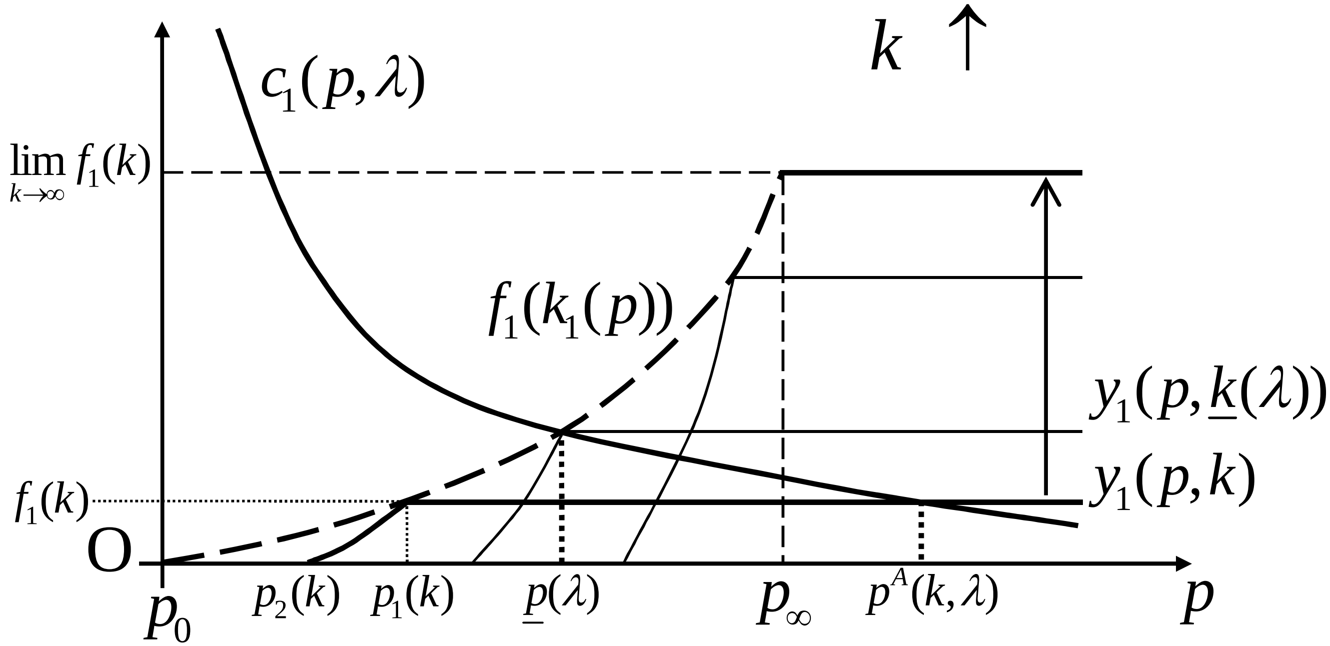
$\tilde{z}_1(\lambda)$
$\hat{\lambda}^1$
$\hat{\lambda}^2$
$\lambda_{\min}(L)$
λ_-^1
λ_-^2
λ_-^3
$\lambda_{\max}(L)$
O
λ_+^1
λ_+^2
λ_+^3
λ
$\tilde{\lambda}^1(L)$
$\tilde{\lambda}^2(L)$
$\tilde{\lambda}^3(L)$

Figure 3



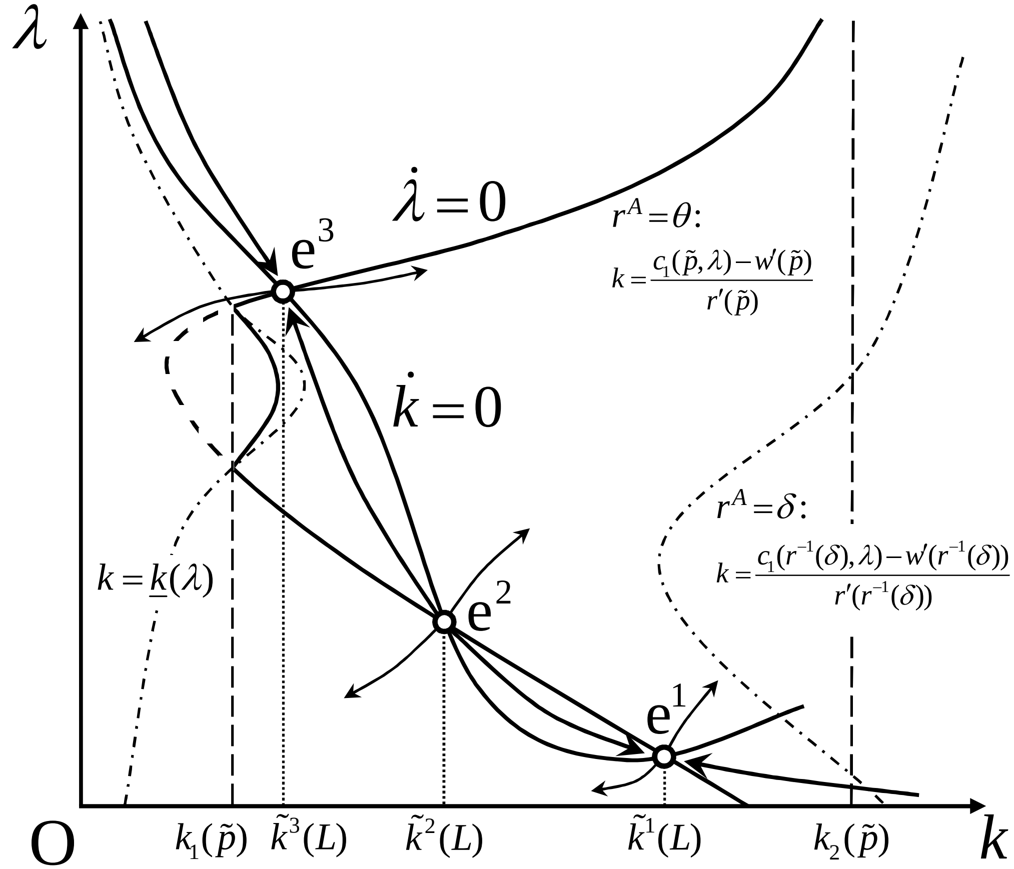
λ^*
$\lambda^* = m\lambda$
The 45° *line*
$\lambda_{\max}(L)$
$\tilde{\lambda}^3(L)$
$\hat{\lambda}^2$
$\tilde{\lambda}^2(L)$
$\hat{\lambda}^1$
$\tilde{\lambda}^1(L)$
$\lambda_{\min}(L)$
N
E^3
e^3
E^2
e^2
E^1
e^1
O
λ
$\tilde{z}_1'(\lambda) < 0$
$\tilde{z}_1'(\lambda) > 0$
$\tilde{z}_1'(\lambda) < 0$

