# Public Capital and International Trade: A Dynamic Analysis *

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**Abstract**

This paper develops a dynamic trade model with a public intermediate good whose stock has a positive effect on private sectors' productivity. Under the assumptions of one primary factor (labor), the national government that determines the level of the public good using Lindahl pricing, and the stock of the public intermediate good as a kind of "unpaid factors", this paper examines the economy's trade pattern and the long-run effects of trade. It is shown that a country with a lower (resp. higher) labor endowment tends to become an exporter of a good which is more (resp. less) dependent on the stock of the public intermediate good. Depending on the country's trade pattern, trade affects the steady-state stock of the public intermediate good and thereby induces a biased technological change. Specifically, in comparison with the autarkic steady state, free trade expands (resp. shrinks) the long-run production possibilities frontier in a country exporting a good that is more (resp. less) dependent on the public intermediate good. This implies that a smaller country unambiguously gains from trade in the long run, whereas a larger country may lose from trade in the long run.

**Key Words:** Public intermediate good; Dynamic Lindahl pricing; Trade pattern; Long-run PPF; Gains/losses from trade

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# 1 Introduction

In economic transactions, including international trade, public intermediate goods (e.g., public infrastructure, information and knowledge, cleaner environment) plays an important role. There has been a number of studies examining trade models with public intermediate goods (Manning and McMillan, 1979; Tawada and Okamoto, 1983; Tawada and Abe, 1984; Ishizawa, 1988; Abe, 1990; Altenburg, 1992; Suga and Tawada, 2007). However, these studies consider static models, in spite of the fact that many real-world examples of the public intermediate goods have a characteristic of durable or capital goods, i.e., stock rather than flow levels of these goods are of significance. In light of this property, dynamic rather than static models are suitable.

McMillan (1978) is an exceptional study that considers stock effect of a public intermediate good in an open economy. He considers a three-sector (two private goods and a public intermediate good), one-factor (labor) small open economy with optimal supply of the public intermediate good. It is shown that the stock of public intermediate good determines the slope of the production possibility frontier and thus determines the pattern of international trade. McMillan's model is recently re-examined by Yanase and Tawada (2010), who show the possibility of multiple steady states and history-dependent dynamic paths. Moreover, they discuss whether the economy gains from trade in the McMillan model.

In McMillan (1978) and Yanase and Tawada (2010), the public intermediate good is assumed to have an impact similar to the "creation-of-atmosphere" type externality classified by Meade (1952). That is, in their models, private sectors' technology exhibits constant returns to scale in primary factors of production only. In one-factor model, this assumption implies that for a give stock of the public intermediate good, the production possibilities frontier becomes linear, as in the standard Ricardian model, and thus the economy hardly diversifies production.

There is another class of public intermediate goods, which can be interpreted as, in Meade's (1952) terminology, the "unpaid factors of production". If the public good is of this type, private sectors' production function is characterized by constant returns to scale in all inputs, including the public intermediate good.[1] This paper focuses on this kind of public intermediate goods and presents a dynamic trade model in which the stock of a public good has a positive effect on private sectors' productivity. As with McMillan (1978) and Yanase and Tawada (2010), we consider an open economy in which three goods (two private consumption goods and a public intermediate good) are produced by one primary input (labor). However, because the private goods are produced under constant returns to scale

[1] Tawada (1980) and Tawada and Okamoto (1983) adopt the alternate term "semi-public input" to describe this kind of intermediate goods.

with respect to both labor and the stock of the public intermediate good, the economy incompletely specializes in both private goods even in the case of one primary factor. Moreover, this alternative specification of the nature of the public intermediate good results in the outcomes on trade patterns and gains from trade (summarized below), which are reversed from those in McMillan (1978) and Yanase and Tawada (2010).

We begin with a dynamic small open economy in which the national government, taking the prices of tradables as given, determines the optimal levels of the public intermediate good by using a Lindahl pricing rule. The dynamic path of the economy and the steady state equilibrium are characterized. It is shown that there exists a unique and saddle-point stable steady state. Comparative static analysis is conducted in order to clarify the properties of the steady state. We then investigate the trade pattern of the economy. It is shown that if a country is initially at the autarkic steady state, after opening trade, the country with a lower (resp. higher) labor endowment tends to be an exporter of a good that is more (resp. less) dependent on the stock of the public intermediate good. We also discuss whether a country gains or losses from trade by comparing the steady-state welfare level under autarky with that under free trade. It is shown that in comparison with the autarkic steady state, free trade increases (resp. reduces) the steady-state stock of the public intermediate good and thereby expands (resp. shrinks) the long-run production possibilities frontier in a country exporting a good that is more (resp. less) dependent on the public intermediate good. This implies that a smaller country, in the sense that it has a lower labor endowment, unambiguously gains from trade in the long run, whereas a larger country, i.e., a country with a higher labor endowment, may lose from trade in the long run.

# 2 The Model

## 2.1 A Dynamic Small Open Economy

We consider a small open economy where two private and one public production sectors and one primary factor exist. The primary factor is supposed to be labor. The public sector produces a public intermediate good with decreasing returns to scale technology with respect to labor. The public intermediate good can be accumulated and its accumulated stock serves in production in the private sectors as a positive external effect without congestion between sectors. The two private sectors are supposed to be sectors 1 and 2 where goods 1 and 2 are produced, respectively, under constant returns to scale technology with respect to labor and the stock of the public intermediate good. Total labor endowment is assumed to be given and constant over time.

**The Production Side** The production function of each private sector is assumed to take the following form:

$$Y_i = R^{\alpha_i} L_i^{1-\alpha_i}, \quad 0 < \alpha_i < 1, \quad i = 1, 2, \tag{1}$$

where $Y_i$ is the output of good $i$, $R$ is the stock of the public intermediate good, and $L_i$ is the labor input in sector $i$. It is clear that the labor productivity in each private sector is exhibited by $\partial Y_i / \partial L_i = (1 - \alpha_i) R^{\alpha_i} L_i^{-\alpha_i}$ and is dependent on the stock of the public intermediate good $R$.

The production function of the public sector is expressed as $G = f(L_R)$, where $L_R$ is a labor input in the public sector. Concerning $f(L_R)$, we assume that

$$f' > 0, \quad f'' < 0, \quad \lim_{L_R \to 0} f' = \infty, \quad \lim_{L_R \to \infty} = 0, \quad f(0) = 0.$$

Given initial stock $R_0 > 0$, the public intermediate good is assumed to accumulate over time according to[2]

$$\dot{R} = f(L_R) - \beta R, \tag{2}$$

where $\beta > 0$ is the depreciation rate of the stock of the public intermediate good.

At each moment of time, the economy must face the following full employment constraint on labor:

$$L_1 + L_2 + L_R = L, \tag{3}$$

where $L > 0$ is labor endowment and is assumed to be given and constant over time.

**The Consumption Side** The consumption side of the economy is described by a representative household, whose lifetime utility is supposed to be:

$$U = \int_0^{\infty} e^{-\rho t} \left[ \gamma \log C_1 + (1 - \gamma) \log C_2 \right] dt, \tag{4}$$

where $C_i$ is consumption of good $i$ ($i = 1, 2$), $\rho$ is the rate of time preference, and $\gamma \in (0, 1)$ is a parameter.

The private goods are traded between countries, while the public intermediate good is assumed to be nontradable. In addition, we assume away the international borrowing. Therefore, the national income must be equal to the total expenditure at any time:

$$pY_1 + Y_2 = pC_1 + C_2, \tag{5}$$

where $p$ is a world price of good 1 in terms of good 2. Because of the assumption of a small open economy, $p$ is assumed to be given and constant over time.

