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Abstract

In this paper, we theoretically look into various features of a chooser flexible cap. The chooser flexible cap is a financial instrument written on an underlying market interest rate index, LIBOR (London Inter-Bank Offer Rate). The chooser flexible cap allows a right for a buyer to exercise a limited and pre-determined number of the interim period caplets in a multiple-period cap agreement. While the chooser flexible cap is more flexible and cheaper instrument than the normal cap, its pricing is more complicated than the cap's because of its flexibility. So it may take long time for its price calculation. We can use the features to cut down the calculation time. At the same time the option holder can use the features for exercise strategies.

Keywords: chooser flexible cap, LIBOR, dynamic programming, exercise strategy.
JEL classification: G13, G15, G21

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1 Introduction

The chooser flexible cap is a financial contract which allows to a buyer the right to exercise dynamically at most l $(1 \leq l \leq N)$ out of N caplets whose i–th one $(i = 0, \cdots, N-1)$ is written on the LIBOR whose setting time and payment time are T_i and T_{i+1}, respectively. There are mainly two merits in the chooser flexible cap compared to a more popular cap. The first merit is that its price is cheaper than the cap's because the chooser flexible cap has less exercise opportunities than the cap. So an option holder can hedge the interest rate risk with the lower cost than the cap. The second merit is its flexibility. The holder of the chooser flexible cap can reflect her/his expectation of the future interest rate change in her/his hedging strategy. For example, if s/he has the expectation that the floating interest rate will increase in the next one year, s/he should exercise the options mainly in one year as s/he observes the real interest rate values. So using the chooser flexible cap, the option holder can flexibly hedge the interest rate risk with the low cost. The chooser flexible cap is less traded than the Bermudan swaption and more traded in Europe than other countries. Extending Pedersen and Sidenius (1998), we proposed a pricing method of the chooser flexible cap in Ito. Ohnishi and Tamba (2004). We used the Hull–White model with one factor model in the paper. We also found that the calculation of the chooser flexible cap with one factor model is very fast. For example, it took only one second for the calculation of even the option with 40 years maturity and 20 times of the exercise opportunities. But with more complex interest rate models, the calculation may take much longer time. Although the chooser flexible cap is the useful financial instrument, it is difficult to decide when the option holder should exercise options because to make an exercise strategy s/he should expect the future change of the LIBOR.

In this paper we do research of various features of the chooser flexible cap. We prove the three features of the chooser flexible cap. Firstly, we show that the option holder easily exercises the option with more exercise opportunities at the same state and period. Secondly, we derive the conditions for non early exercise of the chooser flexible cap. Thirdly, we show that if the option holder exercises the i–caplet at a node, s/he also exercises the i–caplet at upper state nodes at the same period. We can use the features to cut down the calculation time of the chooser flexible cap price. At the same time the option holder can use the features for more theoretically appropriate exercise strategies.

The paper is organized as follows. In Section 2, we show the pricing method of the chooser flexible cap. Section 3 shows that the option holder easily exercises the option with more exercise opportunities. In Section 4, we derive the conditions for non early exercise of the chooser flexible cap. Section 5 is devoted to show that if the option holder exercises the i–caplet at a node, s/he also exercises the i–caplet at upper state nodes. Section 6 concludes the paper.

2 Pricing the Chooser Flexible Cap

2.1 Various Notations about Interest Rates

For $N \in \mathbb{Z}_+$ and $T^* \in \mathbb{R}_{++}$, let

$$0 \leq T_0 < T_1 < \cdots < T_i < T_{i+1} < \cdots < T_{N-1} < T_N \leq T^* \tag{1}$$

be the sequence of setting times and payment times of floating interest rates, that is, for $i = 0, \cdots, N-1$, the floating interest rate which covers time interval $(T_i, T_{i+1}]$, is set at time T_i and paid at time T_{i+1}. For convenience, we let

$$T_{i+1} - T_i = \delta \ (= \text{constant} \in \mathbb{R}_{++}), \quad i = 0, \cdots, N-1. \tag{2}$$

