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Daisuke ITO ^{*†} Masamitsu OHNISHI [‡] Yasuhiro TAMBA [§]

Abstract

In this paper, we deal with no-arbitrage pricing problems of a chooser flexible cap (floor) written on an underlying LIBOR. The chooser flexible cap (floor) allows a right for a buyer to exercise a limited and pre-determined number of the interim period caplets (floorlets) in a multiple-period cap (floor) agreement. Assuming a common diffusion short rate dynamics, e.g., Hull–White model, we propose a dynamic programming approach for their risk neutral evaluation. This framework is suited to a calibration from an observed initial yield curve and market price data of discount bonds, caplets, and floorlets.

Keywords: chooser flexible cap, LIBOR, dynamic programming, Hull–White model, calibration.

JEL classification: G13, G15, G21

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1 Introduction

Various types of exotic interest rate derivatives have appeared since the middle of 1990s. One of the most traded interest rate derivatives is a Bermudan swaption. The Bermudan swaption is an option, which at each date in a schedule of exercise dates gives the holder the right to enter an interest swap, provided that this right has not been exercised at any previous time in the schedule. Because of its usefulness as hedges for callable bonds, the Bermudan swaption is probably the most liquid interest rate instrument with a built-in early exercise feature. There is another type of exotic interest rate derivatives, a chooser flexible cap. The chooser flexible cap is a financial contract which allows to a buyer the right to exercise dynamically at most l ($1 \leq l \leq N$) out of N caplets whose i -th one ($i = 0, \dots, N-1$) is written on the LIBOR whose setting time and payment time are T_i and T_{i+1} , respectively. The chooser flexible cap is less traded than the Bermudan swaption and more traded in Europe than other countries. Although there are some different features between the Bermudan swaption and the chooser flexible cap, these are the same type of instruments in terms of having the early exercise feature. There are mainly two merits in the chooser flexible cap compared to a more popular cap. The first merit is that its price is cheaper than the cap's because the chooser flexible cap has less exercise opportunities than the cap. So an option holder can hedge the interest rate risk with the lower cost than the cap. The second merit is its flexibility. The holder of the chooser flexible cap can reflect her/his expectation of the future interest rate change in her/his hedging strategy. For example, if s/he has the expectation that the floating interest rate will increase in the next one year, s/he should exercise the option mainly in one year as s/he observes the real interest rate values. So using the chooser flexible cap, the option holder can flexibly hedge the interest rate risk with the low cost.

There are a lot of papers for pricing the Bermudan swaption because it is popular in the market. The proposed pricing method of most papers is a Monte Carlo simulation because of its simplicity and applicability for a multifactor model with long maturity instruments. In spite of its usefulness, there are some weaknesses of the Monte Carlo simulation to be applied for pricing exotic derivatives. The most important drawback is that the Monte Carlo simulation has difficulty in dealing with derivatives that contain early exercise features, like an American option and the Bermudan option. So we need some improvement of the Monte Carlo method for pricing the above early exercise derivatives. Longstaff and Schwartz (1998) developed the least square method to overcome the weakness of the Monte Carlo simulation for pricing American option. Andersen (1999) used the Monte Carlo simulation for pricing the Bermudan swaption, and derived a lower bound on the Bermudan swaption prices considering less advantage strategies. Broadie and Glasserman (1997a, 1997b) also coped with the difficulties of the early exercise problem of the Monte Carlo simulation.

Though the Monte Carlo simulation may be used for pricing the chooser flexible cap with some improvement of the method, there are different features between the Bermudan swaption and the chooser flexible cap. While the number of the exercise opportunity of the Bermudan swaption is only one, the chooser flexible cap's is generally greater than two. Another difference is that while payoffs follow sequentially after the exercise in the case of the Bermudan swaption, in the case of the chooser flexible cap the option holder needs to choose exercise time dates, that is, the payoff dates. These different features of the instruments affect a choice of pricing methods applied. The dynamic programming approach may be more appropriate than the Monte Carlo simulation to price the chooser flexible cap. The research of the chooser flexible cap pricing has been

only seen in Pedersen and Sidenius (1998) utilizing an optimality equation of the dynamic programming approach. This method is rather appropriate for pricing exotic derivatives because we can solve the problem by a backward induction setting up a recombining tree. Although Pedersen and Sidenius (1998) made an important contribution in terms of using the optimality equation for the pricing, they did not describe much about a calibration. The other feature of their paper is that they used the spot LIBOR for the tree construction.

In this paper we focus on the pricing problem of exotic interest rate derivatives, in particular the chooser flexible cap and the chooser flexible floor, based on the observed market prices of rather simple interest rate derivatives, e.g., caplets and the floorlets, etc. We extend the pricing method in some points compared with Pedersen and Sidenius (1998). One of them is that deriving the theoretical prices of bond, the caplet and the floorlet we use these prices for a calibration. Another difference is that we use the short rate for the tree construction instead of the LIBOR. This point has advantage in that the short rate model like Hull–White has some convinced features matching the interest rate movement in the real world such as a mean reversion property. We also show numerical examples and comparative statics.

The paper is organized as follows. In Section 2, we introduce the various notations about interest rate. In Section 3, we derive theoretical no-arbitrage prices of discount bonds in an affine term structure. In Section 4 and 5, we discuss the pricing method of the chooser flexible cap and the chooser flexible floor with the optimality equations. Section 6 shows how to calculate theoretical no-arbitrage prices of the caplet and the floorlet. Section 7 is devoted to a calibration of model parameters. In Section 8, we describe a construction of a trinomial tree for the chooser flexible cap pricing. In Section 9, we show numerical examples and discuss comparative statics. Section 10 concludes the paper.

2 Various Notations about Interest Rates

In this section we explain notations of various types of interest rates.

Let $D(t, T)$ $0 \leq t \leq T \leq T^*$ be the time t price of the discount bond (or zero-coupon bond) with maturity T , in brief T -bond, which pays 1-unit of money at the maturity T (where $D(T, T) = 1$ for any $T \in \mathbb{T}^*$). For $0 \leq t \leq S < T \leq T^*$,

$$R(t; S, T) := -\frac{\ln D(t, T) - \ln D(t, S)}{T - S} \left(\iff \exp\{R(t; S, T)(T - S)\} = \frac{D(t, S)}{D(t, T)} \right) \quad (1)$$

is the (continuous compounding based) forward rate prevailing at time t which covers time interval $(S, T]$. For $0 \leq t < T \leq T^*$,

$$Y(t, T) := R(t; t, T) = -\frac{1}{T - t} \ln D(t, T) \left(\iff \exp\{Y(t, T)(T - t)\} = \frac{1}{D(t, T)} \right) \quad (2)$$

is the (continuous compounding based) spot rate prevailing at current time t or yield which covers time interval $(t, T]$. The map $T \mapsto Y(t, T)$ is called the yield curve at time t . For $0 \leq t \leq T \leq T^*$,

$$f(t, T) := \lim_{U \downarrow T} R(t; T, U) = -\frac{\partial}{\partial T} \ln D(t, T) \quad (3)$$

is the (instantaneous) forward rate prevailing at current time t with the maturity time T . The map $T \mapsto f(t, T)$ is called the forward rate curve at time t . For $t \in \mathbb{T}^*$,

$$r(t) := f(t, t) = \lim_{T \downarrow t} Y(t, T) = - \left. \frac{\partial}{\partial T} \ln D(t, T) \right|_{T=t} \quad (4)$$

is the short rate at time t . For $0 \leq t \leq T \leq T^*$,

$$B(t, T) := \exp \left\{ \int_t^T r(s) ds \right\} \quad (5)$$

is the risk-free bank account at time T with unit investment capital at time t (where $B(t, t) = 1$).