[2] A dot over a variable denotes time derivative. To reduce the complexity of notation, we may omit time arguments when no confusion is caused by doing so.

## 2.2 The Optimal Resource Allocation

We characterize the economy's resource allocation as a dynamic optimization problem faced by a social planner. The same solution can be obtained by competitive equilibrium with appropriate Lindahl pricing for the public intermediate good.

Consider a social planner who seeks to maximize the representative household's lifetime utility (4) by choosing appropriate levels of consumptions, factor inputs, and outputs subject to the constraints (1), (2), (3), and (5), taking $p$ as given.

Let us define the current-value Hamiltonian as

$$\begin{aligned} H = & \gamma \log C_1 + (1-\gamma) \log C_2 + \theta \left[ f(L_R) - \beta R \right] \\ & + \pi \left[ pR^{\alpha_1} L_1^{1-\alpha_1} + R^{\alpha_2} L_2^{1-\alpha_2} - pC_1 - C_2 \right] + w \left[ L - L_1 - L_2 - L_R \right]. \end{aligned}$$

Then the optimal controls must satisfy

$$\frac{\partial H}{\partial C_1} = 0 \quad \Rightarrow \quad \frac{\gamma}{C_1} = \pi p, \tag{6}$$

$$\frac{\partial H}{\partial C_2} = 0 \quad \Rightarrow \quad \frac{1-\gamma}{C_2} = \pi, \tag{7}$$

$$\frac{\partial H}{\partial L_1} = 0 \quad \Rightarrow \quad \pi(1-\alpha_1)pY_1 = wL_1, \tag{8}$$

$$\frac{\partial H}{\partial L_2} = 0 \quad \Rightarrow \quad \pi(1-\alpha_2)Y_2 = wL_2, \tag{9}$$

$$\frac{\partial H}{\partial L_R} = 0 \quad \Rightarrow \quad \theta f'(L_R) = w. \tag{10}$$

Moreover, the adjoint equation and the transversality condition are, respectively, expressed as

$$\dot{\theta} = \rho\theta - \frac{\partial H}{\partial R} = (\rho + \beta)\theta - \frac{\pi}{R}(\alpha_1 pY_1 + \alpha_2 Y_2), \tag{11}$$

$$\lim_{t \to \infty} e^{-\rho t} \theta(t) R(t) = 0. \tag{12}$$

# 3 Temporary Equilibrium

In view of (5), (6), and (7), we have $pY_1 + Y_2 = 1/\pi$. In light of this equation, (8) and (9) are respectively rewritten as

$$(1-\alpha_1)y = wL_1, \tag{13}$$

$$(1-\alpha_2)(1-y) = wL_2, \tag{14}$$

where

$$y = \frac{pY_1}{pY_1 + Y_2} \tag{15}$$

is the share of sector 1 in national income.

The temporary equilibrium is thus characterized as a vector $(y, Y_1, Y_2, w, L_1, L_2, L_R)$, which is derived from eqs.(1), (3), (10), (13), (14), and (15), and is dependent on the state variable $R$, co-state variable $\theta$, and parameters $L$ and $p$.[3]

In the following analysis, we make the following assumption regarding the impact of the public intermediate good to industries:

**Assumption 1** $\alpha_1 > \alpha_2$.

Because $\alpha_i$ is the production elasticity of the public intermediate good stock in sector $i$, i.e. $\alpha_i = (\partial Y_i/\partial R) \cdot (R/Y_i)$, Assumption 1 can be interpreted that *sector 1 is more dependent on the stock of the public intermediate good than sector 2*. At the same time, Assumption 1 also implies that sector 2 is more labor intensive than sector 1, as usual in the standard Heckscher-Ohlin-Samuelson model.

**Comparative Statics** The temporary equilibrium solutions of, among others, $y$ and $L_R$ are important for the subsequent analysis, and let us denote them by $y(R, \theta; L, p)$ and $L^R(R, \theta; L, p)$. As shown in the Appendix, we have the following comparative static results. With regard to the share of sector 1, we have

$$\frac{\partial y}{\partial R} = \frac{(\alpha_1 - \alpha_2)y(1-y)[w - (L - L_R)\theta f'']}{R\Delta}, \tag{16a}$$

$$\frac{\partial y}{\partial \theta} = \frac{(\alpha_1 - \alpha_2)y(1-y)f'}{\Delta}, \tag{16b}$$

$$\frac{\partial y}{\partial L} = \frac{(\alpha_1 - \alpha_2)y(1-y)\theta f''}{\Delta}, \tag{16c}$$

$$\frac{\partial y}{\partial p} = \frac{y(1-y)[w - (L - L_R)\theta f'']}{p\Delta} > 0, \tag{16d}$$

where $\Delta \equiv \alpha_1[w(1-y) - L_2\theta f''] + \alpha_2[wy - L_1\theta f''] > 0$. An increase in the relative price of good 1 increases the relative supply of this good and thus $y$. In addition, under Assumption 1, it follows that $\partial y/\partial R > 0$, $\partial y/\partial \theta > 0$, and $\partial y/\partial L < 0$. The mechanism behind the signs of these derivatives is similar to the Rybczynski theorem in the standard Heckscher-Ohlin-Samuelson model. Moreover, because $\theta$ is the shadow price of the public intermediate good, an increase in $\theta$ raises $y$ under Assumption 1.

[3]The temporary equilibrium solutions for $\pi$, $C_1$, and $C_2$ are obtained by substituting the temporary equilibrium solution of $Y_1$ and $Y_2$ into (6), (7), and $pY_1 + Y_2 = 1/\pi$.

With regard to the allocation of labor into public production, we have

$$\frac{\partial L^R}{\partial R} = \frac{(\alpha_1 - \alpha_2)^2 y(1-y)}{R\Delta} > 0, \tag{17a}$$

$$\frac{\partial L^R}{\partial \theta} = \frac{(\alpha_2 L_1 + \alpha_1 L_2) f'}{\Delta} > 0, \tag{17b}$$

$$\frac{\partial L^R}{\partial L} = \frac{w[\alpha_1(1-y) + \alpha_2 y]}{\Delta} > 0, \tag{17c}$$

$$\frac{\partial L^R}{\partial p} = \frac{(\alpha_1 - \alpha_2) y(1-y)}{p\Delta}. \tag{17d}$$

Because an increase in the stock of the public intermediate good saves labor demand in each private sector, the public sector absorbs labor. An increase in $\theta$ raises the value of marginal product of labor in the public sector and thereby boosts labor demand in this sector. An increase in the labor endowment lowers wage, which also boosts labor demand. Finally, under Assumption 1, it follows that $\partial L^R/\partial p > 0$. This is because an increase in $p$ expands sector 1 and shrinks sector 2, which implies $dL_1 > 0 > dL_2$, but from Assumption 1 the decrease in $L_2$ outweighs the increase in $L_1$;[4] the total labor demand in the private sector decreases.

# 4 The Dynamic System

The dynamic system of the small open economy is described by the following differential equations:

$$\dot{R} = f(L^R(R, \theta; L, p)) - \beta R, \tag{18}$$

$$\dot{\theta} = (\rho + \beta)\theta - \frac{\alpha_1 y(R, \theta; L, p) + \alpha_2[1 - y(R, \theta; L, p)]}{R}. \tag{19}$$

## 4.1 The Steady State

Let us denote a steady-state solution of a variable $z$ by $\bar{z}$. The following theorem states that the steady state equilibrium, which satisfies $\dot{R} = \dot{\theta} = 0$, is uniquely determined, and that the equilibrium path is also unique.