Let $D(t,T)$ $0 \le t \le T \le T^*$ be the time t price of the discount bond (or zero–coupon bond) with maturity T, in brief T–bond, which pays 1–unit of money at the maturity T (where $D(T,T) = 1$ for any $T \in \mathbb{T}^* := [0, T^*]$). For $0 \le t \le S < T \le T^*$,

$$r(t) = -\left.\frac{\partial}{\partial T} \ln D(t,T)\right|_{T=t} \tag{3}$$

is the short rate at time t. For $0 \le t \le T \le T^*$,

$$B(t,T) := \exp\left\{\int_t^T r(s)\mathrm{d}s\right\} \tag{4}$$

is the risk–free bank account at time T with unit investment capital at time t (where $B(t,t) = 1$). For $i = 0, \cdots, N-1$, we define the simple (or simple compounding based) interest rate which covers time interval $(T_i, T_{i+1}]$ by

$$L_{T_i}(T_i) := \frac{1}{\delta}\left\{\frac{1}{D(T_i, T_{i+1})} - 1\right\}. \tag{5}$$

This amount is set at time T_i, paid at time T_{i+1}, and is conventionally called as a spot LIBOR (London Inter–Bank Offer Rate). For $i = 0, \cdots, N-1$,

$$L_{T_i}(t) := \frac{1}{\delta}\left\{\frac{D(t,T_i)}{D(t,T_{i+1})} - 1\right\} \tag{6}$$

is the simple (or simple compounding based) interest rate prevailing at time t ($\in [0, T_i]$) which covers time interval $(T_i, T_{i+1}]$, and is called as a forward LIBOR. For $i = 0, \cdots, N-1$,

$$L(t; T_i, T_{i+k}) := \frac{1}{k\delta}\left\{\frac{D(t,T_i)}{D(t,T_{i+k})} - 1\right\} \tag{7}$$

is the simple (or simple compounding based) interest rate prevailing at time t ($\in [0, T_i]$) which covers time interval $(T_i, T_{i+k}]$, and is called as the forward LIBOR for multiple–period.

The i–caplet is a financial contract in which, at time T_{i+1}, the seller pays to the buyer the amount of money corresponding the difference between the spot LIBOR $L_{T_i}(T_i)$ and a predetermined upper–limit exercise rate K ($\in \mathbb{R}$) if the former exceeds the latter:

$$\delta[L_{T_i}(T_i) - K]_+ \ \ (= \delta \max\{L_{T_i}(T_i) - K, 0\}). \tag{8}$$

It could be considered as a call option written on the underlying LIBOR for hedging its upside risk. A cap is a financial instrument of the collection of n caplets.

2.2 Pricing the Chooser Flexible Cap under the Risk Neutral Probability $\mathbb{P}^*$

We consider a continuous trading economy with a finite time horizon given by $\mathbb{T}^* := [0, T^*]$ ($T^* \in \mathbb{R}_{++}$). The uncertainty is modelled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$. In this notation, Ω denotes a sample space with elements $\omega \in \Omega$; $\mathcal{F}$ denotes a σ-algebra on Ω ; and $\mathbb{P}$ denotes a probability measure on $(\Omega, \mathcal{F})$. The uncertainty is resolved over $\mathbb{T}^*$ according to a $N-1$–dimensional Brownian (motion) filtration $\mathbb{F} := (\mathcal{F}(t) : t \in \mathbb{T}^*)$ satisfying the usual conditions. $W := (W(t) : t \in \mathbb{T}^*)$ denotes a $N-1$–dimensional standard $(\mathbb{P};\mathbb{F})$–Brownian motion. Consistent with the no-arbitrage and complete market paradigm, we assume the existence of the risk neutral equivalent martingale measure $\mathbb{P}^*$ with a bank account as a numéraire in this economy. We assume that the forward LIBOR of each period follows a geometric Brownian motion under the each forward neutral probability.