For $N \in \mathbb{Z}_+$, let

$$0 \leq T_0 < T_1 < \cdots < T_i < T_{i+1} < \cdots < T_{N-1} < T_N \leq T^* \quad (6)$$

be the sequence of setting times and payment times of floating interest rates, that is, for $i = 0, \dots, N-1$, the floating interest rate which covers time interval $(T_i, T_{i+1}]$, is set at time T_i and paid at time T_{i+1} . For convenience, we let

$$T_{i+1} - T_i = \delta \quad (= \text{constant} \in \mathbb{R}_{++}), \quad i = 0, \dots, N-1. \quad (7)$$

For $i = 0, \dots, N-1$, we define the simple (or simple compounding based) interest rate which covers time interval $(T_i, T_{i+1}]$ by

$$L_{T_i}(T_i) := \frac{1}{\delta} \left\{ \frac{1}{D(T_i, T_{i+1})} - 1 \right\}. \quad (8)$$

This amount is set at time T_i , paid at time T_{i+1} , and is conventionally called as a spot LIBOR (London Inter-Bank Offer Rate). For $i = 0, \dots, N-1$,

$$L_{T_i}(t) := \frac{1}{\delta} \left\{ \frac{D(t, T_i)}{D(t, T_{i+1})} - 1 \right\} \quad (9)$$

is the simple (or simple compounding based) interest rate prevailing at time $t \in [0, T_i]$ which covers time interval $(T_i, T_{i+1}]$, and is called as a forward LIBOR.

The i -caplet is a financial contract in which, at time T_{i+1} , the seller pays to the buyer the amount of money corresponding the difference between the spot LIBOR $L_{T_i}(T_i)$ and a predetermined upper-limit exercise rate $K \in \mathbb{R}$ if the former exceeds the latter:

$$\delta [L_{T_i}(T_i) - K]_+ \quad (= \delta \max\{L_{T_i}(T_i) - K, 0\}). \quad (10)$$

It could be considered as a call option written on the underlying LIBOR for hedging its upside risk. The i -floorlet is a financial contract in which, at time T_{i+1} , the seller pays to the buyer the amount of money corresponding the difference between the spot LIBOR $L_{T_i}(T_i)$ and a predetermined lower-limit exercise rate $K \in \mathbb{R}$ if the former falls below the latter:

$$\delta [K - L_{T_i}(T_i)]_+ \quad (= \delta \max\{K - L_{T_i}(T_i), 0\}). \quad (11)$$

It could be considered as a put option written on the underlying LIBOR for hedging its downside risk.

3 No-Arbitrage Prices of Discount Bonds

3.1 Affine Term Structure Model

We consider a continuous trading economy with a finite time horizon given by $\mathbb{T}^* := [0, T^*]$ ($T^* \in \mathbb{R}_{++}$). The uncertainty is modelled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$. In this notation, Ω denotes a sample space with elements $\omega \in \Omega$; \mathcal{F} denotes a σ -algebra on Ω ; and \mathbb{P} denotes a probability measure on (Ω, \mathcal{F}) . The uncertainty is resolved over \mathbb{T}^* according to a 1-dimensional Brownian (motion) filtration $\mathbb{F} := (\mathcal{F}(t) : t \in \mathbb{T}^*)$ satisfying the usual conditions. $W := (W(t) : t \in \mathbb{T}^*)$ denotes a 1-dimensional standard $(\mathbb{P}; \mathbb{F})$ -Brownian motion.

Consistent with the no-arbitrage and complete market paradigm, we assume the existence of the risk neutral equivalent martingale measure \mathbb{P}^* with a bank account as a numéraire in this economy. Assume that, under the risk neutral probability measure \mathbb{P}^* , the short rate process $r = (r(t) : t \in \mathbb{T}^*)$ follows a diffusion type (Markovian) SDE:

$$dr(t) = \mu(r(t), t)dt + \sigma(r(t), t)dW^*(t), \quad t \in \mathbb{T}^*, \quad (12)$$

where the drift coefficient $\mu(r(t), t)$ and the diffusion coefficient $\sigma(r(t), t)$ are deterministic functions of r and t with suitable regularity conditions. Since the short rate process $r = (r(t) : t \in \mathbb{T}^*)$ is Markovian, we have, by the risk neutral evaluation method,

$$\begin{aligned} D(t, T) &= \mathbb{E}^* \left[\frac{1}{B(t, T)} \middle| \mathcal{F}(t) \right] \\ &= \mathbb{E}^* \left[\exp \left\{ - \int_t^T r(s) ds \right\} \middle| r(t) \right] \\ &=: D(t, T; r(t)), \quad 0 \leq t \leq T \leq T^*. \end{aligned} \quad (13)$$

In equation (12), we assume that the drift coefficient function $\mu(r(t), t)$ and the diffusion coefficient function $\sigma(r(t), t)$ have the forms:

$$\mu(r(t), t) = m_0(t) + m_1(t)r(t); \quad (14)$$

$$\frac{\sigma^2(r(t), t)}{2} = s_0(t) + s_1(t)r(t). \quad (15)$$

Then, at time t , no-arbitrage price $D(t, T; r(t))$ of T -bond has the following the Affine Term Structure (ATS), see Filipović (2002):

$$D(t, T; r(t)) = \exp\{-a(t, T) - b(t, T)r(t)\}, \quad 0 \leq t \leq T \leq T^*, \quad (16)$$

where $a(t, T)$ and $b(t, T)$ are solution functions of the following system of simultaneous differential equations:

$$\frac{\partial}{\partial t} a(t, T) = -m_0(t)b(t, T) + s_0(t)\{b(t, T)\}^2; \quad (17)$$

$$\frac{\partial}{\partial t} b(t, T) = -1 - m_1(t)b(t, T) + s_1(t)\{b(t, T)\}^2 \quad (18)$$

with the terminal conditions:

$$a(T, T) = 0, \quad b(T, T) = 0. \quad (19)$$

In this case, the forward rate satisfies

$$\begin{aligned} f(t, T; r(t)) &:= -\frac{\partial}{\partial T} \ln D(t, T; r(t)) \\ &= \frac{\partial}{\partial T} a(t, T) + r(t) \frac{\partial}{\partial T} b(t, T), \quad 0 \leq t \leq T \leq T^*. \end{aligned} \quad (20)$$

3.2 Hull–White Model

The Hull–White model, Hull and White (1990), is one of the most popular short rate model with the ATS feature in practice because it has desirable characters of the interest rate such a mean reversion property. It can also be fitted with an observable initial term structure. The Hull–White model assumes that, under the risk neutral probability measure \mathbb{P}^* , the short rate process $r = (r(t) : t \in \mathbb{T}^*)$ satisfies the following special form of SDE with the ATS property:

$$dr(t) = \{\alpha(t) - \beta r(t)\}dt + \sigma dW^*(t), \quad t \in \mathbb{T}^*, \quad (21)$$

that is,

$$m_0(t) = \alpha(t), \quad m_1(t) = -\beta; \quad (22)$$

$$s_0(t) = \frac{\sigma^2}{2}, \quad s_1(t) = 0. \quad (23)$$

Under the Hull–White model, the time t price of T –bond can be explicitly derived as follows:

$$D(t, T; r(t)) = \exp\{-a(t, T) - b(t, T)r(t)\}, \quad 0 \leq t \leq T \leq T^*, \quad (24)$$

where

$$a(t, T) = -\frac{\sigma^2}{2} \int_t^T \{b(s, T)\}^2 ds + \int_t^T \alpha(s)b(s, T) ds; \quad (25)$$

$$b(t, T) = \frac{1 - e^{-\beta(T-t)}}{\beta}. \quad (26)$$

Then, the initial forward rate can be derived explicitly as:

$$\begin{aligned} f(0, T; r(0)) &= \frac{\partial}{\partial T} a(0, T) + r(0) \frac{\partial}{\partial T} b(0, T) \\ &= -\frac{\sigma^2}{2\beta^2} (e^{-\beta T} - 1)^2 + \int_0^T \alpha(s) \frac{\partial}{\partial T} b(s, T) ds + r(0) \frac{\partial}{\partial T} b(0, T), \\ 0 \leq T \leq T^*. \end{aligned} \quad (27)$$