**Theorem 1** *There exists a unique steady-state equilibrium with incomplete specialization in the dynamic small open economy. Moreover, the steady state is a local saddle point.*

(Proof) Because the marginal utility of each good goes to infinity if $C_i \to 0$, $C_i > 0$ always holds. In light of the production function (1), which implies that both inputs are necessary, $\bar{R} = 0$ with $\bar{C}_i > 0$ is impossible. It is also

[4]Assumption 1 implies that $1 - \alpha_1 < 1 - \alpha_2$.

impossible that both $\bar{L}_1$ and $\bar{L}_2$ are zero. From the steady-state condition $\dot{R} = 0$, we have $R = f(L_R)/\beta$. Substituting this, (10), (13), and (14) into the production function (1), it follows that

$$Y_1 = \left[\frac{f(L_R)}{\beta}\right]^{\alpha_1}\left[\frac{(1-\alpha_1)y}{\theta f'(L_R)}\right]^{1-\alpha_1}, \quad Y_2 = \left[\frac{f(L_R)}{\beta}\right]^{\alpha_2}\left[\frac{(1-\alpha_2)(1-y)}{\theta f'(L_R)}\right]^{1-\alpha_2}. \tag{20}$$

In addition, in view of (10), (13), and (14), the labor-market clearing condition (3) can be rewritten as

$$(1-\alpha_1)y + (1-\alpha_2)(1-y) = \theta f'(L_R)(L - L_R). \tag{21}$$

Substituting (20) and (21) into (15) and rearranging, we have

$$\frac{1-\alpha_2}{(1-\alpha_1)p}\left[\frac{\beta(L-L_R)}{f(L_R)}\right]^{\alpha_1-\alpha_2} = \frac{[(1-\alpha_2)(1-y)]^{\alpha_2}}{[(1-\alpha_1)y]^{\alpha_1}}[(1-\alpha_1)y+(1-\alpha_2)(1-y)]^{\alpha_1-\alpha_2}, \tag{22}$$

which implicitly determines $y$ as a function of $L_R$: $y = \phi(L_R)$. Under Assumption 1, it is verified that the function $\phi(L_R)$ satisfies $\phi(0) = 0$, $\phi(L) = 1$, and $\phi'(L_R) > 0$. Next, let us consider the steady-state condition $\dot{\theta} = 0$, which, after substituting $R = f(L_R)/\beta$, can be rewritten as

$$(\rho+\beta)\theta = \frac{\beta[\alpha_1 y + \alpha_2(1-y)]}{f(L_R)}. \tag{23}$$

From (21) and (23), another functional relation between $y$ and $L_R$ can be obtained:

$$y = \frac{1}{\alpha_1-\alpha_2}\left\{\frac{(\rho+\beta)f(L_R)}{\beta f'(L_R)(L-L_R) + (\rho+\beta)f(L_R)} - \alpha_2\right\} \equiv \psi(L_R). \tag{24}$$

Under Assumption 1, it holds that

$$\psi(0) = -\frac{\alpha_2}{\alpha_1-\alpha_2} < 0, \quad \psi(L) = \frac{1-\alpha_2}{\alpha_1-\alpha_2} > 1, \quad \psi'(L_R) > 0.$$

Then, as illustrated in Figure 1, there exists a unique pair of steady-state solutions $(\bar{L}_R, \bar{y}) \in (0, L) \times (0, 1)$. Once these solutions are determined, the steady-state solutions of the other variables are also obtained.

What remains to show is the local saddle-point stability of the steady state. By linearizing the dynamic system (18) and (19) around the steady state, we have[5]

$$\begin{bmatrix}\dot{R}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}R-\bar{R}\\ \theta-\bar{\theta}\end{bmatrix}, \tag{25}$$

[5]See Appendix for the derivation of $a_{ij}$'s.

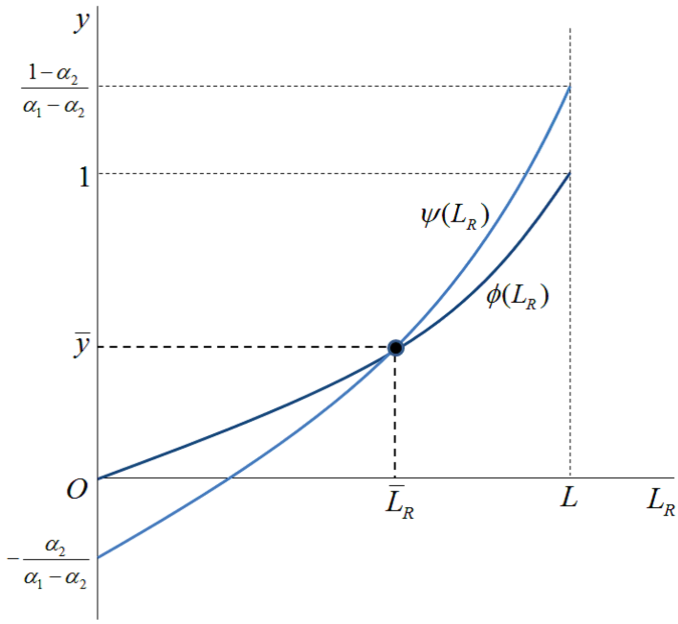


Figure 1: Existence and Uniqueness of the Steady State

where

$$a_{11} \equiv \frac{f(\bar{L}_R)}{\bar{R}} \left\{ \frac{f'(\bar{L}_R)\bar{L}_R}{f(\bar{L}_R)} \cdot \frac{\partial L_R/\partial R}{\bar{L}_R/\bar{R}} - 1 \right\} < 0,$$

$$a_{12} \equiv f'(\bar{L}_R)\frac{\partial L^R}{\partial \theta} > 0,$$

$$a_{21} \equiv \frac{1}{\bar{R}} \left\{ \frac{\alpha_1 \bar{y} + \alpha_2(1-\bar{y})}{\bar{R}} - (\alpha_1 - \alpha_2)\frac{\partial y}{\partial R} \right\} > 0,$$

$$a_{22} \equiv \frac{\alpha_1\alpha_2\bar{w} - \theta f''(\bar{L}_R)[\alpha_1\bar{y} + \alpha_2(1-\bar{y})](\alpha_2\bar{L}_1 + \alpha_1\bar{L}_2)}{\bar{R}\bar{\theta}\Delta} > 0.$$

It is clear that the characteristic roots of the above system have opposite signs. This means that the steady state is a local saddle point. □

## 4.2 Properties of the Steady State Equilibrium

The steady state solutions depend on the exogenous parameters $L$ and $p$. In this section, we examine how these parameters affects the steady state equilibrium.

**Effects of a Change in $L$** We can show that an increase in the labor endowment $L$ increases the steady-state stock of the public intermediate good $\bar{R}$:[6]

$$\frac{\partial \bar{R}}{\partial L} = -\frac{1}{J}\left(a_{22}f'\frac{\partial L^R}{\partial L} + a_{12}\frac{\alpha_1-\alpha_2}{\bar{R}}\frac{\partial y}{\partial L}\right)$$
$$= -\frac{\alpha_1\alpha_2(f')^2}{R\Delta^2 J}\left\{w[\alpha_1(1-y)+\alpha_2 y] - (\alpha_2 L_1 + \alpha_1 L_2)\theta f''\right\} > 0, \quad (26)$$

where $J \equiv a_{11}a_{22} - a_{12}a_{21}$, which is negative, as shown in Theorem 1. Basically, a higher labor endowment leads to more labor input in the public production, which results in the more stock of the public intermediate good in the long run.

**Proposition 1** *An increase in the labor endowment unambiguously increases the stock of the public intermediate good in the long run.*

With regard to the shadow price of the public intermediate good, we can show the following result:

$$\frac{\partial \bar{\theta}}{\partial L} = \frac{1}{J}\left(a_{21}f'\frac{\partial L^R}{\partial L} + a_{11}\frac{\alpha_1-\alpha_2}{\bar{R}}\frac{\partial y}{\partial L}\right) < 0. \quad (27)$$

An increase in the labor endowment reduces the steady-state value of $\theta$.