Let $W(T_i, L_{T_i}(T_i), l), i = 0, \cdots, N-1, l = 1, \cdots, M$ be the fair (no–arbitrage) price of the chooser flexible cap when, at time T_i, at most l exercises are remained to the buyer. The optimality equation can be derived by using the Bellman Principle. In this subsection, we derive the optimality equation under the risk neutral probability measure $\mathbb{P}^*$ with a bank account as a numéraire.

Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$W(T_{N-1}, L_{T_{N-1}}(T_{N-1}), l) = D(T_{N-1}, T_N)\delta[L_{T_{N-1}}(T_{N-1}) - K]_+, \quad l = 1, \quad \cdots, M; \tag{9}$$

(ii) For $i = N-2, \cdots, 0$:

$$\begin{aligned} W(T_i, L_{T_i}(T_i), l) &= \max\Big\{ D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \\ &+ \mathbb{E}^*\Big[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)}{B(T_i, T_{i+1})}\Big| L_{T_i}(T_i)\Big], \\ &\mathbb{E}^*\Big[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l)}{B(T_i, T_{i+1})}\Big| L_{T_i}(T_i)\Big]\Big\}, \quad l = 1, \cdots, M, \end{aligned} \tag{10}$$

where

$$W(T_i, L_{T_i}(T_i), 0) = 0, \quad i = 0, \cdots, N-1. \tag{11}$$

2.3 Pricing the Chooser Flexible Cap under the Forward Neutral Probability $\mathbb{P}^{T_N}$

We can also derive the optimality equation under the forward neutral probability $\mathbb{P}^{T_N}$ with a T_N–bond as a numéraire.

Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$W(T_{N-1}, L_{T_{N-1}}(T_{N-1}), l) = D(T_{N-1}, T_N)\delta[L_{T_{N-1}}(T_{N-1}) - K]_+, \quad l = 1, \cdots, M; \tag{12}$$

(ii) For $i = N-2, \cdots, 0$:

$$\begin{aligned} W(T_i, L_{T_i}(T_i), l) &= \max\Big\{ D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \\ &+ D(T_i, T_N)\mathbb{E}^{T_N}\Big[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)}{D(T_{i+1}, T_N)}\Big| L_{T_i}(T_i)\Big], \\ &D(T_i, T_N)\mathbb{E}^{T_N}\Big[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l)}{D(T_{i+1}, T_N)}\Big| L_{T_i}(T_i) L_{T_i}(T_i)\Big]\Big\}, \quad l = 1, \cdots, M, \end{aligned} \tag{13}$$

where

$$W(T_i, L_{T_i}(T_i), 0) = 0, \quad i = 0, \cdots, N-1. \tag{14}$$

2.4 Pricing the Chooser Flexible Cap under Varying Forward Neutral Probabilities $\mathbb{P}^{T_i}(1 \le i \le N)$

In this subsection we write the optimality equation under forward neutral probabilities $\mathbb{P}^{T_i}$ varying at each period with a T_i–bond as a numéraire. This optimality equation is different from the both equations of Subsection 2.2 and 2.3 that have the fixed probability measures at all periods.

Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$W(T_{N-1}, L_{T_{N-1}}(T_{N-1}), l) = D(T_{N-1}, T_N)\delta[L_{T_{N-1}}(T_{N-1}) - K]_+, \quad l = 1, \cdots, M; \tag{15}$$

(ii) For $i = N-2, \cdots, 0$:

$$\begin{aligned} &W(T_i, L_{T_i}(T_i), l) = \max\Big\{ D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \\ &+ D(T_i, T_{i+1})\mathbb{E}^{T_{i+1}}[W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)|L_{T_i}(T_i)], \\ &D(T_i, T_{i+1})\mathbb{E}^{T_{i+1}}[W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l)|L_{T_i}(T_i)]\Big\}, \quad l = 1, \cdots, M, \end{aligned} \tag{16}$$

where

$$W(T_i, L_{T_i}(T_i), 0) = 0, \quad i = 0, \cdots, N-1. \tag{17}$$

3 Easily Exercise the Option with l if s/he Exercises it with $l-1$ at T_i

In this section we theoretically prove that the option holder of the chooser flexible cap with more exercise opportunities easily exercises the i–caplet at T_i.

