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The Comparative Statics on Asset Prices
Based on Bull and Bear Market Measure[*]

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Abstract– For single–period complete financial asset markets with representative investors, we introduce a bull market measure for uncertain state occurrence and its associated ordering between representative investors in markets based on their marginal rate of substitution between equilibrium consumption allocations among possible states. These concepts combine and generalize the likelihood–ratio–dominance relation between probability prospects of state occurrence and the Arrow–Pratt ordering of risk aversion in expected utility settings. By analyzing the comparative statics for bull market effects on equilibrium asset prices, we derive some monotone properties of the risk–free rate and discounted prices of dividend–monotone assets.

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1. Introduction

A great number of studies on the comparative statics for economic behaviors in uncertain economic environments have been done under expected utility settings. The most common problems are portfolio selections discussed in financial economics and the effects of stochastic dominance shifts on probability prospects of asset returns or shifts in risk aversion of investors on optimal portfolios, see e.g. Gollier [4].

One of the reasons why expected utility settings have been widely used for analysis of decision making under uncertainty is their easy tractability due to the decomposability of expected utilities into probability prospects (probability distributions) of risks and attitudes toward risks (von Neumann and Morgenstern (vN–M) or Bernoulli utility functions, [19]). However, there are many critical discussions regarding the validity of expected utility settings, see e.g. Machina [12, 13].

When we adopt the expected utility settings for modeling of decision making under uncertain economic environments, comparative static analysis for the effects of changes in probability prospects and/or attitudes toward risks become of great interests. There have been many studies on these respective subjects since the publication of the seminal papers by Rothschild and Stiglitz [17, 18] for the former and Arrow [1] and Pratt [16] for the latter. However, only few papers are devoted to comparative static studies in the frameworks of non–expected utilities.

In this paper, under non–expected utility settings, we analyze comparative statics for single–period complete financial asset markets with representative investors. We introduce a bull market measure for uncertain state occurrence and its associated ordering between representative investors in markets based on their marginal rate of substitution between equilibrium consumption allocations among possible states by using the TP_2 (Totally Positive of Order 2) concept[4]. This concept can be combined and generalized with the likelihood–ratio–dominance (LRD) relation between probability prospects of state occurrence (Milgrom [14]) and the Arrow–Pratt ordering of risk aversion (Arrow [1] and Pratt [16]) in expected utility settings. Therefore, our results are extensions of the previous results obtained by Milgrom [14] and Ohnishi [15]. Milgrom [14] examined the effects of LRD shifts in the probability prospect of the representative investor on the risky asset in a financial market with one risk–free and one risky assets, that is, a market where the 1–fund separation theorem holds. Ohnishi [15] derived some monotone properties for the effects of the LRD shifts in probability prospect or changes in the Arrow–Pratt risk aversion of the representative investor in a complete financial market with multiple assets.

This paper is organized as follows. In Section 2, we consider a single period asset market model with the representative investor, and examine equilibrium asset prices in this market. In Section 3, we introduce a bull and bear market measure and its associated ordering by using the TP_2 concept. In Section 4, we analyze the comparative statics for the bull market effects on equilibrium asset prices, and show that dividend–monotone asset prices have some monotone properties.

2. Asset Market

Consider a standard single–period asset market model described as follows, see e.g. Duffie [3] and LeRoy and Werner [11]. This model has two dates indexed by $t = 0, 1$. The uncertainty is revealed at $t = 1$ and is classified into one of S states, indexed by $s = 1, \ldots, S$. In the asset market, there are I investors indexed by $i = 1, \cdots, I$. Each investor i is characterized by his/her own utility function $V^i : \mathbb{R}^S_+ \to \mathbb{R}$ and an initial endowment $e^i = (e^i_1, \ldots, e^i_S)^\top \in \mathbb{R}^S_+$. The utility function V^i exhibits investor i's preference $\succsim^i$

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[4] TP_2 is frequently called Log–Supermodularity in the theory of economics of uncertainty (see, e.g., Athey [2] and Gollier [4]). As for the details of the theory of Total Positivity (TP), see Karlin [7], and for its economic and financial applications, see Jewitt [5, 6], Kijima and Ohnishi [8, 9, 10].

concerning a consumption allocation $\boldsymbol{c}^i = (c_1^i, \dots, c_S^i)^\top \in \mathbb{R}_+^S$ for $t = 1$ which is planned at $t = 0$. We assume that the utility function V^i is strictly increasing and strictly concave[5]. Notice that we generally consider non–expected utility functions, some comments on a reduction to the cases of expected utilities are considered in Section 3.2.