4 Pricing the Chooser Flexible Cap

4.1 Pricing the Chooser Flexible Cap under the Risk Neutral Probability \mathbb{P}^*

The chooser flexible cap is a financial contract which allows to a buyer the right to exercise dynamically at most l ($1 \leq l \leq N$) out of N caplets whose i –th one ($i = 0, \dots, N - 1$) is

written on the LIBOR whose setting time and payment time are T_i and T_{i+1} , respectively. Let $W(T_i, r(T_i), l), i = 0, \dots, N-1, l = 1, \dots, M$ be the fair (no-arbitrage) price of the chooser flexible cap when, at time T_i , at most l exercises are remained to the buyer. The optimal equation can be derived by using the Bellman Principle. We can use the short rate as the state variable instead of the LIBOR. This is possible because the LIBOR is the increasing function in the short rate and these are one to one relation as we see later in equation (40). In this subsection, we derive the optimality equation under the risk neutral probability measure \mathbb{P}^* with a bank account as a numéraire.

Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$\begin{aligned} W(T_{N-1}, r(T_{N-1}), l) &= D(T_{N-1}, T_N) \delta [L_{T_{N-1}}(T_{N-1}) - K]_+, \\ l &= 1, \dots, M; \end{aligned} \quad (28)$$

(ii) For $i = N-2, \dots, 0$:

$$\begin{aligned} W(T_i, r(T_i), l) &= \max \left\{ D(T_i, T_{i+1}) \delta (L_{T_i}(T_i) - K)_+ + \mathbb{E}^* \left[\frac{W(T_{i+1}, r(T_{i+1}), l-1)}{B(T_i, T_{i+1})} \middle| r(T_i) \right], \right. \\ &\quad \left. \mathbb{E}^* \left[\frac{W(T_{i+1}, r(T_{i+1}), l)}{B(T_i, T_{i+1})} \middle| r(T_i) \right] \right\}, \quad l = 1, \dots, M, \end{aligned} \quad (29)$$

where

$$W(T_i, r(T_i), 0) = 0, \quad i = 0, \dots, N-1. \quad (30)$$

4.2 Pricing the Chooser Flexible Cap under the Forward Neutral Probability \mathbb{P}^{T_N}

We can also derive the optimality equation under the forward neutral probability \mathbb{P}^{T_N} with a T_N -bond as a numéraire.

Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$\begin{aligned} W(T_{N-1}, r(T_{N-1}), l) &= D(T_{N-1}, T_N) \delta [L_{T_{N-1}}(T_{N-1}) - K]_+, \\ l &= 1, \dots, M; \end{aligned} \quad (31)$$

(ii) For $i = N-2, \dots, 0$:

$$\begin{aligned} W(T_i, r(T_i), l) &= \max \left\{ D(T_i, T_{i+1}) \delta (L_{T_i}(T_i) - K)_+ \right. \\ &\quad \left. + D(T_i, T_N) \mathbb{E}^{T_N} \left[\frac{W(T_{i+1}, r(T_{i+1}), l-1)}{D(T_{i+1}, T_N)} \middle| r(T_i) \right], \right. \\ &\quad \left. D(T_i, T_N) \mathbb{E}^{T_N} \left[\frac{W(T_{i+1}, r(T_{i+1}), l)}{D(T_{i+1}, T_N)} \middle| r(T_i) \right] \right\}, \\ l &= 1, \dots, M, \end{aligned} \quad (32)$$

where

$$W(T_i, r(T_i), 0) = 0, \quad i = 0, \dots, N-1. \quad (33)$$

4.3 Pricing the Chooser Flexible Cap under Varying Forward Neutral Probabilities $\mathbb{P}^{T_i}(1 \leq i \leq N)$

In this subsection we write the optimality equation under forward neutral probabilities \mathbb{P}^{T_i} varying at each period with a T_i -bond as a numéraire. This optimality equation is different from the both optimality equations of Subsection 4.1 and 4.2 that have the fixed probability measures at all periods.

Optimality Equation:

(i) For $i = N - 1$ (Terminal Condition):

$$\begin{aligned} W(T_{N-1}, r(T_{N-1}), l) &= D(T_{N-1}, T_N) \delta[L_{T_{N-1}}(T_{N-1}) - K]_+, \\ l &= 1, \dots, M; \end{aligned} \quad (34)$$

(ii) For $i = N - 2, \dots, 0$:

$$\begin{aligned} W(T_i, r(T_i), l) &= \max \left\{ D(T_i, T_{i+1}) \delta(L_{T_i}(T_i) - K)_+ \right. \\ &\quad \left. + D(T_i, T_{i+1}) \mathbb{E}^{T_{i+1}} [W(T_{i+1}, r(T_{i+1}), l - 1) | r(T_i)], \right. \\ &\quad \left. D(T_i, T_{i+1}) \mathbb{E}^{T_{i+1}} [W(T_{i+1}, r(T_{i+1}), l) | r(T_i)] \right\}, \\ l &= 1, \dots, M, \end{aligned} \quad (35)$$

where

$$W(T_i, r(T_i), 0) = 0, \quad i = 0, \dots, N - 1. \quad (36)$$

5 Pricing the Chooser Flexible Floor

The chooser flexible floor is a financial contract which allows to a buyer the right to exercise dynamically at most l ($1 \leq l \leq N$) out of N floorlets whose i -th one ($i = 0, \dots, N - 1$) is written on the LIBOR whose setting time and payment time are T_i and T_{i+1} , respectively. Let $V(T_i, r(T_i), l)$, $i = 0, \dots, N - 1$, $l = 1, \dots, M$ be the fair (no-arbitrage) price of the chooser flexible floor when, at time T_i , at most l exercises are remained to the buyer. We derive the optimality equation for the chooser flexible floor under the risk neutral probability measure \mathbb{P}^* by the same way as Subsection 4.1. The optimality equations under \mathbb{P}^{T_N} and \mathbb{P}^{T_i} can be obtained by the same ways as Subsection 4.2 and 4.3.

Optimality Equation:

(i) For $i = N - 1$ (Terminal Condition):

$$V(T_{N-1}, r(T_{N-1}), l) = D(T_{N-1}, T_N) \delta[K - L_{T_{N-1}}(T_{N-1})]_+, \quad l = 1, \dots, M; \quad (37)$$

(ii) For $i = N - 2, \dots, 0$:

$$\begin{aligned} V(T_i, r(T_i), l) &= \max \left\{ D(T_i, T_{i+1}) \delta(K - L_{T_i}(T_i))_+ + \mathbb{E}^* \left[\frac{V(T_{i+1}, r(T_{i+1}), l - 1)}{B(T_i, T_{i+1})} \middle| r(T_i) \right], \right. \\ &\quad \left. \mathbb{E}^* \left[\frac{V(T_{i+1}, r(T_{i+1}), l)}{B(T_i, T_{i+1})} \middle| r(T_i) \right] \right\}, \quad l = 1, \dots, M, \end{aligned} \quad (38)$$

where

$$V(T_i, r(T_i), 0) = 0, \quad i = 0, \dots, N - 1. \quad (39)$$

6 No-Arbitrage Prices of the Caplet and the Floorlet

To calculate the price of the chooser flexible cap using the Hull–White model, we need to decide the values of the model parameters. The parameters are decided as observable simple derivatives (the caplet and the floorlet) in the market fit theoretical prices derived by the model. So, for the calibration we need to derive theoretical prices of the caplet and the floorlet based on the Hull–White model.