Proposition 3.1. *The holder of the chooser flexible cap exercises the option, i–caplet, with the exercise opportunity l at T_i if s/he exercises it with $l-1$ at the same state and T_i for $\forall i, i = 0, \cdots, N-1$.*

Corollary 3.1. *The holder of the chooser flexible cap does not exercise the option, i–caplet, with the exercise opportunity $l-1$ at T_i if s/he does not exercise it with l at the same state and T_i for $\forall i, i = 0, \cdots, N-1$.*

4 Conditions for Non Early Exercise of the Chooser Flexible Cap

In this section we derive theoretical conditions under which the option holder does not exercise i–caplet at T_i.

4.1 Conditions of $l = 1$

In this subsection we derive theoretical conditions under which we do not early exercise a $N-2$–caplet of the chooser flexible cap with $l = 1$ at $t = T_{N-2}$. Using the induction, we will derive the conditions under which we do not early exercise the chooser flexible cap with $l = 1$ at T_i for $i = 0, \cdots, N-2$.

Proposition 4.1. *The holder of the chooser flexible cap with $l = 1$ does not exercise the $N-2$–caplet at $t = T_{N-2}$ under the condition*

$$L_{T_{N-2}}(T_{N-2}) < \frac{L_{T_{N-1}}(T_{N-2})}{1 + \delta L_{T_{N-1}}(T_{N-2})}. \tag{18}$$

Proposition 4.2. *The holder of the chooser flexible cap with $l = 1$ does not exercise the option, i–caplet, at $t = T_i$ under the condition*

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)}, \quad i = 0, \cdots, N-2. \tag{19}$$

4.2 Conditions of $2 \leq l < N - i$

In this subsection we derive theoretical conditions under which we do not early exercise a $N-3$–caplet of the chooser flexible cap with $2 \leq l$ at $t = T_{N-3}$. Using the induction, we will derive the conditions under which we do not early exercise the chooser flexible cap with $2 \leq l < N - i$ at T_i for $i = 0, \cdots, N-1$.

Proposition 4.3. *The holder of the chooser flexible cap with $l = 2$ does not exercise the i–caplet at $t = T_{N-3}$ under the conditions*

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-2}}(T_{N-3})}{1 + \delta L_{T_{N-2}}(T_{N-3})}; \tag{20}$$

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-1}}(T_{N-3})}{1 + 2L(T_{N-3}; T_{N-2}, T_N)}. \tag{21}$$

Proposition 4.4. *The holder of the chooser flexible cap with $2 \leq l < N - i$ does not exercise the option, i–caplet, at $t = T_i$ under the conditions*

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)}; \tag{22}$$

$$L_{T_i}(T_i) < \frac{L_{T_{i+2}}(T_i)}{1 + 2L(T_i; T_{i+1}, T_{i+3})}, \quad i = 0, \cdots, N-3. \tag{23}$$

5 Exercise the Option at higher state variables

According to our numerical examples in Ito, Ohnishi and Tamba (2004), it seems that the option holder exercises the i-caplet at higher state than a state at which s/he exercises the option. In this section we prove that this proposition is satisfied.

Proposition 5.1. *The option holder exercises the i-caplet of the chooser flexible cap at higher state than a state at which s/he exercises the option at T_i, if s/he exercises the $i+1$-caplet at the same state on $t = T_{i+1}$ for $i = 0, \cdots, N-2$.*

6 Conclusion

We do research of various features of the chooser flexible cap. We find the three features of the chooser flexible cap in this paper. Firstly, we show that the option holder easily exercises the option with more exercise opportunities. Secondly, we derive the conditions for non early exercise of the chooser flexible cap. Thirdly, we prove that if the option holder exercises the i–caplet at a node, s/he also exercises the i–caplet at upper state nodes. We can use these features to cut down computational time of the chooser flexible cap price. We can also use the features for profitable exercise strategies. We derive the features assuming that LIBOR of each period follows the process of the geometric Brownian motion. So the chooser flexible cap has the feature not only in the Hull–White model of Hull and White (1990) as we see in Ito. Ohnishi and Tamba (2004) but also in Brace, Gątarek and Musiela (1997) frame work.