The asset market trades N assets indexed by $n = 1, \cdots, N$, and they are characterized by an $S \times N$ dividend matrix $D = (d_{sn}; s = 1, \cdots, S, n = 1, \cdots, N)$, where d_{sn} is the amount of the dividend which asset n pays in state s at $t = 1$. We assume that the asset market is frictionless and competitive. Further, we assume that the asset market is complete. That is, the rank of the dividend matrix equals the number of states: $\operatorname{rank}(D) = S$. We also assume that the asset market does not admit any arbitrage opportunities. That is, there are no portfolios $\boldsymbol{h} = (h_1, \dots, h_N)^\top \in \mathbb{R}^N$ satisfying either of the following two conditions:

$$\langle \boldsymbol{q}, \boldsymbol{h} \rangle \leq 0; \quad D\boldsymbol{h} \geq \boldsymbol{0} \ \text{ and } \ D\boldsymbol{h} \neq \boldsymbol{0} \tag{1}$$

or

$$\langle \boldsymbol{q}, \boldsymbol{h} \rangle < 0; \quad D\boldsymbol{h} \geq \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{q} = (q_1, \dots, q_N)^\top \in \mathbb{R}_{++}^N$ are asset prices. In the theory of asset pricing, it is widely known that if the asset market does not admit arbitrage opportunities and is complete, then there exists a unique price system, Arrow–Debreu securities or state securities. The Arrow–Debreu security $s \, (= 1, \dots, S)$ or state s security pays one unit of dividend when state s occurs at $t = 1$ and nothing elsewhere. We call the prices of the Arrow–Debreu securities, state prices and denote them as $\boldsymbol{\psi} = (\psi_1, \dots, \psi_S)^\top \in \mathbb{R}_{++}^S$.

In the asset market, every investor i will maximize his/her utility from a consumption allocation $\boldsymbol{c}^i = (c_1^i, \dots, c_S^i)^\top$ for $t = 1$ which is planned at $t = 0$ by purchasing a portfolio $\boldsymbol{h}^i = (h_1^i, \dots, h_N^i)^\top$. Then, the utility maximization problem of agent i is represented as follows:

$$\underset{\boldsymbol{c}^i, \boldsymbol{h}^i}{\text{Maximize}} \quad V^i(\boldsymbol{c}^i) \tag{3}$$

$$\text{subject to} \quad \boldsymbol{e}^i + D\boldsymbol{h}^i = \boldsymbol{c}^i, \tag{4}$$

$$\langle \boldsymbol{q}, \boldsymbol{h}^i \rangle = 0. \tag{5}$$

We can rewrite the above problem by state prices as follows:

$$\underset{\boldsymbol{c}^i}{\text{Maximize}} \quad V^i(\boldsymbol{c}^i) \tag{6}$$

$$\text{subject to} \quad \langle \boldsymbol{\psi}, \boldsymbol{c}^i \rangle = \langle \boldsymbol{\psi}, \boldsymbol{e}^i \rangle. \tag{7}$$

The constraints of the above optimization problems are equalities because the utility function V^i is assumed to be strictly increasing.

Next, we define the representative investor in the asset market. The representative investor has the utility function U defined as follows:

$$U(\boldsymbol{c}) := \sup_{\boldsymbol{c}^1, \cdots, \boldsymbol{c}^I} \left\{ \sum_{i=1}^I \lambda^i V^i(\boldsymbol{c}^i) \, \middle| \, \sum_{i=1}^I \boldsymbol{c}^i = \boldsymbol{c} \right\}, \quad \boldsymbol{c} \in \mathbb{R}_+^S \tag{8}$$

where $\lambda^i \geq 0$, $i = 1, \cdots, I$; is investor i's weight. At this time, it is widely known that the representative investor's utility function is strictly increasing and strictly concave when