We assume that the short rate $r(t)$ follows the Hull–White model (21). From the results (24), (25) and (26) the forward LIBOR $L_{T_i}(t)$ can be represented as

$$\begin{aligned} L_{T_i}(t) &:= \frac{D(t, T_i) - D(t, T_{i+1})}{\delta D(t, T_{i+1})} \\ &= -\frac{1}{\delta} + \frac{1}{\delta} \exp\left\{-a(t, T_i) + a(t, T_{i+1})\right\} \exp\left\{-b(t, T_i) + b(t, T_{i+1})\right\} r(t). \end{aligned} \quad (40)$$

Defining

$$g(t, T_i, T_{i+1}) := \frac{1}{\delta} \exp\left\{-a(t, T_i) + a(t, T_{i+1})\right\}, \quad (41)$$

$$h(t, T_i, T_{i+1}) := -b(t, T_i) + b(t, T_{i+1}), \quad (42)$$

$$\bar{L}_{T_i}(t) := L_{T_i}(t) + \frac{1}{\delta}, \quad (43)$$

we can represent $\bar{L}_{T_i}(t)$ as

$$\bar{L}_{T_i}(t) = g(t, T_i, T_{i+1}) e^{h(t, T_i, T_{i+1})r(t)}. \quad (44)$$

From the Itô's Lemma $\bar{L}_{T_i}(t)$ follows the SDE

$$\begin{aligned} \frac{d\bar{L}_{T_i}(t)}{\bar{L}_{T_i}(t)} &= \left\{ \frac{g(t, T_i, T_{i+1})_t}{g(t, T_i, T_{i+1})} + h(t, T_i, T_{i+1})_t r(t) + h(t, T_i, T_{i+1})(\alpha(t) - \beta r(t)) \right. \\ &\quad \left. + \frac{1}{2} h(t, T_i, T_{i+1})^2 \sigma^2 \right\} dt \\ &\quad + h(t, T_i, T_{i+1}) \sigma dW^*(t), \end{aligned} \quad (45)$$

where $g(t, T_i, T_{i+1})_t$ and $h(t, T_i, T_{i+1})_t$ are partial differentials of $g(t, T_i, T_{i+1})$ and $h(t, T_i, T_{i+1})$ with respect to t . $g(t, T_i, T_{i+1})_t$ and $h(t, T_i, T_{i+1})_t$ can be calculated as

$$g(t, T_i, T_{i+1})_t = \left[\frac{\sigma^2}{2} \left\{ b(t, T_i) + b(t, T_{i+1}) \right\} - \alpha(t) \right] h(t, T_i, T_{i+1}) g(t, T_i, T_{i+1}); \quad (46)$$

$$h(t, T_i, T_{i+1})_t = h(t, T_i, T_{i+1}) \beta. \quad (47)$$

Hence, we obtain

$$\begin{aligned} \frac{d\bar{L}_{T_i}(t)}{\bar{L}_{T_i}(t)} &= \left[\frac{h(t, T_i, T_{i+1}) \sigma^2}{2} \left\{ b(t, T_i) + b(t, T_{i+1}) \right\} + \frac{1}{2} h(t, T_i, T_{i+1})^2 \sigma^2 \right] dt + h(t, T_i, T_{i+1}) \sigma dW^*(t) \\ &= h(t, T_i, T_{i+1}) \sigma \left[\left\{ \frac{\sigma}{2} \left(b(t, T_i) + b(t, T_{i+1}) \right) + \frac{1}{2} h(t, T_i, T_{i+1}) \sigma \right\} dt + dW^*(t) \right] \\ &= h(t, T_i, T_{i+1}) \sigma \left\{ \sigma b(t, T_{i+1}) dt + dW^*(t) \right\}. \end{aligned} \quad (48)$$

Changing the probability measure by

$$dW^{T_{i+1}}(t) = \sigma b(t, T_{i+1})dt + dW^*(t), \quad (49)$$

we obtain that $\bar{L}_{T_i}(t)$ follows the SDE

$$\frac{d\bar{L}_{T_i}(t)}{\bar{L}_{T_i}(t)} = h(t, T_i, T_{i+1})\sigma dW^{T_{i+1}}(t), \quad 0 \leq t \leq T_i, \quad (50)$$

where $W^{T_{i+1}}(t)$ is the Brownian Motion under the forward neutral probability $\mathbb{P}^{T_{i+1}}$.

By utilizing the risk neutral evaluation method, we know that the fair (no-arbitrage) price of i -caplet at time t ($\in [0, T_i]$), $Cpl_{T_i}(t)$, is given by the expectation under the risk neutral probability measure \mathbb{P}^* :

$$Cpl_{T_i}(t) = \mathbb{E}^* \left[\frac{\delta[L_{T_i}(T_i) - K]_+}{B(t, T_{i+1})} \middle| \mathcal{F}(t) \right], \quad 0 \leq t \leq T_i. \quad (51)$$

So under the forward neutral probability measure $\mathbb{P}^{T_{i+1}}$ we can show it as

$$\begin{aligned} Cpl_{T_i}(t) &= \delta D(t, T_{i+1}) \mathbb{E}^{T_{i+1}}[(L_{T_i}(T_i) - K)_+ | \mathcal{F}(t)] \\ &= \delta D(t, T_{i+1}) \mathbb{E}^{T_{i+1}}[(\bar{L}_{T_i}(T_i) - K')_+ | \mathcal{F}(t)], \quad 0 \leq t \leq T_i. \end{aligned} \quad (52)$$

Using the same calculation as the Black-Scholes Formula, we obtain

$$Cpl_{T_i}(t) = \delta D(t, T_{i+1}) [\bar{L}_{T_i}(t) \Phi(d(t, T_i)) - K' \Phi(d(t, T_i) - \nu(t, T_i))], \quad (53)$$

where $\bar{L}_{T_i}(t)$, $\Phi(d(t, T_i))$, K' , $d(t, T_i)$, and $\nu(t, T_i)$ are defined as follows:

$$\bar{L}_{T_i}(t) := L_{T_i}(t) + \frac{1}{\delta}, \quad (54)$$

$$\Phi(d(t, T_i)) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-x^2/2} dx, \quad (55)$$

$$K' := K + \frac{1}{\delta}, \quad (56)$$

$$d(t, T_i) := \frac{\ln(\bar{L}_{T_i}(t)/K')}{\nu(t, T_i)} + \frac{\nu(t, T_i)}{2}, \quad (57)$$

$$\begin{aligned} \nu(t, T_i) &:= \frac{\sigma}{\sqrt{2\beta^3}} [1 - 2e^{-\beta(T_{i+1}-T_i)} + e^{-2\beta(T_{i+1}-T_i)} - e^{-2\beta(T_i-t)} \\ &\quad + 2e^{-\beta(T_{i+1}+T_i-2t)} - e^{-2\beta(T_{i+1}-t)}]^{1/2}. \end{aligned} \quad (58)$$

Similarly, by the risk neutral evaluation method, the fair (no-arbitrage) price $Fll_{T_i}(t)$ of i -floorlet at time t ($\in [0, T_i]$) is given by:

$$Fll_{T_i}(t) = \mathbb{E}^* \left[\frac{\delta[K - L_{T_i}(T_i)]_+}{B(t, T_{i+1})} \middle| \mathcal{F}(t) \right], \quad 0 \leq t \leq T_i, \quad (59)$$

so that

$$Fll_{T_i}(t) = \delta D(t, T_{i+1}) \mathbb{E}^{T_{i+1}}[(K' - \bar{L}_{T_i}(T_i))_+ | \mathcal{F}(t)], \quad 0 \leq t \leq T_i. \quad (60)$$