Let us turn to the steady-state share of sector 1, $\bar{y}$. As shown in the previous section, an increase in $L$ lowers the temporary-equilibrium value of $y$ under Assumption 1. However, $y$ is also dependent on $R$ and $\theta$, which are endogenously determined in the long run and are dependent on $L$. Under Assumption 1, the long-run effect of an increase in $L$ on the share of sector 1 is given by

$$\frac{\partial \bar{y}}{\partial L} = \underset{(-)}{\frac{\partial y}{\partial L}} + \underset{(+)}{\frac{\partial y}{\partial R}}\underset{(+)}{\frac{\partial \bar{R}}{\partial L}} + \underset{(+)}{\frac{\partial y}{\partial \theta}}\underset{(-)}{\frac{\partial \bar{\theta}}{\partial L}}$$
$$= \underset{(-)}{\frac{\partial y}{\partial L}}\left\{1 + \frac{\alpha_1-\alpha_2}{\bar{R}J}\left(\frac{\partial y}{\partial \theta}a_{11} - \frac{\partial y}{\partial R}a_{12}\right)\right\} + \underset{(+)}{\frac{\partial L^R}{\partial L}}\frac{f'}{J}\left(\frac{\partial y}{\partial \theta}a_{21} - \frac{\partial y}{\partial R}a_{22}\right). \quad (28)$$

Given $a_{11} < 0$, $a_{12} > 0$, $J < 0$, and the fact that $\partial y/\partial \theta$ and $\partial y/\partial R$ have the same signs as $\alpha_1 - \alpha_2$, the sign of the first term in the above equation is the same as that of $\partial y/\partial L$. With regard to the second term, computations yield

$$\frac{f'}{J}\left(\frac{\partial y}{\partial \theta}a_{21} - \frac{\partial y}{\partial R}a_{22}\right) = \frac{(\alpha_1-\alpha_2)\bar{y}(1-\bar{y})(L-\bar{L}_R)f'f''[\alpha\bar{y} + \alpha_2(1-\bar{y})]}{\Delta\bar{R}^2 J}, \quad (29)$$

[6] See Appendix for derivation.

the sign of which is the same as that of $\alpha_1 - \alpha_2$. This means that the sign of the second term of (28) is positive under Assumption 1. Therefore, the two terms in the last equation in (28) have the opposite signs. This result suggests that the long-run effect of a change in $L$ can be opposite of the short-run effect.

**Effects of a Change in** $p$ We next examine the long-run effects of a change in the world price of good 1, $p$. The effect on the steady-state stock of the public intermediate good is

$$\frac{\partial \bar{R}}{\partial p} = -\frac{(\alpha_1 - \alpha_2)\bar{y}(1-\bar{y})}{Jp\Delta}\left\{a_{22}f' + a_{12}\frac{\bar{w} - (L - \bar{L}_R)\bar{\theta}f''}{\bar{R}}\right\}, \tag{30}$$

the sign of which is the same as that of $\alpha_1 - \alpha_2$. Thus, under Assumption 1, an increase in $p$ augments $\bar{R}$. The intuition is basically the same as the comparative static result on the temporary equilibrium solution $L^R(R, \theta; L, p)$; an increase in $p$ reduces the total labor demand in the private sector if $\alpha_1 > \alpha_2$, and thus more labor is allocated in the public sector at each moment in time.

The effect on the steady-state value of the co-state variable is given by

$$\frac{\partial \bar{\theta}}{\partial p} = \frac{(\alpha_1 - \alpha_2)\bar{y}(1-\bar{y})}{Jp\Delta}\left\{a_{21}f' + a_{11}\frac{\bar{w} - (L - \bar{L}_R)\bar{\theta}f''}{\bar{R}}\right\}, \tag{31}$$

the sign of which is ambiguous. However, from (30) and (31), the long-run effect of on the share of sector 1 is given by

$$\begin{aligned}\frac{\partial \bar{y}}{\partial p} &= \frac{\partial y}{\partial p} + \frac{\partial y}{\partial R}\frac{\partial \bar{R}}{\partial p} + \frac{\partial y}{\partial \theta}\frac{\partial \bar{\theta}}{\partial p} \\ &= \frac{\partial y}{\partial p}\left\{1 + \frac{\alpha_1 - \alpha_2}{\bar{R}J}\left(\frac{\partial y}{\partial \theta}a_{11} - \frac{\partial y}{\partial R}a_{12}\right)\right\} + \frac{\partial L^R}{\partial p}\frac{f'}{J}\left(\frac{\partial y}{\partial \theta}a_{21} - \frac{\partial y}{\partial R}a_{22}\right).\end{aligned} \tag{32}$$

The sign of the first term of (32) is unambiguously positive. Moreover, in light of (29), the second term of (32) is also unambiguously positive. Therefore, we can confirm that the share of sector 1 is positively related to the relative price of good in the long-run steady state as well as in the temporary equilibrium.

To summarize the long-run effects of a change in the relative price, we obtain the following proposition.

**Proposition 2** *Suppose that sector 1 is more dependent on the stock of the public intermediate good. Then, an increase in the relative price of good 1 increases the stock of the public intermediate good and the share of good 1 in the long run.*

# 5 Trade Pattern

In this section, we present an analysis of trade pattern of an economy when the economy begins to trade. In order to examine the trade pattern, we begin with the derivation of the economy's autarkic equilibrium.

## 5.1 The Autarkic Equilibrium

Under autarky $C_i = Y_i$, $i = 1, 2$, holds at each moment in time. In other words, the relative price of good 1 at each moment is determined by

$$y(R, \theta; L, p) = \gamma. \tag{33}$$

Substituting (33) into (21), we obtain the optimal static allocation as

$$(1 - \alpha_1)\gamma + (1 - \alpha_2)(1 - \gamma) = (L - L_R)\theta f'(L_R), \tag{34}$$

which determines the optimal allocation of labor into public production as $L_R = L^{Ra}(\theta; L)$, with the following properties:

$$\frac{\partial L^{Ra}}{\partial \theta} = \frac{(L - L_R)f'}{\theta[f' - (L - L_R)f'']} > 0, \quad \frac{\partial L^{Ra}}{\partial L} = \frac{f'}{f' - (L - L_R)f''} > 0. \tag{35}$$

Moreover, substituting (33) into (19) yields the dynamic equation for the costate variable $\theta$:

$$\dot{\theta} = (\rho + \beta)\theta - \frac{\alpha_1 \gamma + \alpha_2(1 - \gamma)}{R}. \tag{36}$$

Therefore, the dynamic system of the economy under autarky is characterized by

$$\dot{R} = f(L^{Ra}(\theta; L)) - \beta R \tag{37}$$

and (36). It is easily verified that there exists a unique steady-state equilibrium, which is a saddle point (see Figure 2).

Let us denote the steady-state solutions of $R$ and $\theta$ under autarky by $\bar{R}_a$ and $\bar{\theta}_a$. The dependence of these steady-state values on $L$ is derived as

$$\frac{\partial \bar{R}_a}{\partial L} = \frac{\bar{R}_a f'}{\beta \bar{R}_a + \bar{\theta}_a f' \frac{\partial L^{Ra}}{\partial \theta}} \frac{\partial L^{Ra}}{\partial L} > 0, \quad \frac{\partial \bar{\theta}_a}{\partial L} = -\frac{\bar{\theta}_a f'}{\beta \bar{R}_a + \bar{\theta}_a f' \frac{\partial L^{Ra}}{\partial \theta}} \frac{\partial L^{Ra}}{\partial L} < 0. \tag{38}$$

As in the case of a small open economy, an increase in labor endowment augments the steady-state stock of the public intermediate good and lowers the shadow price of it.