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[5] In this paper, the terms "increasing" and "decreasing" are used in the weak sense, that is, "increasing" means "nondecreasing" and "decreasing" means "nonincreasing".

every investor i's utility function V^i is strictly increasing and strictly concave. Thus, the constraint $\sum_{i=1}^{I} \boldsymbol{c}^i = \boldsymbol{c}$ is an equality. We denote the aggregated endowment: $\boldsymbol{e} = \sum_{i=1}^{I} \boldsymbol{e}^i$. Then the representative investor is characterized by his/her utility function $U : \mathbb{R}_+^S \to \mathbb{R}$ and the initial endowment $\boldsymbol{e} \in \mathbb{R}_+^S$. In the equilibrium, we consider the utility maximization problem of the representative investor in the asset market as follows:

$$\underset{\boldsymbol{c}}{\text{Maximize}} \quad U(\boldsymbol{c}) \tag{9}$$

$$\text{subject to} \quad \langle \boldsymbol{\psi}, \boldsymbol{c} \rangle = \langle \boldsymbol{\psi}, \boldsymbol{e} \rangle. \tag{10}$$

Under this set–up, the representative investor's consumption allocation is equal to the aggregated endowment: $\boldsymbol{c} = \boldsymbol{e}$, in the equilibrium of asset market. Then, by the first order conditions for optimality of the optimization problem (9), (10), state prices ψ_s, $s = 1, \ldots, S$ in the equilibrium are represented as follows:

$$\psi_s = \frac{\partial_s U(\boldsymbol{e})}{\sum_{s=1}^{S} e_s \partial_s U(\boldsymbol{e})}, \quad s = 1, \ldots, S, \tag{11}$$

where

$$\partial_s U(\boldsymbol{c}) := \frac{\partial U}{\partial c_s}(\boldsymbol{c}), \quad \boldsymbol{c} \in \mathbb{R}_+^S \tag{12}$$

is the marginal utility function from consumption c_s in state s $(= 1, \ldots, S)$ at $t = 1$.

The equilibrium price q of an arbitrary asset or asset portfolio whose dividend in state s $(= 1, \ldots, S)$ at $t = 1$ is d_s, is represented as follows:

$$q = \sum_{s=1}^{S} d_s \psi_s. \tag{13}$$

The equilibrium price q_f of the risk–free asset and the risk–free rate R_f are evaluated as follows:

$$q_f = \sum_{s=1}^{S} \psi_s = \frac{\sum_{s=1}^{S} \partial_s U(\boldsymbol{e})}{\sum_{s=1}^{S} e_s \partial_s U(\boldsymbol{e})}; \quad R_f = \frac{1}{q_f} = \frac{1}{\sum_{s=1}^{S} \psi_s} = \frac{\sum_{s=1}^{S} e_s \partial_s U(\boldsymbol{e})}{\sum_{s=1}^{S} \partial_s U(\boldsymbol{e})}. \tag{14}$$

If we define

$$\beta_s := \frac{\psi_s}{\sum_{s=1}^{S} \psi_s} = R_f \psi_s = \frac{\partial_s U(\boldsymbol{e})}{\sum_{s=1}^{S} \partial_s U(\boldsymbol{e})}, \quad s = 1, \ldots, S, \tag{15}$$

then we can consider $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_S)^\top$ as a probability vector (or a probability mass function) since $\beta_s \geq 0$, $s = 1, \ldots, S$; and $\sum_{s=1}^{S} \beta_s = 1$. In the arbitrage pricing theory, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_S)^\top$ is called the risk neutral probability measure or equivalent martingale measure. By rewriting eq. (13), we obtained the risk neutral evaluation formula for the equilibrium price of an arbitrary asset whose value in state s $(= 1, \cdots, S)$ is d_s, as follows:

$$q = \frac{1}{R_f} \sum_{s=1}^{S} \beta_s d_s = q_f \sum_{s=1}^{S} \beta_s d_s. \tag{16}$$