To calculate the theoretical price of the floorlet, we derive the parity relation of the caplet and the floorlet. From the relation

$$(x - K)_+ - (K - x)_+ = x - K, \quad (61)$$

we take the expectation of both sides under the forward neutral probability \mathbb{P}^{T_i} and multiply both sides by $\delta D(t, T_{i+1})$. Then we have

$$\begin{aligned} LHS &= \delta D(t, T_{i+1}) \mathbb{E}^{T_{i+1}}[(L_{T_i}(T_i) - K)_+ | \mathcal{F}(t)] - \delta D(t, T_{i+1}) \mathbb{E}^{T_{i+1}}[(K - L_{T_i}(T_i))_+ | \mathcal{F}(t)] \\ &= Cpl_{T_i}(t) - Fll_{T_i}(t), \end{aligned} \quad (62)$$

$$\begin{aligned} RHS &= \delta D(t, T_{i+1}) \mathbb{E}^{T_{i+1}}[(L_{T_i}(T_i) - K) | \mathcal{F}(t)] \\ &= \delta D(t, T_{i+1}) \mathbb{E}^{T_{i+1}}[(\bar{L}_{T_i}(T_i) - K') | \mathcal{F}(t)] \\ &= \delta D(t, T_{i+1}) (\bar{L}_{T_i}(t) - K'). \end{aligned} \quad (63)$$

Using the parity relation

$$Cpl_{T_i}(t) - Fll_{T_i}(t) = \delta D(t, T_{i+1}) (\bar{L}_{T_i}(t) - K'), \quad (64)$$

we can calculate the floorlet price as follows:

$$\begin{aligned} Fll_{T_i}(t) &= Cpl_{T_i}(t) - \delta D(t, T_{i+1}) (\bar{L}_{T_i}(t) - K') \\ &= \delta D(t, T_{i+1}) \{ \bar{L}_{T_i}(t) \Phi(d(t, T_i)) - K' \Phi(d(t, T_i) - \nu(t, T_i)) - (\bar{L}_{T_i}(t) - K') \} \\ &= \delta D(t, T_{i+1}) \{ -\bar{L}_{T_i}(t) (1 - \Phi(d(t, T_i))) + K' (1 - \Phi(d(t, T_i) - \nu(t, T_i))) \} \\ &= \delta D(t, T_{i+1}) \{ -\bar{L}_{T_i}(t) \Phi(-d(t, T_i)) + K' \Phi(\nu(t, T_i) - d(t, T_i)) \}. \end{aligned} \quad (65)$$

7 Calibration

From the results of Section 6 unknown parameters $\alpha(\cdot)$, β , σ could be estimated as follows. In the Hull–White model by solving the theoretical value formula of the forward rate:

$$\begin{aligned} f(0, T; r(0)) &= \frac{\partial}{\partial T} a(0, T) + r(0) \frac{\partial}{\partial T} b(0, T) \\ &= -\frac{\sigma^2}{2\beta^2} (e^{-\beta T} - 1)^2 + \int_0^T \alpha(s) \frac{\partial}{\partial T} b(s, T) ds + r(0) \frac{\partial}{\partial T} b(0, T), \\ 0 \leq T \leq T^*, \end{aligned} \quad (66)$$

with respect to the function $\alpha(\cdot)$, we have

$$\begin{aligned} \alpha(T) &= \frac{\partial}{\partial T} \left[f(0, T) + \frac{\sigma^2}{2\beta^2} (e^{-\beta T} - 1)^2 \right] - \beta \left[f(0, T) + \frac{\sigma^2}{2\beta^2} (e^{-\beta T} - 1)^2 \right], \\ 0 \leq T \leq T^*. \end{aligned} \quad (67)$$

If we let the observed forward rate curve at current time 0 as

$$f_{\text{mkt}}(0, T), \quad 0 \leq T \leq T^*, \quad (68)$$

then the next includes unknown parameters β and σ , only:

$$\begin{aligned} \alpha_{\text{mkt}}(T) &:= \frac{\partial}{\partial T} \left[f_{\text{mkt}}(0, T) + \frac{\sigma^2}{2\beta^2} (e^{-\beta T} - 1)^2 \right] - \beta \left[f_{\text{mkt}}(0, T) + \frac{\sigma^2}{2\beta^2} (e^{-\beta T} - 1)^2 \right], \\ 0 \leq T \leq T^*. \end{aligned} \quad (69)$$

Accordingly, the remaining unknown parameters β and σ could be determined, for example, by the minimizers of the following criterion function:

$$\begin{aligned}
C(\beta, \sigma) &:= w_1 \sum_i |D(t, T_i)_{\text{mkt}} - D(t, T_i)_{\text{mdl}}|^2 \\
&\quad + w_2 \sum_i |Cpl_{T_i}(t)_{\text{mkt}} - Cpl_{T_i}(t)_{\text{mdl}}|^2 \\
&\quad + w_3 \sum_i |Fll_{T_i}(t)_{\text{mkt}} - Fll_{T_i}(t)_{\text{mdl}}|^2, \tag{70}
\end{aligned}$$

where $D(t, T_i)_{\text{mkt}}$, $Cpl_{T_i}(t)_{\text{mkt}}$, $Fll_{T_i}(t)_{\text{mkt}}$: the observed prices of i -th discount bond, caplet, and floorlet, respectively; $D(t, T_i)_{\text{mdl}}$, $Cpl_{T_i}(t)_{\text{mdl}}$, $Fll_{T_i}(t)_{\text{mdl}}$: the theoretical prices of i -th discount bond, caplet, and floorlet, respectively; w_1, w_2, w_3 ($\in \mathbb{R}_+$) ($w_1 + w_2 + w_3 = 1$): weighting coefficients. We can take weighting coefficients in other ways like taking summation after we put weighting coefficients on each difference between the market data and the theoretical price of each i -th instrument. We can also use other financial instruments for this calibration.

The real processes of financial asset prices (bond, simple interest derivatives, etc.) and financial variables (yield, forward rate, LIBOR, etc.) are governed by the original market probability measure \mathbb{P} (Observation Measure). However, in order to price exotic interest rate derivatives, we need the probability laws of the above financial assets and/or variables under the risk neutral probability measure \mathbb{P}^* . These measures are not directly nor obviously related but only partially related. Roughly speaking, we can not fully infer the probability laws under the risk neutral probability measure \mathbb{P}^* from the historical data of the above financial assets and variables by utilizing various tools of statistical data analyses. Instead, we must learn from the observed market prices of related financial assets, in which information about the probability laws under the risk neutral probability measure \mathbb{P}^* are implicitly involved, that is a calibration. In Financial Engineering, various peculiar methodologies have been developed and accumulated.

8 Construction of the Trinomial Tree for Chooser Flexible Cap Pricing

Using a trinomial tree of the short rate $r(t)$ that follows the Hull–White model, we construct the trinomial tree of the short rate and the chooser flexible cap price in the discrete time setting under the risk neutral probability \mathbb{P}^* in this section. We can also construct the trees under the forward neutral probabilities \mathbb{P}^{T_N} and \mathbb{P}^{T_i} as the same way. We construct the short rate trinomial tree based on Hull–White (1994). For the simplicity of the calculation, we set $\Delta t = \delta = T_{i+1} - T_i$ as constant for all i . (i, j) represents a node $t = i\Delta t$ and $r = j\Delta r$. We define $D(i, i+1, j)$ as the $(i+1)\Delta t$ -bond price, $L(i, i+1, j)$ as the LIBOR with the payment time $(i+1)\Delta t$, and $r_{i,j}$ as the short rate value at the node (i, j) .