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7 Appendix

7.1 Proof of the Proposition 3.1

Proof. We prove that the holder of the chooser flexible cap exercises the option, i–caplet, with the exercise opportunities l at T_i if s/he exercises it with $l-1$ at T_i for $i = 0, \cdots, N-1$. We define $W(T_i, j, l; E)$ as an option holder exercises the i-caplet of the chooser flexible cap with the exercise opportunity of l times at a state j. Under the assumption, $W(T_i, j, l-1; E)$ at T_i, we want to prove $W(T_i, j, l; E)$. $W(T_i, j, l-1; E)$ means

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ + D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l-2)}{D(T_{i+1},T_{i+2})}|L(T_i)]$$
$$> D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l-1)}{D(T_{i+1},T_{i+2})}|L(T_i)]. \tag{24}$$

This inequality can be rewritten as

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+$$
$$> D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2)}{D(T_{i+1},T_{i+2})}|L(T_i)]. \tag{25}$$

On the other hand, $W(T_i,j,l;E)$ means

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ + D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l-1)}{D(T_{i+1},T_{i+2})}|L(T_i)]$$
$$> D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l)}{D(T_{i+1},T_{i+2})}|L(T_i)]. \tag{26}$$

This inequality can be rewritten as

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+$$
$$> D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l)-W(T_{i+1},j,l-1)}{D(T_{i+1},T_{i+2})}|L(T_i)]. \tag{27}$$

Hence, to prove (27) we should prove

$$W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2) \geq W(T_{i+1},j,l)-W(T_{i+1},j,l-1). \tag{28}$$

$W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2)$ is considered as that the option holder buys the chooser flexible cap with the exercise opportunity $l-1$ and sells the chooser flexible cap with the exercise opportunity $l-2$ at T_{i+1}. The option holder of $W(T_{i+1},j,l-1)$ can mimic the exercise strategy of $W(T_{i+1},j,l-2)$. After finishing the mimic of the strategy of $W(T_{i+1},j,l-2)$, the option holder of $W(T_{i+1},j,l-1)$ can exercise one time at one of the remaining time periods. $W(T_{i+1},j,l)-W(T_{i+1},j,l-1)$ is considered as that the option holder buys the chooser flexible cap with the exercise opportunity l and sells the chooser flexible cap with the exercise opportunity $l-1$ at T_{i+1}. The option holder of $W(T_{i+1},j,l)$ can mimic the exercise strategy of $W(T_{i+1},j,l-1)$. After finishing the mimic of the strategy of $W(T_{i+1},j,l-1)$, the option holder of $W(T_{i+1},j,l)$ can exercise one time at one of the remaining time periods. Comparing the both sides of (28), $W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2)$ is at least more worth than $W(T_{i+1},j,l)-W(T_{i+1},j,l-1)$ because the more periods are remained to exercise the remaining one time option. □

7.2 Proof of the Corollary 3.1

Proof. We prove that the holder of the chooser flexible cap does not exercise the option, i–caplet, with the exercise opportunity $l-1$ at T_i if s/he does not exercise it with l at T_i for $i=0,\cdots,N-1$. We define $W(T_i,j,l;X)$ as the option holder does not exercise the i-caplet of the chooser flexible cap with the exercise opportunity of l times at a state j

and $t = T_i$. Under the assumption, $W(T_i, j, l; X)$, we want to prove $W(T_i, j, l-1; X)$ at T_i. We assume

$$D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ + D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, j, l-1)}{D(T_{i+1}, T_{i+2})}|L(T_i)]$$
$$< D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, j, l)}{D(T_{i+1}, T_{i+2})}|L(T_i)]. \tag{29}$$

This inequality can be rewritten as

$$D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+$$
$$< D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, j, l) - W(T_{i+1}, j, l-1)}{D(T_{i+1}, T_{i+2})}|L(T_i)]. \tag{30}$$