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3. Bull and Bear Market

3.1 Definition

In the previous section, we see that an asset market is characterized by a representative investor j. So we call an asset market with a representative investor j $(=1,2)$, as market j, where representative investor j is characterized by his/her utility function $U^j : \mathbb{R}_+^S \to \mathbb{R}$ and an aggregated endowment $\boldsymbol{e} = \sum_{i=1}^{I} \boldsymbol{e}^i$. In this and following sections, without any loss of generality, we assume that an aggregated endowment is monotone increasing in the state number:

$$e_1 \leq \ldots \leq e_S. \tag{17}$$

Now we characterize different attitude to the asset market to introduce the desirability to states. We reinterpret the marginal utility in state s: $\partial_s U(\boldsymbol{e})$ to be the measure of the desirability to state s. We call the marginal utility in state s to be reinterpreted, bull and bear market measure. Note that the consumption allocation $\boldsymbol{c}$ of the representative investor with strictly increasing utility function is equal to the aggregated endowment $\boldsymbol{e}$. Note also that the aggregated endowment is assumed to be monotone increasing in the state numbers: $e_1 \leq \ldots \leq e_S$. We now introduce an ordering relation between asset markets. We say that the asset market is more bullish (or more bearish), if the more bullish (more bearish) asset market prefer the state with higher (lower) consumption allocation than the less bullish (less bearish) asset market. To get more precise expression, we use the common economic concept of marginal rate of substitution of consumption allocation to state m for consumption allocation to state l, which is defined as the ratio of their marginal utilities:

$$\mathrm{MRS}_{ml}(\boldsymbol{e}) := \frac{\partial_m U(\boldsymbol{e})}{\partial_l U(\boldsymbol{e})}, \quad 1 \leq l \leq m \leq S; \ \boldsymbol{e} \in \mathbb{R}_+^S. \tag{18}$$

Note that $e_l \leq e_m$ for $1 \leq l \leq m \leq S$ by the assumption on the aggregated endowment. Then eq. (18) expresses the amount of lower consumption allocation that the investor must be given to compensate him/her for one unit marginal reduction in higher consumption allocation m. That is, eq. (18) is the relative evaluation of an additional consumption allocation with higher consumption state m measured by that at state l with lower consumption. In this preparation, we define that for two markets 1 and 2, market 2 is more bullish than market 1 if the following inequality holds:

$$\mathrm{MRS}_{ml}^1(\boldsymbol{e}) := \frac{\partial_m U^1(\boldsymbol{e})}{\partial_l U^1(\boldsymbol{e})} \leq \frac{\partial_m U^2(\boldsymbol{e})}{\partial_l U^2(\boldsymbol{e})} =: \mathrm{MRS}_{ml}^2(\boldsymbol{e}), \quad 1 \leq l \leq m \leq S; \ \boldsymbol{e} \in \mathbb{R}_+^S. \tag{19}$$

Hence, eq. (19) tells us relative evaluations in more bullish markets at the higher state m are higher than those in less bullish markets. We redefine the ordering of bull market, eq. (19) by the use of TP_2 (Totally Positive of Order 2) concept.

Definition 3.1. Market 2 is said to be more bullish (less bearish) than market 1 (or market 1 is said to be more bearish (less bullish) than market 2) at aggregated endowment $\boldsymbol{e}$ $(\in \mathbb{R}_+^S)$ if and only if the function $\partial_s U^j(\boldsymbol{e})$ is TP_2 with respect to $s = 1, \cdots, S$ and $j = 1, 2$, that is,

$$\begin{vmatrix} \partial_l U^1(\boldsymbol{e}) & \partial_m U^1(\boldsymbol{e}) \\ \partial_l U^2(\boldsymbol{e}) & \partial_m U^2(\boldsymbol{e}) \end{vmatrix} \geq 0, \quad 1 \leq l \leq m \leq S. \tag{20}$$