In the case of the Hull–White model from (25) and (26) we have

$$\begin{aligned}
a(T_i, T_{i+1}) &= -\frac{\sigma^2}{2} \int_{T_i}^{T_{i+1}} \{b(s, T_{i+1})\}^2 ds + \int_{T_i}^{T_{i+1}} \alpha(s) b(s, T_{i+1}) ds \\
&\cong -\frac{\sigma^2}{2} \int_{T_i}^{T_{i+1}} \{b(s, T_{i+1})\}^2 ds + \alpha_i \int_{T_i}^{T_{i+1}} b(s, T_{i+1}) ds \\
&= -\frac{\sigma^2}{2\beta^2} \left((T_{i+1} - T_i) + \frac{2}{\beta} e^{-\beta(T_{i+1}-T_i)} - \frac{1}{2\beta} e^{-2\beta(T_{i+1}-T_i)} - \frac{3}{2\beta} \right) \\
&\quad + \alpha_i \frac{1}{\beta^2} \left(\beta(T_{i+1} - T_i) + e^{-\beta(T_{i+1}-T_i)} - 1 \right) \\
&=: a(i, i+1), \tag{71}
\end{aligned}$$

$$b(T_i, T_{i+1}) = \frac{1 - e^{-\beta(T_{i+1}-T_i)}}{\beta} =: b(i, i+1). \tag{72}$$

Then the bond price $D(i, i+1, j)$ and the LIBOR $L(i, i+1, j)$ are represented respectively as

$$D(i, i+1, j) = e^{-a(i, i+1) - b(i, i+1)r_{i, j}}, \tag{73}$$

$$L(i, i+1, j) = -\frac{1}{\delta} + \frac{1}{\delta D(i, i+1, j)}. \tag{74}$$

The bank account $B(T_i, T_{i+1})$ with $r(t)$ transiting from a node (i, j) to a node $(i+1, j')$ can be approximated utilizing the trapezoidal rule as

$$B(T_i, T_{i+1}) = \exp\left\{ \int_{T_i}^{T_{i+1}} r(s) ds \right\} \cong \exp\left\{ \frac{r_{i, j} + r_{i+1, j'}}{2} \delta \right\}. \tag{75}$$

We define $p_u(i, j), p_m(i, j), p_d(i, j)$ as the transition probabilities from the node (i, j) to up, same, and down states at $t = (i+1)\Delta t$ respectively, $W(i, j, l)$ as the chooser flexible cap price at the node (i, j) with the exercise opportunity l , and $W(i+1, j+1, l), W(i+1, j, l), W(i+1, j-1, l)$ as the chooser flexible cap prices at each state, $j+1, j$, and $j-1$, at time $(i+1)\Delta t$ with above corresponding transition probabilities from (i, j) , and $B_u(i, j), B_m(i, j), B_d(i, j)$ as the bank account values at each state, $j+1, j$, and $j-1$, at time $(i+1)\Delta t$ with above corresponding transition probabilities from (i, j) . The chooser flexible cap price with the exercisable number $l (1 \leq l < N-i)$ at (i, j) , $W(i, j, l)$, can be derived as

$$\begin{aligned}
W(i, j, l) &= \max \left\{ D(i, i+1, j) \delta (L(i, i+1, j) - K)_+ + p_u(i, j) \frac{W(i+1, j+1, l-1)}{B_u(i, j)} \right. \\
&\quad + p_m(i, j) \frac{W(i+1, j, l-1)}{B_m(i, j)} + p_d(i, j) \frac{W(i+1, j-1, l-1)}{B_d(i, j)}, \\
&\quad \left. p_u(i, j) \frac{W(i+1, j+1, l)}{B_u(i, j)} + p_m(i, j) \frac{W(i+1, j, l)}{B_m(i, j)} + p_d(i, j) \frac{W(i+1, j-1, l)}{B_d(i, j)} \right\}. \tag{76}
\end{aligned}$$

By the backward induction from the terminal condition, we can finally calculate the current chooser flexible cap price at $(0, 0)$.

9 Numerical Examples

9.1 Construction of the Trinomial Tree of the Short Rate and the Chooser Flexible Cap

Following Hull and White (1994), we construct the trinomial tree of the short rate in Figure 1 which fits the term structure with a gently increasing initial yield curve, where we set the parameters as $\beta = 0.3$ and $\sigma = 0.01$. We also set the time interval $\delta = 0.25$ for both of the short rate and the spot LIBOR in this example. But we can set different time intervals for the tree construction of each rate. For example, we can set the shorter time interval for the short rate tree than the spot LIBOR. We make a programming for the calculation with MATLAB. We also construct the trinomial tree with $K = 0.7$ and $l = 1$ in Figure 2 and $l = 2$ in Figure 3.

9.2 Comparative Statics

In this subsection, we discuss how each parameter and an initial condition affect the price of the chooser flexible cap. The Table 1 shows the chooser flexible cap price calculated with only one parameter changed where other parameters and initial conditions are fixed on the bench mark values, $\beta=0.3$, R : increasing 0.0025 in each period starting from 0.1, $\sigma=0.01$, $N=8$, $T=2$, $\delta=0.25$, $K=0.8$, and $l=2$.

The result shows that the chooser flexible cap price is larger as l (Exercise opportunity) is bigger. This is because the payoff increases as the option holder has more exercise opportunities. The chooser flexible cap price is larger as β (Parameter of HW model) is bigger. The value of β affects the level and speed of the mean reversion of the short rate. The larger β is, the smaller mean reversion level is. This keeps the values of the short rate and LIBOR small, and causes the small chooser flexible cap price. The chooser flexible cap price is larger as σ (Parameter of HW model) is bigger. The reason is the same as the result of the simple Black-Scholes formula. As σ is bigger, the value of the instruments, with which we can hedge the floating interest rate risk, is highly evaluated by buyers of the option. The chooser flexible cap price is smaller as K (Exercise rate) is bigger. This result is caused by the payoff function, $\delta(L_{T_i}(T_i) - K)_+$, at each period. As the value of K increases, the payment value gets smaller. The chooser flexible cap price is larger as T (Option maturity) is bigger. Because we can replicate the chooser flexible cap with the short maturity by the longer one, the longer one is at least more expensive than the shorter one. Finally, we examine six patterns of the initial yield curve such as increasing curves and decreasing curves at different yield value levels. Among the six initial yield curve examples, the price of the case of R5(the increasing, 0.01 at each period, curve from 0.1) is most expensive, and the prices of decreasing and flat yield curve are zero in this example. But the prices with other parameter set cases do not necessarily result in the same. Although the increasing initial yield curves cause the bigger values of the spot LIBOR than the decreasing one, the bigger values cause not only the bigger values of payments but also the bigger values of the discount bonds, which are the discount values used on the backward calculation. As we calculate the values of the chooser flexible cap backwardly, the chooser flexible cap values are discounted more in the case of the increasing case than the decreasing case. As the result, the price differences between the increasing and decreasing cases get smaller and smaller on the backward calculation. In some cases, the option price with the decreasing yield curve may be more expensive than the increasing case.