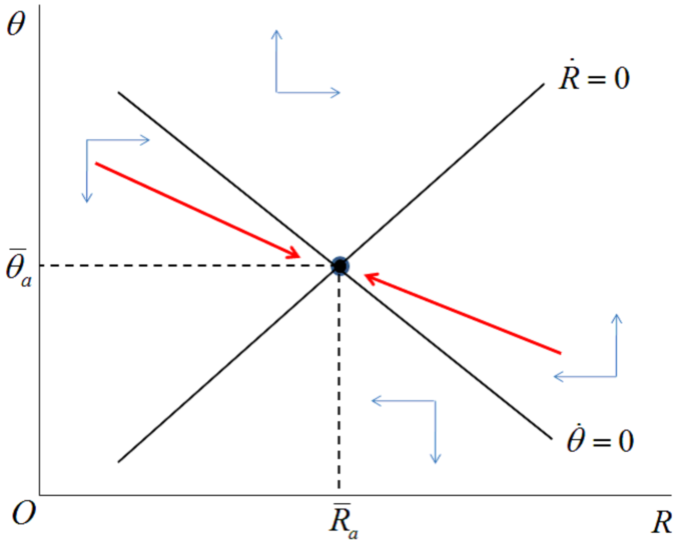


Figure 2: The Steady State under Autarky

### 5.2 Trade Pattern

Suppose that the economy is initially at the autarkic steady state, where the relative price of good 1, $\bar{p}_a$, is determined by

$$y(\bar{R}_a, \bar{\theta}_a; L, \bar{p}_a) = \gamma. \tag{39}$$

The effect of an increase in the labor endowment $L$ on the autarkic steady-state price $\bar{p}_a$ is derived as follows (see Appendix):

$$\frac{d\bar{p}_a}{dL} = -\frac{(\alpha_1 - \alpha_2)f''\beta\bar{R}_a\bar{p}_a}{\left(\beta\bar{R}_a + \bar{\theta}_a f' \frac{\partial L^{Ra}}{\partial\theta}\right)[f' - (L - L_R)f'']}. \tag{40}$$

Under Assumption 1, the sign of (40) becomes positive. That is, a country with a lower labor endowment has a lower relative price of good 1 under the autarkic steady-state equilibrium. This implies the following theorem.

**Theorem 2** *Suppose that the country is initially in an autarkic steady state. Then, after opening trade, if the country's labor endowment is low (resp. high), the country tends to face a lower (resp. higher) international relative price of a good that is more (resp. less) dependent on the stock of the public intermediate good, and thus tends to become an exporter of that good.*

McMillan (1978) and Yanase and Tawada (2010) consider a dynamic model of a small open economy with a stock of public intermediate good in which the production technology of each private good exhibits a Ricardian property, i.e., constant returns to scale with respect to labor. They show that if the labor endowment is sufficiently large (small), a small open country completely specializes in a good whose productivity is more (less) sensitive to the public intermediate good. This implies that after opening of trade, a country with a higher labor endowment becomes an exporter of a good whose productivity is more sensitive to the public intermediate good. However, in the present model with a constant-returns technology with respect to labor and the public capital, the result is reversed. This suggests that the specialization patterns in the presence of a stock of public intermediate good analyzed in McMillan (1978) and Yanase and Tawada (2010) are not robust and are dependent on the property of the public intermediate good with respect to its impact on private sectors' production technology.

# 6 Gains or Losses from Trade

In this section, we discuss whether an economy gains or loses from trade in the long run by comparing the steady-state solutions under free trade with those under autarky.

## 6.1 The Long-run Effect of Trade on the Stock of Public Intermediate Good

In light of Proposition 2, it follows that $\bar{R} > \bar{R}_a$ (resp. $\bar{R} < \bar{R}_a$) if the world relative price of good 1 is higher (resp. lower) than $\bar{p}_a$. Given this and Theorem 2, we obtain the following result.

**Proposition 3** *If a country exports a good that is more (resp. less) dependent on the stock of the public intermediate good, the steady-state stock of the public good is higher (resp. lower) under free trade than under autarky.*

An intuition behind Proposition 3 is as follows. Consider an economy that imports good 1. Because the effectiveness of the public good is higher in sector 1, the economy will reduce the need for accumulating $R$ under free trade.

Proposition 3 has an implication for how country size (measured by the labor endowment) and the effect of trade liberalization on the stock of public intermediate good are related. Let us consider two countries, home and foreign, and denote the foreign country's variables by asterisks (*). From (40), it follows that under Assumption 1, $L > L^*$ leads to $\bar{p}_a > \bar{p}_a^*$. If, in addition, the world relative price of good 1 lies between $\bar{p}_a$ and $\bar{R}_a^*$, the home (resp. foreign) country exports good 2 (resp. good 1). Then, from Proposition 3, it follows that $\bar{R} < \bar{R}_a$ and $\bar{R}^* > \bar{R}_a^*$.

## 6.2 The Long-run PPF

From (1) and (3), the production possibilities frontier (PPF) is characterized by the following equation:

$$Y_1^{A_1} R^{1-A_1} + Y_2^{A_2} R^{1-A_2} = L - L_R, \tag{41}$$

where $A_i \equiv 1/(1-\alpha_i) > 1$, $i = 1, 2$. Let us consider the steady-state equilibrium, where $\dot{R} = f(L_R) - \beta R = 0$, and thus the long-run PPF. The intercept of the long-run PPF on the $Y_i$–axis is given by $\left\{\bar{R}^{A_i-1}[L - f^{-1}(\beta\bar{R})]\right\}^{1/A_i} \equiv \tilde{Y}_i$. It is easily verified that the long-run PPF is strictly concave to the origin.[7] In addition, the following result is obtained.

**Lemma 1** *(i) For a given level of $L$, an increase in $\bar{R}$ expands the long-run PPF. (ii) An increase in $L$ expands the long-run PPF.*

(Proof) (i) By computation, it follows that

$$\begin{aligned} \frac{\partial\left\{\bar{R}^{A_i-1}[L - f^{-1}(\beta\bar{R})]\right\}}{\partial\bar{R}} &= (A_i - 1)\bar{R}^{A_i-2}[L - f^{-1}(\beta\bar{R})] - \bar{R}^{A_i-1}\beta(f^{-1})' \\ &= \frac{\bar{R}^{A_i-2}}{f'(\bar{L}_R)}\left\{(A_i - 1)(L - \bar{L}_R)f'(\bar{L}_R) - \beta\bar{R}\right\}. \end{aligned} \tag{42}$$

Because we focus on the intercept on the $Y_i$–axis, it holds that $y = 1$ for $i = 1$ and $y = 0$ for $i = 2$. In each case, (21) is rewritten as $(L - L_R)f'(L_R) = (1 - \alpha_i)/\theta$. Moreover, the steady-state condition $\dot{\theta} = 0$ is rewritten as $1/\theta = (\rho + \beta)R/\alpha_i$. Substituting these expressions into (42), we have $(A_i - 1)(L - \bar{L}_R)f'(\bar{L}_R) - \beta\bar{R} = \rho R > 0$, and thus $\partial\tilde{Y}_i/\partial\bar{R}$ is unambiguously positive for $i = 1, 2$. Because both intercepts increases, the long-run PPF expands in response to an increase in $\bar{R}$.
(ii) Given that the steady-state stock of $R$ depends on $L$ and in view of (42), it follows that

$$\begin{aligned} \frac{d\left\{\bar{R}^{A_i-1}[L - f^{-1}(\beta\bar{R})]\right\}}{d\bar{L}} &= \frac{\partial\left\{\bar{R}^{A_i-1}[L - f^{-1}(\beta\bar{R})]\right\}}{\partial\bar{L}} + \frac{\partial\bar{R}}{\partial L}\frac{\partial\left\{\bar{R}^{A_i-1}[L - f^{-1}(\beta\bar{R})]\right\}}{\partial\bar{R}} \\ &= \bar{R}^{A_i-1}\left\{1 + \frac{\partial\bar{R}}{\partial L}\frac{\rho}{f'(\bar{L}_R)}\right\} > 0. \end{aligned} \tag{43}$$

Therefore, it follows that $d\tilde{Y}_i/dL > 0$ for $i = 1, 2$, indicating an expansion of the long-run PPF. □

Putting Lemma 1 (i) and Proposition 3 together, we obtain the following proposition regarding the effect of trade on the long-run PPF.