$\square$

3.2 The Case of Expected Utility

Until now, we consider $U : \mathbb{R}_+^S \to \mathbb{R}$ to be a non–expected utility function. In this subsection, we assume, for a special case, $U : \mathbb{R}_+^S \to \mathbb{R}$ has an expected utility representation:

$$U(\boldsymbol{c}) = \sum_{s=1}^{S} \pi_s u(c_s), \boldsymbol{c} \in \mathbb{R}_+^S, \tag{21}$$

where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_S)^\top$ is a probability vector consisting of a probability prospect π_s that state s occurs at $t = 1$ with $\pi_s \geq 0$, $s = 1, \ldots, S$, $\sum_{s=1}^{S} \pi_s = 1$, and $u : \mathbb{R}_+ \to \mathbb{R}$ is a von Neumann–Morgenstern (vN–M) (or Bernoulli) utility function. We further assume that an investor is risk–averse: $u' > 0, u'' \leq 0$. Since a utility function $U : \mathbb{R}_+^S \to \mathbb{R}$ with an expected utility representation is linear in probability, we can decompose it into two parts: a probability prospect π and an attitude toward risk represented by a vN–M utility function u. The effects of the former are investigated with the application of the theory of stochastic dominances, one of which is the likelihood–ratio–dominance (LRD). For the latter, Arrow [1] and Pratt [16] consider that the attitude toward risk is represented by the degree of concavity of vN–M utility function. This is called the Arrow–Pratt Absolute Risk Aversion (ARA). As discussed later, since effects of shifts in probability prospect π and changes in attitude toward risk on decision making, work in the same (opposite) direction, these concepts could be considered dual in the sense of TP$_2$ (see Gollier [4]). The observation motivates our introduction of new bull and bear market concepts

We introduce the definition and properties of stochastic dominances used in this and following sections:

Definition 3.2 (Stochastic Dominances). Let $\boldsymbol{\alpha}^j = (\alpha_1^j, \ldots, \alpha_S^j)$, $j = 1, 2$ be two probability vectors (or probability mass functions) on $\{1, \ldots, S\}$.

(1) $\boldsymbol{\alpha}^2$ dominates $\boldsymbol{\alpha}^1$ in the sense of likelihood–ratio–dominance (LRD) if α_s^j is TP$_2$ (Totally Positive of Order 2) with respect to $s = 1, \ldots, S$ and $j = 1, 2$, that is,

$$\begin{vmatrix} \alpha_l^1 & \alpha_m^1 \\ \alpha_l^2 & \alpha_m^2 \end{vmatrix} \geq 0, \quad 1 \leq l \leq m \leq S. \tag{22}$$

In this case, we write this as $\boldsymbol{\alpha}^1 \leq_{\text{LRD}} \boldsymbol{\alpha}^2$.

(2) $\boldsymbol{\alpha}^2$ dominates $\boldsymbol{\alpha}^1$ in the sense of First order Stochastic Dominance (FSD) if

$$\sum_{s=k}^{S} \alpha_s^1 \geq \sum_{s=k}^{S} \alpha_s^2, \quad k = 1, \ldots, S. \tag{23}$$

In this case, we write this as $\boldsymbol{\alpha}^1 \leq_{\text{FSD}} \boldsymbol{\alpha}^2$. $\qquad\square$

The following theorem is well known in the theory of stochastic dominances. For details see, e.g., Kijima and Ohnishi [8].

Theorem 3.1. Let $\boldsymbol{\alpha}^j = (\alpha_1^j, \ldots, \alpha_S^j)$, $j = 1, 2$ be two probability vectors (or probability mass functions) on $\{1, \ldots, S\}$. We have the followings:

(1) $\boldsymbol{\alpha}^1 \leq_{\text{LRD}} \boldsymbol{\alpha}^2$ implies $\boldsymbol{\alpha}^1 \leq_{\text{FSD}} \boldsymbol{\alpha}^2$.

(2) $\boldsymbol{\alpha}^1 \leq_{\text{FSD}} \boldsymbol{\alpha}^2$ holds if and only if, for any S–dimensional vector $\boldsymbol{f} = (f_1, \ldots, f_S)$ with increasing components ($f_1 \leq \ldots \leq f_S$), we have

$$\sum_{s=1}^{S} \alpha_s^1 f_s \leq \sum_{s=1}^{S} \alpha_s^2 f_s. \tag{24}$$

$\qquad\square$

Next, introduce the definition and properties of Arrow–Pratt ordering of risk aversion. Define the Absolute Risk Aversion (ARA) of u in the Arrow–Pratt sense by