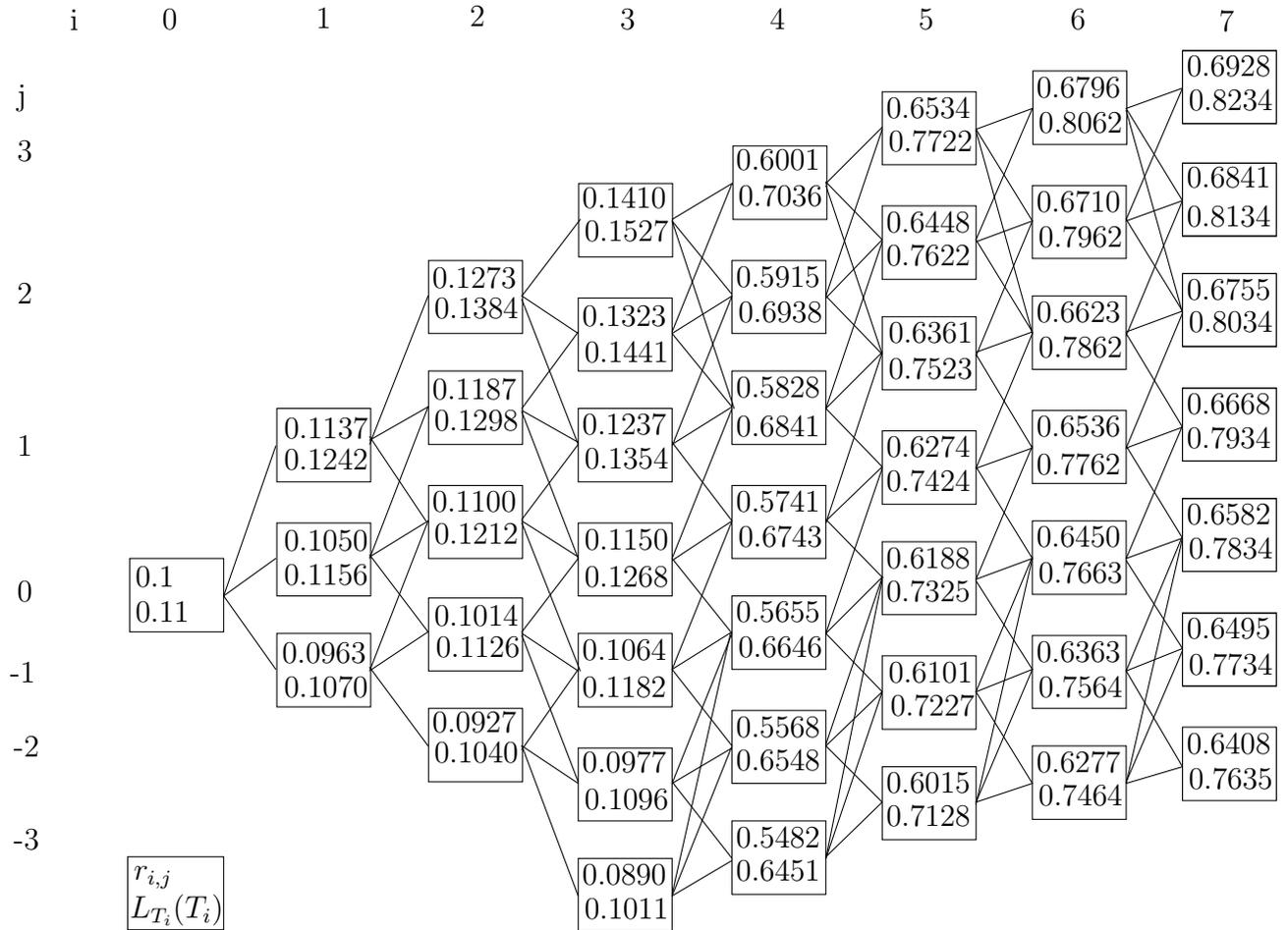


Figure 1: The Trinomial Tree of the Short Rate and the Spot LIBOR. We construct the tree following Hull–White (1994) based on the parameter set, $\beta = 0.3$, R : increasing 0.0025 in each period starting from 0.1, $\sigma = 0.01$, $N = 8$, $T = 2$, $\delta = 0.25$, $K = 0.7$, and $l = 2$. The upper nodal values, $r_{i,j}$, are the short rates and lower values, $L_{T_i}(T_i)$, are the spot LIBORs at each node (i, j) .

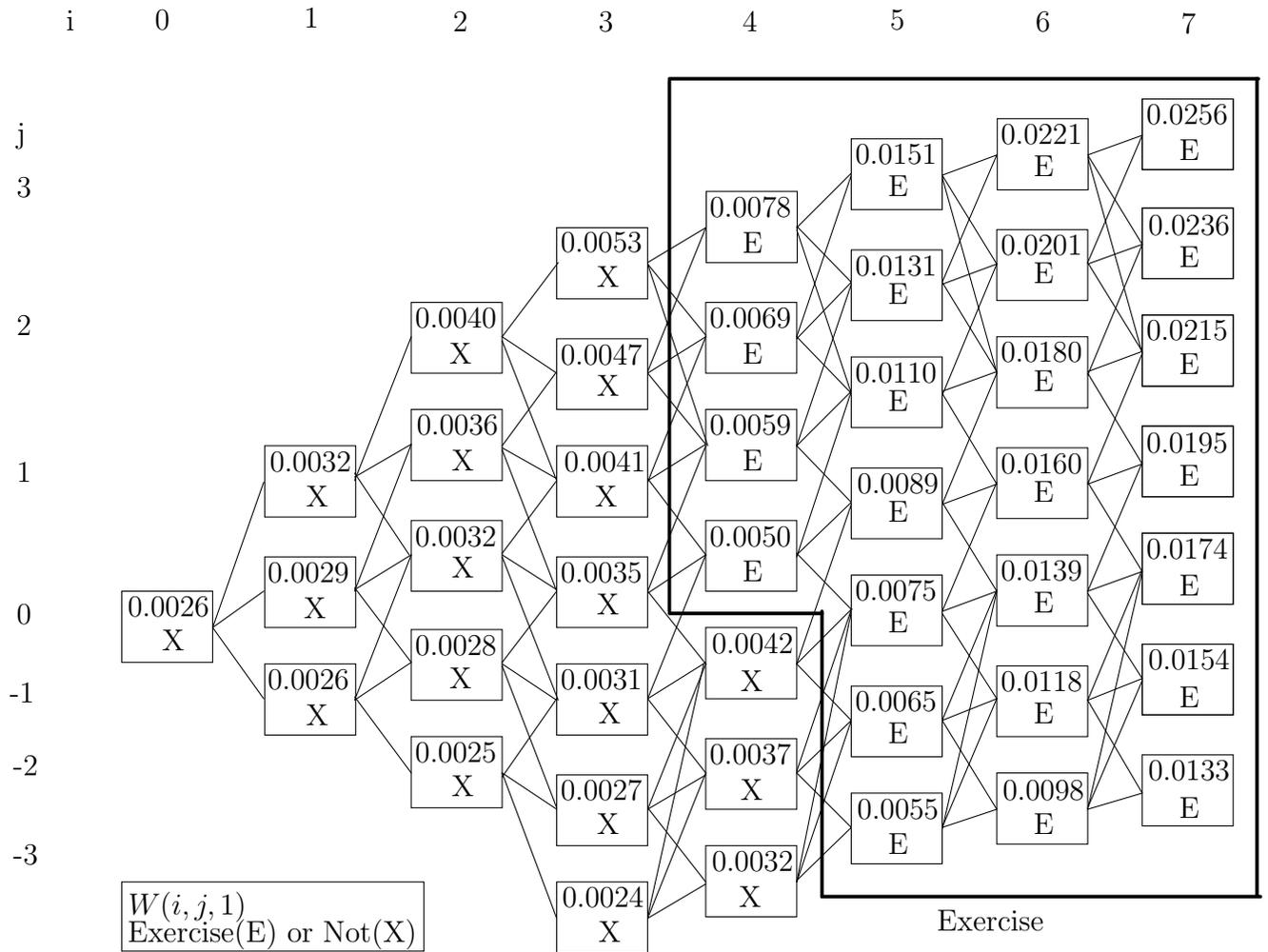


Figure 2: The Trinomial Tree of Chooser Flexible Cap Values with the Exercise Opportunity 1. The tree is built calculating the chooser flexible cap backwardly based on the LIBOR tree of the Figure 1. The upper nodal values, $W(i, j, 1)$, are the chooser flexible cap prices with the exercise opportunity 1 at each node (i, j) . The lower rows show whether the option holder exercises the i -caplet, E, at each node or not, X. Furthermore, the surrounded area with the thick line represents the exercise nodes.