[7] The strict concavity of the production possibility frontier was proven by Tawada (1980) for the case of a static economy with many factors, many final goods, and many public inputs.

**Proposition 4** *In comparison with the autarkic steady state, free trade expands (resp. shrinks) the long-run PPF in a country exporting a good that is more (resp. less) dependent on the stock of the public intermediate good.*

Because the equal amount of public-good stock affects production in both sectors (to a greater or lesser extent), an increase (resp. a decrease) in the stock of public intermediate good induces a outward (resp. inward) shift of the long-run PPF. However, the shift is biased toward good 2, which is less dependent on the public-good stock, as shown in Figure 3. This is verified as follows. The percentage change in $\tilde{Y}_i$ in response to a change in $\bar{R}$ is derived as follows:

$$\frac{d\tilde{Y}_i}{\tilde{Y}_i} = \frac{d\left\{\bar{R}^{A_i-1}[L - f^{-1}(\beta\bar{R})]\right\}}{A_i\left\{\bar{R}^{A_i-1}[L - f^{-1}(\beta\bar{R})]\right\}} = \frac{\rho}{A_i[L - f^{-1}(\beta\bar{R})]f'(\bar{L}_R)}d\bar{R}.$$

Because $1/A_1 - 1/A_2 = \alpha_2 - \alpha_1 < 0$, it follows that $d\tilde{Y}_1/\tilde{Y}_1 < d\tilde{Y}_2/\tilde{Y}_2$. Thus, trade liberalization induces a kind of *biased technological change toward the sector which is less dependent on the stock of public intermediate good.* Let us consider two countries, with $L > L^*$, and assume that the home (resp. foreign) country exports good 2 (resp. good 1). Then, trade induces the inward shift of the home country's long-run PPF (from $\tilde{Y}_2^a\tilde{Y}_1^a$ to $\tilde{Y}_2^f\tilde{Y}_1^f$) and the outward shift of the foreign country's (from $\tilde{Y}_2^{a*}\tilde{Y}_1^{a*}$ to $\tilde{Y}_2^{f*}\tilde{Y}_1^{f*}$).[8] Note also that the same holds for the percentage change in $\tilde{Y}_i$ in response to a change in $L$. Therefore, under Assumption 1, the home country's long-run PPF under autarky, $\tilde{Y}_2^a\tilde{Y}_1^a$, lies lateral to the foreign country's, $\tilde{Y}_2^{a*}\tilde{Y}_1^{a*}$, with a bias toward sector 2.

### 6.3 Gains or Losses from Trade

Now, we are in a position to discuss whether a country gains or loses from trade in the long run. We begin with a country that is assumed to be labor-scarce and thus exports good 1 under free trade. From Proposition 4 and Corollary **??**, the long-run PPF of that country expands. The world relative price of good 1 is higher than the autarkic equilibrium price in this country, i.e., $\bar{p}_a^* < p$. In addition, the country's consumption point, $C_f^*$, lies northwest of the production point $X_f^*$. Thus, as shown in Figure 4, it follows the indifference curve under free trade lies above the indifference curve under autarky. Because the consumer's consumption possibility improves and because the national income increases under free trade, the country unambiguously gains from trade.

We next consider the labor-abundant country, whose long-run PPF shrinks under free trade in comparison with autarky. The consumer benefits from

[8] Figure 3 also shows that the long-run relative price of good in the autarkic equilibrium is higher under the home country than under the foreign country. This is consistent with Theorem 2.

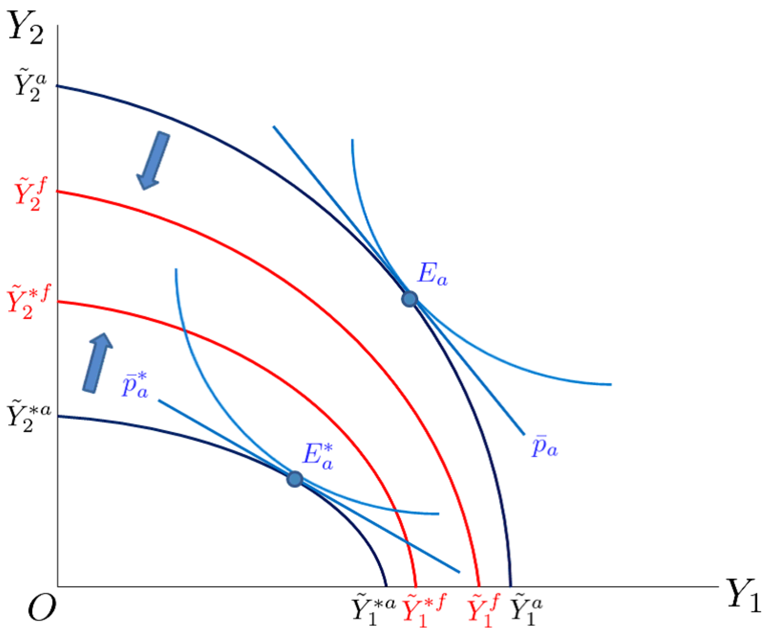


Figure 3: The effect of trade on long-run PPF ($L > L^*$)

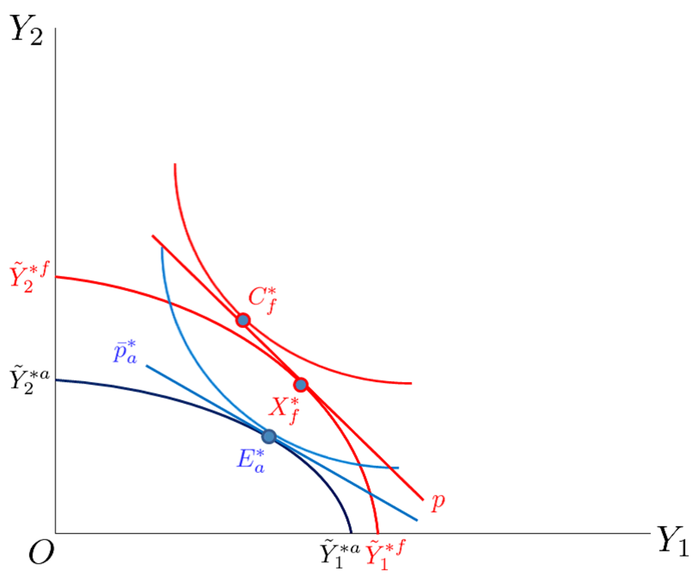


Figure 4: The gain from trade in the labor-scarce country

the improvement of consumption possibility under free trade, but national income in this country decreases. Then, the country may lose from trade, as shown in Figure 5.