$$A(x; u) := -\frac{u''(x)}{u'(x)} \quad (\geq 0). \tag{25}$$

Definition 3.3. Let u^j ($u^{j'} > 0$, $u^{j''} \leq 0$), $j = 1, 2$ be twice differentiable vN–M utility functions of two risk averters, defined on a common open interval of the real line $\mathbb{R}$. If it holds that

$$A(x; u^1) = -\frac{u^{1''}(x)}{u^{1'}(x)} \geq -\frac{u^{2''}(x)}{u^{2'}(x)} = A(x; u^2), \quad \forall x, \tag{26}$$

then u^1 is said to be more risk averse than u^2 (u^2 is said to be more risk tolerant than u^1) in the sense of APRA (Arrow–Pratt Risk Aversion) and in this case, we write this as $u^1 \geq_{\text{APRA}} u^2$. $\qquad \square$

In previous works (e.g., Jewitt [5, 6], Kijima and Ohnishi [9]), the ordering of ARA in the Arrow–Pratt sense is characterized by the TP_2 concept: $u^1 \geq_{\text{APRA}} u^2$ if and only if:

$$\begin{vmatrix} u^{1'}(x) & u^{1'}(y) \\ u^{2'}(x) & u^{2'}(y) \end{vmatrix} \geq 0, \quad x \leq y, \tag{27}$$

that is, $u^{j'}(x)$ is TP_2 with respect to $j = 1, 2$ and x.

In the case of expected utility, marginal utility in state s is represented by

$$\partial_s U(\boldsymbol{c}) = \pi_s u'(c_s), \quad s = 1, \dots, S; \ \boldsymbol{c} \in \mathbb{R}_+^S, \tag{28}$$

therefore the condition of Definition 3.1 is reduced to the followings: $\pi_s^j u^{j'}(e_s)$ is TP_2 with respect to $s = 1, \cdots, S$ and $j = 1, 2$, i.e.,

$$\begin{vmatrix} \pi_l^1 u^{1'}(e_l) & \pi_m^1 u^{1'}(e_m) \\ \pi_l^2 u^{2'}(e_l) & \pi_m^2 u^{2'}(e_m) \end{vmatrix} = \pi_l^1 u^{1'}(e_l) \pi_m^2 u^{2'}(e_m) - \pi_m^1 u^{1'}(e_m) \pi_l^2 u^{2'}(e_l) \geq 0, \quad 1 \leq l \leq m \leq S. \tag{29}$$

Each of the conditions $\boldsymbol{\pi}^1 \leq_{\text{LRD}} \boldsymbol{\pi}^2$ and $u^1 \geq_{\text{APRA}} u^2$ is generalized to

$$\pi_l^1 u^{1'}(e_l) \pi_m^2 u^{2'}(e_m) - \pi_m^1 u^{1'}(e_m) \pi_l^2 u^{2'}(e_l) = \begin{vmatrix} \pi_l^1 u^{1'}(e_l) & \pi_m^1 u^{1'}(e_m) \\ \pi_l^2 u^{2'}(e_l) & \pi_m^2 u^{2'}(e_m) \end{vmatrix} \geq 0, \quad 1 \leq l \leq m \leq S. \tag{30}$$

In other words, eq. (30) suggests a set of sufficient condition of Definition 3.2:

Proposition 3.1. Suppose that two probability prospects $\boldsymbol{\pi}^j$, $j = 1, 2$ and two vN–M utility functions $u^j : \mathbb{R}_+ \to \mathbb{R}$ satisfy the following two conditions:

(SD) In the sense of LRD, $\boldsymbol{\pi}^2$ dominates $\boldsymbol{\pi}^1$, i.e., $\boldsymbol{\pi}^1 \leq_{\text{LRD}} \boldsymbol{\pi}^2$;

(RA) In the sense of APRA, market 1 is more risk averse than market 2, i.e., $u^1 \geq_{\text{APRA}} u^2$.

Then the conditions of Definition 3.1. are satisfied. $\qquad \square$

4. Comparative Statics

In this section, we examine some comparative statics on equilibrium prices of three types of assets or asset portfolios as follows:

Definition 4.1.