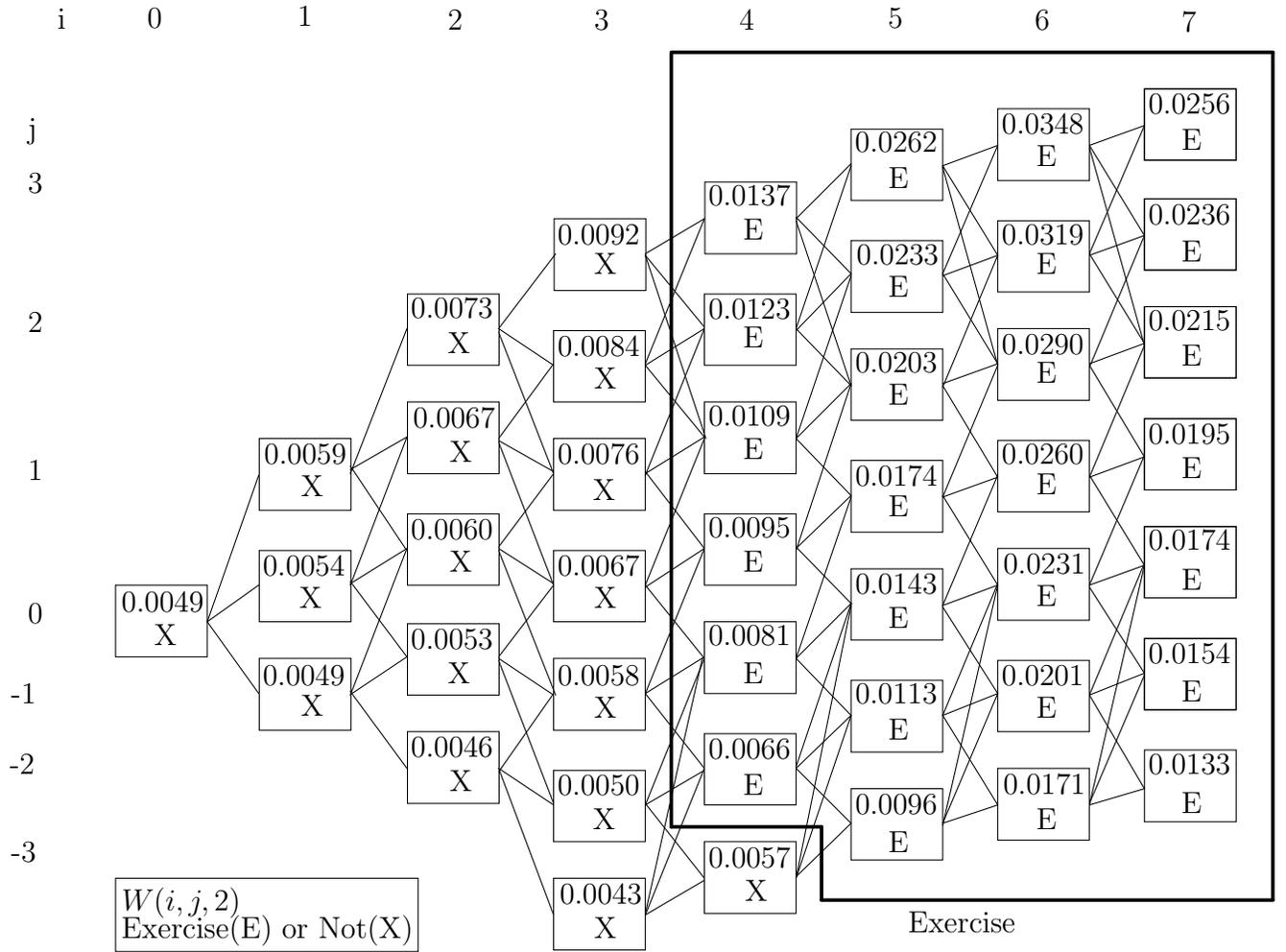


Figure 3: The Trinomial Tree of Chooser Flexible Cap Values with the Exercise Opportunity 2. The tree is built calculating the chooser flexible cap backwardly based on the LIBOR tree of the Figure 1. The upper nodal values, $W(i, j, 2)$, are the chooser flexible cap prices with the exercise opportunity 2 at each node (i, j) . The lower rows show whether the option holder exercises the i -caplet, E, at each node, X. Furthermore, the surrounded area with the thick line represents the exercise nodes.

Table 1. Comparative Statics^a

	$l = 1$	$l = 2$	$l = 3$	$l = 4$	Change of the Price
$l = 2$	0.0703	0.1222			l bigger \Rightarrow price bigger
$l = 4$	0.0703	0.1222	0.1591	0.1859	
$\beta = 0.3$	0.0703	0.1222			β bigger \Rightarrow price smaller
$\beta = 0.6$	0	0			
$\sigma = 0.01$	0.0703	0.1222			σ bigger \Rightarrow price bigger
$\sigma = 0.5$	18.7	26.8			
$K = 0.8$	0.0703	0.1222			K bigger \Rightarrow price smaller
$K = 0.1$	63.0	101.2			
$T = 2, N = 8, \delta = 0.25$	0.03645	0.03949			$T(or N)$ bigger \Rightarrow price bigger
$T = 4, N = 16, \delta = 0.25$	0.0703	0.1222			
$R0$: gently increasing from 0.1	0.0703	0.1222			flat and decreasing \Rightarrow price 0 $R5 \Rightarrow$ price biggest
$R1$: flat at 0.1	0	0			
$R2$: decreasing from 0.14	0	0			
$R3$: increasing from 0.06	0.0954	0.1771			
$R4$: decreasing from 0.1	0	0			
$R5$: increasing from 0.1	0.6	1.1			

^a The Table 1 shows the chooser flexible cap prices (the table values $\times 10^{-3}$) calculated with only one parameter changed where other parameters and initial conditions are fixed on the bench mark values, l (Exercise opportunity)= 2, β (Parameter of HW model)= 0.3, σ (Parameter of HW model)= 0.01, K (Exercise rate)= 0.8, T (Option maturity)= 2, N (Number of time periods)= 8, δ (Time interval)= 0.25, and R (Initial Yield Curve): increasing 0.0025 in each period starting from 0.1. The column of "Change of the Price" shows the change of the chooser flexible cap price when each corresponding parameter is changed. $R0$ is the gently increasing, 0.005 at each period, initial yield curve from 0.1. $R1$ is the flat curve at 0.1. $R2$ is the decreasing, 0.01 at each period, curve from 0.14. $R3$ is the increasing, 0.01 at each period, curve from 0.06. $R4$ is the decreasing, 0.01 at each period, curve from 0.1. $R5$ is the increasing, 0.01 at each period, curve from 0.1.

10 Conclusion

In this paper we propose the pricing method of the chooser flexible cap. We have mainly three contributions in this paper. Firstly, we utilize the dynamic programming approach with the short rate model, in particular the Hull–White model. Secondly, deriving the theoretical prices of bond, the caplet and the floorlet, we use the theoretical prices for the calibration. Thirdly, we show the numerical examples and discuss comparative statics.

The future plan of the research is to price the chooser flexible cap after the calibration thorough real market data. Furthermore we should verify if prices derived by the model suit the chooser flexible cap prices of the real market. We can analytically prove the results of comparative statics shown in Subsection 9.2. Some of them are under way in our future research paper. One factor model like this paper has only one driving Brownian motion and it implies perfect correlation of all forward rates with different maturity dates. Furthermore, by working with the one factor short rate as the model primitive, it is difficult to allow for a precise fit to processes of quoted instruments (bond, caplet and floorlet) because of low degrees of freedom. We can extend the one factor model of this paper to a multi factor model.

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