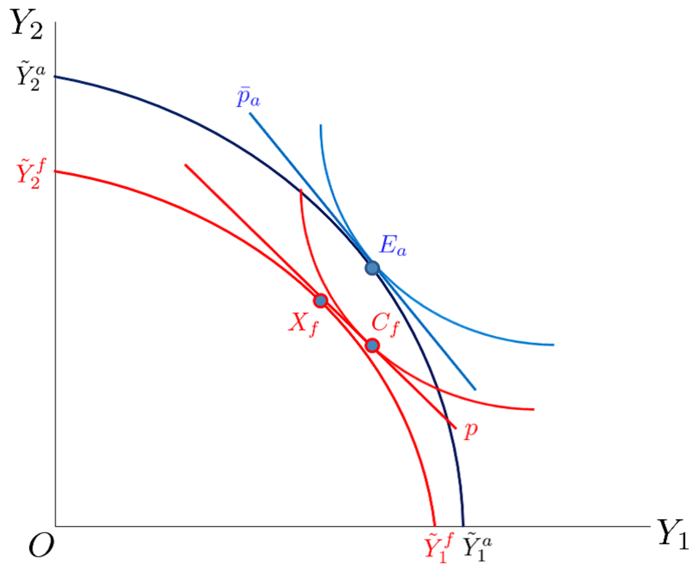


Figure 5: The case of the loss from trade in the labor-abundant country

To sum up, the following theorem is esablished.

**Theorem 3** *If a country exports a good that is more dependent on the stock of the public intermediate good, the country unambiguously gains from trade in the long run. However, if a country exports a good that is less dependent on the stock of the public intermediate good, the country may lose from trade in the long run.*

Yanase and Tawada (2010) add the gains-from-trade analysis to McMillan's (1978) model and show that a country unambiguously gains from trade in the long run only if the country has a comparative advantage in a good with productivity more sensitive to the public intermediate good; if the country has a comparative advantage in a good with productivity less sensitive to the public intermediate good, the economy may lose from trade in the long run. In the present model, we obtain the similar result. However, in their model, the country that gains (resp. may lose) from trade is the larger (resp. smaller) country measured in labor endowments. By contrast, as implied by Theorem 2, the country that gains (resp. may lose) from trade is the *smaller (resp. larger)* country in the present model. In this sense, we can conclude that the results on gains/losses from trade obtained in Yanase and Tawada (2010) are not robust and are dependent on the property of the public intermediate good.

# 7 Concluding Remarks

In this paper, we have analyzed a dynamic trade model with a public intermediate good that has a stock externality on private sectors' productivities. It was shown that there exists a unique and saddle-point stable steady state. We showed that if a country is initially at the autarkic steady state, after opening trade, a country with a lower (resp. higher) labor endowment tends to become an exporter of a good that is more (resp. less) dependent on the stock of the public intermediate good. Moreover, a country exporting a good that is more dependent on the stock of the public intermediate good unambiguously gains from trade in the long run, whereas a country exporting a good that is less dependent on the stock of the public intermediate good may lose from trade in the long run.

Throughout the paper, we have assumed a small country in which the national government is a price taker in the world commodity markets and the public intermediate good is provided by the government using the Lindahl pricing rule. Our model can be extended to that of a world economy consisting of two or more large countries. Under such environment, the government in each country may act strategically in providing the public good; i.e., the governments noncooperatively determine the volumes of the public goods, recognizing that they can affect the international prices. Then, how does such strategic behavior affect the pattern of trade in each country? The case of noncooperative supply of public intermediate good also has a significant implication for normative side of international trade. In a static model, Shimomura (2007) proves that if governments determine the levels of public goods noncooperatively, free trade is beneficial. Then, in our dynamic framework, are there gains from trade when we consider Nash instead of Lindahl pricing for public intermediate goods? These are interesting topics for future research.

# 8 Appendix

## 8.1 Comparative Statics for the Temporary Equilibrium

Totally differentiating eqs.(1), (3), (10), (13), (14), and (15), we have

$$\begin{bmatrix} \frac{1}{\pi} & -(1-y)p & y & 0 & 0 & 0 & 0 \\ -(1-\alpha_1) & 0 & 0 & L_1 & w & 0 & 0 \\ 1-\alpha_2 & 0 & 0 & L_2 & 0 & w & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\theta f'' \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -\frac{w}{\pi p} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{w}{\pi} & 0 \end{bmatrix} \begin{bmatrix} dy \\ dY_1 \\ dY_2 \\ dw \\ dL_1 \\ dL_2 \\ dL_R \end{bmatrix} = \begin{bmatrix} \frac{y(1-y)}{\pi p}dp \\ 0 \\ 0 \\ f'd\theta \\ dL \\ \alpha_1 \frac{Y_1}{R} dR \\ \alpha_2 \frac{Y_2}{R} dR \end{bmatrix}. \tag{44}$$

Solving (44) for $dy$ and $dL_R$, respectively, we obtain

$$dy = \frac{y(1-y)}{\Delta} \left\{ \frac{(\alpha_1-\alpha_2)[w-(L-L_R)\theta f'']}{R} dR + (\alpha_1-\alpha_2) f' d\theta \right. \\ \left. + (\alpha_1-\alpha_2)\theta f'' dL + \frac{w-(L_1+L_2)\theta f''}{p} dp \right\}, \tag{45}$$

$$dL_R = \frac{1}{\Delta} \left\{ \frac{(\alpha_1-\alpha_2)^2 y(1-y)}{R} dR + (\alpha_2 L_1 + \alpha_1 L_2) f' d\theta \right. \\ \left. + w[\alpha_1(1-y)+\alpha_2 y] dL + \frac{(\alpha_1-\alpha_2)y(1-y)}{p} dp \right\}, \tag{46}$$

where

$$\Delta \equiv \alpha_1[w(1-y) - L_2\theta f''] + \alpha_2[wy - L_1\theta f''] > 0.$$

## 8.2 An Analysis of the Dynamic System

Before analyzing the dynamic system (18) and (19), we show the following lemma.

**Lemma 2** $\epsilon_{LR} \equiv (\partial L^R/\partial R)\cdot(R/L^R) < 1.$

(Proof) From (17), we have

$$\epsilon_{LR} - 1 = \frac{(\alpha_1-\alpha_2)^2 y(1-y) - \Delta L_R}{\Delta L_R}, \tag{47}$$

where the numerator of the right-hand side of the above equation can be calculated as

$$(\alpha_1-\alpha_2)^2 y(1-y) - \Delta L_R = (\alpha_1 L_2 + \alpha_2 L_1)\theta f'' L_R \\ + wL\{-\alpha_1(1-y-\lambda_2) + \alpha_2(\lambda_1 - y)\}, \tag{48}$$

where $\lambda_i \equiv L_i/L$, $i = 1, 2$, and it must hold that $0 \leq \lambda_i \leq 1$ and $0 < \lambda_1 + \lambda_2 < 1$. It is clear that $1 - y - \lambda_2 - (\lambda_1 - y) = 1 - (\lambda_1 + \lambda_2) > 0$. Then, either $1 - y - \lambda_2 > \lambda_1 - y > 0$ or $0 > 1 - y - \lambda_2 > \lambda_1 - y$ holds. Given (13) and (14), it holds that $[(1-\alpha_1)/(1-\alpha_2)]y/(1-y) = \lambda_1/\lambda_2$. This implies that $1 - y - \lambda_2 > \lambda_1 - y > 0$ holds if $\alpha_1 > \alpha_2$ and $0 > 1 - y - \lambda_2 > \lambda_1 - y$ holds if $\alpha_1 < \alpha_2$. In either case, the sign of $-\alpha_1(1 - y - \lambda_2) + \alpha_2(\lambda_1 - y)$ is negative. Because $f'' < 0$, the sign of (48) becomes unambiguously negative. □