We want to prove

$$D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ + D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, j, l-2)}{D(T_{i+1}, T_{i+2})}|L(T_i)]$$
$$< D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, j, l-1)}{D(T_{i+1}, T_{i+2})}|L(T_i)]. \tag{31}$$

This inequality can be rewritten as

$$D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+$$
$$< D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, j, l-1) - W(T_{i+1}, j, l-2)}{D(T_{i+1}, T_{i+2})}|L(T_i)]. \tag{32}$$

Hence, to prove (32) we should prove

$$W(T_{i+1}, j, l-1) - W(T_{i+1}, j, l-2) \geq W(T_{i+1}, j, l) - W(T_{i+1}, j, l-1). \tag{33}$$

This inequality is satisfied as we see in Proposition 1. □

7.3 Proof of the Proposition 4.1

Proof. We derive theoretical conditions under which the option holder does not exercise $N-2$–caplet at T_{N-2}. The condition under which the option holder does not exercises the $N-2$–caplet is

$$D(T_{N-2}, T_N)\delta(L_{T_{N-2}}(T_{N-2}) - K)_+$$
$$< D(T_{N-2}, T_N)\mathbb{E}^{T_N}[\frac{W(T_{N-1}, L_{N-1}(T_{N-1}), 1)}{D(T_{N-1}, T_N)}|L(T_{N-2})]$$
$$= D(T_{N-2}, T_N)\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})]. \tag{34}$$

$$RHS \geq D(T_{N-2}, T_N)\delta(\mathbb{E}^{T_N}[L_{T_{N-1}}(T_{N-1})|L(T_{N-2})] - K)_+$$
$$= D(T_{N-2}, T_{N-1})\frac{1}{1 + \delta L_{T_{N-1}}(T_{N-2})}\delta(L_{T_{N-1}}(T_{N-2}) - K)_+$$
$$\geq D(T_{N-2}, T_{N-1})\delta(\frac{L_{T_{N-1}}(T_{N-2})}{1 + \delta L_{T_{N-1}}(T_{N-2})} - K)_+ \tag{35}$$

Hence the sufficient condition to satisfy this proposition is

$$L_{T_{N-2}}(T_{N-2}) < \frac{L_{T_{N-1}}(T_{N-2})}{1 + \delta L_{T_{N-1}}(T_{N-2})}. \tag{36}$$

□

7.4 Proof of the Proposition 4.2

Proof. In the case of $l = 1$ and $t = T_{N-2}$, from the result of Proposition 3 we prove that we do not exercise the option under the conditions (36). In the case of $l = 1$ and $t = T_{i+1}$, we suppose that we do not exercise the option under the condition

$$L_{T_{i+1}}(T_{i+1}) < \frac{L_{T_{i+2}}(T_{i+1})}{1 + \delta L_{T_{i+2}}(T_{i+1})}, \tag{37}$$

that is,

$$\begin{aligned} &D(T_{i+1}, T_{i+2})\delta(L_{T_{i+1}}(T_{i+1}) - K)_+ \\ &< D(T_{i+1}, T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), 1)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})] = W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), 1). \end{aligned} \tag{38}$$

In the case of $l = 1$ and $t = T_i$ we would like to show that, under the condition

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)}, \tag{39}$$

we do not exercise the option at $t = T_i$ by the induction. The optimality equation at $t = T_i$ is

$$\begin{aligned} W(T_i, L_{T_i}(T_i), 1) = \max\{&D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+, \\ &D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), 1)}{D(T_{i+1}, T_{i+2})}|L(T_i)]\}. \end{aligned} \tag{40}$$

From the hypothesis, (38), substituting LHS of (38) for the second term of (40) we obtain

$$\begin{aligned} &D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, L_{T_i+1}(T_{i+1}), 1)}{D(T_{i+1}, T_{i+2})}|L(T_i)] \\ &> D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{1}{D(T_{i+1}, T_{i+2})}\{D(T_{i+1}, T_{i+2})\delta(L_{T_{i+1}}(T_{i+1}) - K)_+\}|L(T_i)] \\ &= D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)]. \end{aligned} \tag{41}$$