Figure 4



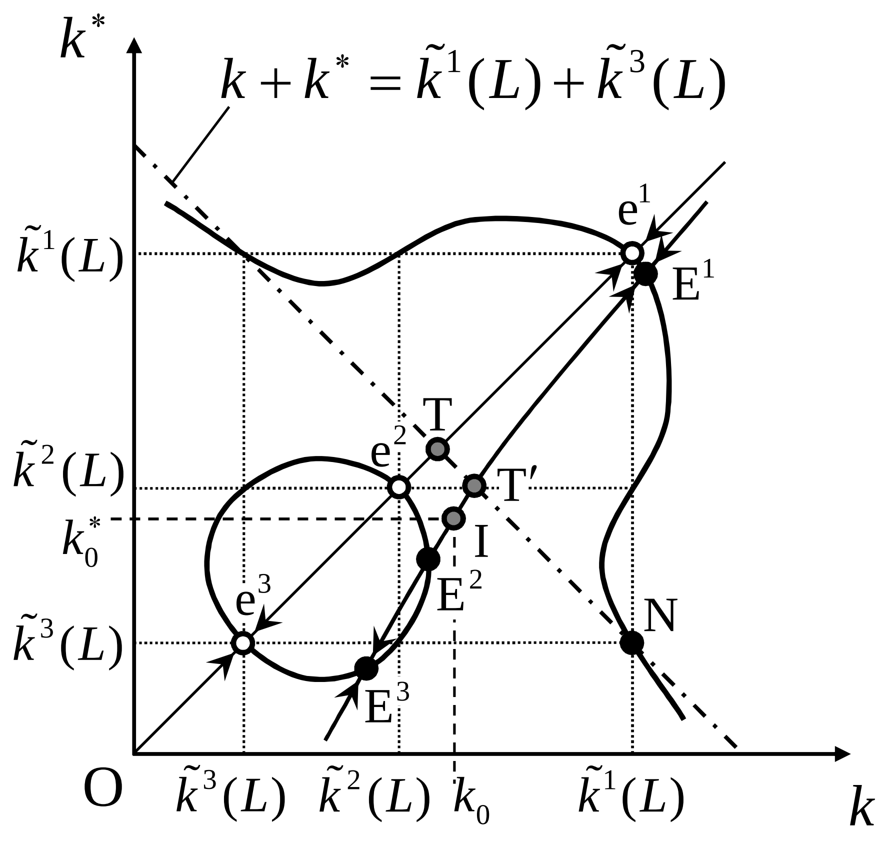
$k \uparrow$
$c_1(p,\lambda)$
$\lim_{k\to\infty} f_1(k)$
$f_1(k_1(p))$
$y_1(p,\underline{k}(\lambda))$
$y_1(p,k)$
$f_1(k)$
O
p_0
$p_2(k)$
$p_1(k)$
$\underline{p}(\lambda)$
p_∞
$p^A(k,\lambda)$
p

Figure 5



λ
λ̇ = 0
e3
r^A = θ:
k = (c1(p̃, λ) − w′(p̃)) / r′(p̃)
k̇ = 0
r^A = δ:
k = (c1(r^−1(δ), λ) − w′(r^−1(δ))) / r′(r^−1(δ))
k = k(λ)
e2
e1
O
k1(p̃)
k̃3(L)
k̃2(L)
k̃1(L)
k2(p̃)
k

Figure 6



k^*
$k + k^* = \tilde{k}^1(L) + \tilde{k}^3(L)$
$\tilde{k}^1(L)$
$\tilde{k}^2(L)$
k_0^*
$\tilde{k}^3(L)$
e1
E1
T
e2
T′
I
E2
e3
N
E3
O
$\tilde{k}^3(L)$
$\tilde{k}^2(L)$
k_0
$\tilde{k}^1(L)$
k