(RF) Assets and/or portfolios that pay one unit of consumption as a dividend in all states. That is, the asset whose dividend d_s in state s is 1 in every state $s\ (=1,\cdots,S)$, i.e., $d_1 = \cdots = d_S = 1$. We call this type of assets risk–free assets;

(MI) Assets and/or portfolios that pay dividends which are monotonic increasing in states, i.e., $d_1 \leq \cdots \leq d_S$. We call this type of assets MI (Monotone Increasing) assets;

(MD) Assets and/or portfolios that pay dividends which are monotonic decreasing in states, i.e., $d_1 \geq \cdots \geq d_S$. We call this type of assets MD (Monotone Decreasing) assets. $\square$

By eq. (14) in Section 2, the equilibrium price q_f of the risk–free asset and the risk–free rate R_f are evaluated as

$$q_f = \sum_{s=1}^{S} \psi_s = \frac{\sum_{s=1}^{S} \partial_s U(\boldsymbol{e})}{\sum_{s=1}^{S} e_s \partial_s U(\boldsymbol{e})}; \quad R_f = \frac{1}{q_f} = \frac{1}{\sum_{s=1}^{S} \psi_s} = \frac{\sum_{s=1}^{S} e_s \partial_s U(\boldsymbol{e})}{\sum_{s=1}^{S} \partial_s U(\boldsymbol{e})}. \tag{31}$$

In the remaining of this subsection, we consider two markets, market 1 and 2 for comparative static analysis and assume that market 2 is more bullish than market 1 (market 1 is more bearish than market 2). In this set–up, we obtain the following theorem:

Theorem 4.1. Suppose two markets and assume that one market is more bullish than the other market. Then, the risk–free rate in a more bullish market is higher than that in a less bullish market. Or equivalently, the equilibrium price of the risk–free asset in a more bullish market is lower than that in a less bullish market. That is, the following relations hold:

$$R_f^1 \leq R_f^2 \quad \text{and} \quad q_f^1 \geq q_f^2. \tag{32}$$

Proof. Define

$$\alpha_s^j := \frac{\partial_s U^j(\boldsymbol{e})}{\sum_{s=1}^{S} \partial_s U^j(\boldsymbol{e})}, \quad j = 1, 2. \tag{33}$$

Then, we can rewrite the risk–free rates by using α_s^j, $s = 1, \cdots, S$ as:

$$R_f^j = \sum_{s=1}^{S} \alpha_s^j e_s, \quad j = 1, 2. \tag{34}$$

Notice that $\boldsymbol{\alpha}^j = (\alpha_1^j, \cdots, \alpha_S^j)^\top$, $j = 1, 2$ are considered as probability vectors (or probability mass functions), since $\alpha_s^j \geq 0$, $s = 1, \cdots, S$; $\quad \sum_{s=1}^{S} \alpha_s^j = 1$.

Now we show that

$$\frac{\alpha_s^2}{\alpha_s^1} \text{ is monotone increasing in } s. \tag{35}$$

By definition of α_s^j, $s = 1, \cdots, S$, $j = 1, 2$, we have

$$\frac{\alpha_s^2}{\alpha_s^1} = \frac{\sum_{s=1}^{S} \partial_s U^1(\boldsymbol{e})}{\sum_{s=1}^{S} \partial_s U^2(\boldsymbol{e})} \cdot \frac{\partial_s U^2(\boldsymbol{e})}{\partial_s U^1(\boldsymbol{e})}. \tag{36}$$

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Since, by Definition 3.1,

$$\frac{\partial_l U^2(\boldsymbol{e})}{\partial_l U^1(\boldsymbol{e})} \leq \frac{\partial_m U^2(\boldsymbol{e})}{\partial_m U^1(\boldsymbol{e})}, \quad 1 \leq l \leq m \leq S, \tag{37}$$

the second factor of the right hand side in eq. (36) is monotone increasing in s so that the likelihood ratio α_s^2/α_s^1 is monotone increasing in s. Since $\boldsymbol{\alpha}^j$, $j = 1, 2$ can be considered as probability vectors (or probability mass functions), $\boldsymbol{\alpha}^2$ dominates $\boldsymbol{\alpha}^1$ in the sense of LRD, i.e., $\boldsymbol{\alpha}^1 \leq_{\text{LRD}} \boldsymbol{\alpha}^2$. In the previous section, we know that the stochastic order of LRD implies the stochastic order of FSD. We assume that $e_1 \leq \ldots \leq e_S$ so that we obtain