Let us denote the right-hand side of (18) and that of (19) by $\Phi(R, \theta; L, p)$ and $\Psi(R, \theta; L, p)$, respectively. Then, we have

$$\frac{\partial \Phi}{\partial R} = f' \frac{\partial L^R}{\partial R} - \beta, \tag{49}$$

$$\frac{\partial \Phi}{\partial \theta} = f' \frac{\partial L^R}{\partial \theta} > 0, \tag{50}$$

$$\frac{\partial \Psi}{\partial R} = \frac{1}{R} \left\{ \frac{\alpha_1 y + \alpha_2(1-y)}{R} - (\alpha_1 - \alpha_2) \frac{\partial y}{\partial R} \right\} > 0, \tag{51}$$

$$\frac{\partial \Psi}{\partial \theta} = \rho + \beta - \frac{1}{R}(\alpha_1 - \alpha_2) \frac{\partial y}{\partial \theta}, \tag{52}$$

where the sign of $\partial \Psi / \partial R$ comes from[9]

$$\begin{aligned}
&\frac{\alpha_1 y + \alpha_2(1-y)}{R} - (\alpha_1 - \alpha_2)\frac{\partial y}{\partial R} \\
&= \frac{\alpha_1\alpha_2 w - \theta f''\{[\alpha_1 y + \alpha_2(1-y)](\alpha_2 L_1 + \alpha_1 L_2) - (\alpha_1-\alpha_2)^2 y(1-y)(L_1+L_2)\}}{R\Delta} \\
&= \frac{\alpha_1\alpha_2 w}{R\Delta} - \frac{\theta f''}{R\Delta w}\{[\alpha_1 y + \alpha_2(1-y)][\alpha_2(1-\alpha_1)y + \alpha_1(1-\alpha_2)(1-y)] \\
&\qquad - (\alpha_1-\alpha_2)^2 y(1-y)[(1-\alpha_1)y + (1-\alpha_2)(1-y)]\} \\
&> 0.
\end{aligned} \tag{53}$$

The signs of $\partial \Phi / \partial R$ and $\partial \Psi / \partial \theta$ are in general ambiguous. However, evaluating at the steady state, we have

$$\left.\frac{\partial \Phi}{\partial R}\right|_{\dot{R}=0} = \frac{f}{\bar{R}}(\epsilon_f \epsilon_{LR} - 1) < 0, \tag{54}$$

[9] We used eqs.(13) and (14). Let us define $h(y) \equiv [\alpha_1 y + \alpha_2(1-y)][\alpha_2(1-\alpha_1)y + \alpha_1(1-\alpha_2)(1-y)] - (\alpha_1-\alpha_2)^2 y(1-y)[(1-\alpha_1)y + (1-\alpha_2)(1-y)]$. Clearly, $h(y)$ is continuous in $y$ and it holds that $h(0) = \alpha_1\alpha_2(1-\alpha_2) > 0$ and $h(1) = \alpha_1\alpha_2(1-\alpha_1) > 0$. Moreover, it is verified that the real root of the equation $h(y) = 0$, if it exists, lies outside the interval $[0, 1]$. Then, it follows that $h(y) > 0$ $\forall y \in [0, 1]$.

where $\epsilon_f \equiv f' L_R / f$ [10] and

$$\left.\frac{\partial \Psi}{\partial \theta}\right|_{\dot{\theta}=0} = \frac{\alpha_1 \bar{y} + \alpha_2(1-\bar{y})}{\bar{R}\bar{\theta}} - \frac{(\alpha_1-\alpha_2)^2 \bar{y}(1-\bar{y}) f'}{\bar{R}\Delta}$$
$$= \frac{\alpha_1\alpha_2 \bar{w} - \theta f''[\alpha_1\bar{y} + \alpha_2(1-\bar{y})](\alpha_2 \bar{L}_1 + \alpha_1 \bar{L}_2)}{\bar{R}\bar{\theta}\Delta} > 0. \quad (55)$$

## 8.3 Comparative Statics for the Steady State Equilibrium

Totally differentiating the steady state conditions, we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} d\bar{R} \\ d\bar{\theta} \end{bmatrix} = -\begin{bmatrix} \Phi_L \\ \Psi_L \end{bmatrix} dL - \begin{bmatrix} \Phi_p \\ \Psi_p \end{bmatrix} dp, \quad (56)$$

where $a_{ij}$'s are defined in (25) and

$$\Phi_L \equiv f' \frac{\partial L^R}{\partial L} = f' \frac{\bar{w}[\alpha_1(1-\bar{y}) + \alpha_2 \bar{y}]}{\Delta} > 0,$$
$$\Psi_L \equiv -\frac{\alpha_1 - \alpha_2}{R} \frac{\partial y}{\partial L} = -\frac{(\alpha_1-\alpha_2)^2 \bar{y}(1-\bar{y})\theta f''}{\bar{R}\Delta} > 0,$$
$$\Phi_p \equiv f' \frac{\partial L^R}{\partial p} = f' \frac{(\alpha_1-\alpha_2) y (1-y)}{p\Delta},$$
$$\Psi_p \equiv -\frac{\alpha_1 - \alpha_2}{R} \frac{\partial y}{\partial p} = -\frac{(\alpha_1-\alpha_2)\bar{y}(1-\bar{y})[\bar{w} - (L - \bar{L}_R)\bar{\theta} f'']}{\bar{R} p \Delta}.$$

Solving (56), we have

$$d\bar{R} = -\frac{\left(a_{22} f' \frac{\partial L^R}{\partial L} + a_{12} \frac{\alpha_1-\alpha_2}{R} \frac{\partial y}{\partial L}\right) dL + \left(a_{22} f' \frac{\partial L^R}{\partial p} + a_{12} \frac{\alpha_1-\alpha_2}{R} \frac{\partial y}{\partial p}\right) dp}{a_{11}a_{22} - a_{12}a_{21}}, \quad (57)$$

$$d\bar{\theta} = \frac{\left(a_{21} f' \frac{\partial L^R}{\partial L} + a_{11} \frac{\alpha_1-\alpha_2}{R} \frac{\partial y}{\partial L}\right) dL + \left(a_{21} f' \frac{\partial L^R}{\partial p} + a_{11} \frac{\alpha_1-\alpha_2}{R} \frac{\partial y}{\partial p}\right) dp}{a_{11}a_{22} - a_{12}a_{21}}. \quad (58)$$

Substituting the values of $a_{ij}$'s and the derivatives into (57), we have the comparative static results for $\bar{R}$, i.e., (26) and (30), and for $\bar{\theta}$, i.e., (27) and (31), respectively.

## 8.4 Derivation of Eq.(40)

Totally differentiating (39), we have

$$\left(\frac{\partial y}{\partial R}\frac{\partial \bar{R}_a}{\partial L} + \frac{\partial y}{\partial \theta}\frac{\partial \bar{\theta}_a}{\partial L} + \frac{\partial y}{\partial L}\right) dL + \frac{\partial y}{\partial p} dp = 0. \quad (59)$$

[10] Because $f(L_R)$ is assumed to be concave, $f' \leq f/L_R$ holds. Given this and Lemma (2), it follows that $\epsilon_f \epsilon_{LR} < 1$.

In light of (16), (35), and (38), the coefficient of $dL$ is rewritten as

$$\frac{\partial y}{\partial R}\frac{\partial \bar{R}_a}{\partial L}+\frac{\partial y}{\partial \theta}\frac{\partial \bar{\theta}_a}{\partial L}+\frac{\partial y}{\partial L}=\frac{(\alpha_1-\alpha_2)\gamma(1-\gamma)\bar{\theta}_a f''\beta\bar{R}_a}{\Delta\left(\beta\bar{R}_a+w\frac{\partial L^{Ra}}{\partial\theta}\right)}. \tag{60}$$

Substituting (16) and (60) into (59) and rearranging terms, we have (40).

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