$$R_f^1 = \sum_{s=1}^{S} \alpha_s^1 e_s \leq \sum_{s=1}^{S} \alpha_s^2 e_s = R_f^2, \tag{38}$$

or equivalently,

$$q_f^1 = \frac{1}{R_f^1} \geq \frac{1}{R_f^2} = q_f^2. \tag{39}$$

$\square$

Theorem 4.2. Let us consider the risk–free asset as a numèrare. The relative price of an MI (MD) asset in a more bullish (more bearish) market is higher than that in a less bullish (less bearish) market. That is, the following inequalities hold:

$$\frac{q_{\text{MI}}^1}{q_f^1} \leq \frac{q_{\text{MI}}^2}{q_f^2} \quad \text{while} \quad \frac{q_{\text{MD}}^1}{q_f^1} \geq \frac{q_{\text{MD}}^2}{q_f^2}, \tag{40}$$

where q_{MI}^j and q_{MD}^j are prices of an MI asset and MD asset, respectively.

Proof. We define

$$\beta_s := \frac{\psi_s}{\sum_{s=1}^{S} \psi_s} = R_f \psi_s = \frac{\partial_s U(\boldsymbol{e})}{\sum_{s=1}^{S} \partial_s U(\boldsymbol{e})}, \quad s = 1, \cdots, S, \tag{41}$$

then we can consider $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_S)^\top$ as a probability vector (a probability mass function) since

$$\beta_s \geq 0, \ s = 1, \cdots, S; \quad \sum_{s=1}^{S} \beta_s = 1. \tag{42}$$

By eq. (16) of Section 2, an equilibrium relative price of an asset whose value in state s $(= 1, \cdots, S)$ is d_s, can be evaluated as

$$\frac{q}{q_f} = R_f q = R_f \sum_{s=1}^{S} d_s \psi_s = \sum_{s=1}^{S} \beta_s d_s. \tag{43}$$

We can prove the theorem in a similar way to the proof of Theorem 4.1. $\square$

Here, MI (MD) assets are characterized below. MI (MD) assets have the property that more high dividends are realized when higher (lower) consumption allocations are realized. In other words, MI (MD) assets pay higher dividends in economic boom (slowdown). Finally, we give an economic interpretation of the result of Theorem 4.2. Definition 3.1 can be rewritten as;

$$\text{MRS}_{ml}^1(\boldsymbol{e}) \leq \text{MRS}_{ml}^2(\boldsymbol{e}) \quad \text{and} \quad \text{MRS}_{lm}^1(\boldsymbol{e}) \geq \text{MRS}_{lm}^2(\boldsymbol{e}), \quad 1 \leq l \leq m \leq S; \ \boldsymbol{e} \in \mathbb{R}_+^S, \tag{44}$$

where $\mathrm{MRS}_{ml(lm)}(\boldsymbol{e})$ is marginal rate of substitution of consumption allocation to state $m(l)$ for consumption allocation to state $l(m)$. In other words, eq. (44) tells us the relative evaluation of an additional consumption allocation with higher (lower) consumption state $m(l)$ measured by that at state $l(m)$ with lower (higher) consumption of more bullish (more bearish) markets is higher than less bullish (less bearish) markets. From the assumption of dividends of MI (MD) assets, the demand of MI (MD) assets in more bullish (more bearish) markets is relatively higher than that in less bullish (less bearish) markets. Therefore the equilibrium price of MI (MD) assets in more bullish (more bearish) markets is higher than that in less bullish (less bearish) markets.

5. Concluding Remarks

For single–period complete financial asset markets with representative investors, we introduced a bull market measure for uncertain state occurrence and its associated ordering between representative investors in markets based on their marginal rate of substitution between equilibrium consumption allocations among possible states. By analyzing the comparative statics for bull market effects on equilibrium asset prices, we derive some monotone properties of the risk–free rate and discounted prices of dividend–monotone assets. These results are extensions of the previous results based on the likelihood–ratio–dominance relation between probability prospects of state occurrence and the Arrow–Pratt ordering of risk aversion under expected utility settings.

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