Utilizing the relation

$$\begin{aligned} &D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)] \\ &\geq D(T_i, T_{i+1})\delta(\frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)} - K)_+, \end{aligned} \tag{42}$$

and comparing the first term of RHS of (40) and (41), we obtain that the non early exercise condition is

$$D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \leq D(T_i, T_{i+1})\delta(\frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)} - K)_+. \tag{43}$$

Then the sufficient condition of non early exercise is

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)}. \tag{44}$$

This condition is satisfied from the hypothesis, (39).

□

7.5 Proof of the Proposition 4.3

Proof. We consider the case of $2 \leq l < N-i$. From the forward neutral evaluation, we have

$$\frac{W(T_i, L_{T_i}(T_i), l)}{D(T_i, T_{i+2})} = \max\Big\{\mathbb{E}^{T_{i+2}}\Big[\frac{\delta(L_{T_i}(T_i) - K)_+ + W(T_{i+1}, L_{T_i+1}(T_{i+1}), l-1)}{D(T_{i+1}, T_{i+2})}\Big| L(T_i)\Big], \\ \mathbb{E}^{T_{i+2}}\Big[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}, l)}{D(T_{i+1}, T_{i+2})}\Big| L(T_i)\Big]\Big\}. \tag{45}$$

This equation can be rewritten as

$$\begin{aligned} W(T_i, L_{T_i}(T_i), l) = \max\Big\{ & D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \\ & + D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)}{D(T_{i+1}, T_{i+2})}|L(T_i)], \\ & D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, L_{T_i+1}(T_{i+1}), l)}{D(T_{i+1}, T_{i+2})}|L(T_i)]\Big\}. \end{aligned} \tag{46}$$

The value of the chooser flexible cap with $2 \leq l$ at $t = T_{N-3}$ is

$$W(T_{N-3}, L_{T_{N-3}}(T_{N-3}), l) = \max\{A, B\}, \tag{47}$$

where

$$\begin{aligned} A = & D(T_{N-3}, T_{N-2})\delta(L_{T_{N-3}}(T_{N-3}) - K)_+ + D(T_{N-3}, T_{N-1}) \\ & \mathbb{E}^{T_{N-1}}[\frac{W(T_{N-2}, L_{T_{N-2}}(T_{N-2}), l-1)}{D(T_{N-2}, T_{N-1})}|L(T_{N-3})]; \end{aligned} \tag{48}$$

$$B = D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{W(T_{N-2}, L_{N-2}(T_{N-2}), l)}{D(T_{N-2}, T_{N-1})}|L(T_{N-3})]. \tag{49}$$

In the case of $3 \leq l$, we exercise the options at all of $t = T_{N-3}, T_{N-2}$and T_{N-1}. In the case of $l = 2$, we want to derive the conditions for non early exercise of the option at $t = T_{N-3}$. A is calculated as

$$\begin{aligned} A = & D(T_{N-3}, T_{N-2})\delta(L_{T_{N-3}}(T_{N-3}) - K)_+ \\ & + D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{W(T_{N-2}, L_{T_{N-2}}(T_{N-2}), 1)}{D(T_{N-2}, T_{N-1})}|L(T_{N-3})] \\ & = D(T_{N-3}, T_{N-2})\delta(L_{T_{N-3}}(T_{N-3}) - K)_+ \\ & + D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\max\Big\{\delta(L_{T_{N-2}}(T_{N-2}) - K)_+, \\ & \frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})]\Big\}|L(T_{N-3})]. \end{aligned} \tag{50}$$

On the other hand, we have

$$\begin{aligned} B = & D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{W(T_{N-2}, L_{N-2}(T_{N-2}), 2)}{D(T_{N-2}, T_{N-1})}|L(T_{N-3})] \\ = & D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2}) - K)_+|L(T_{N-3})] \\ + & D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})]|L(T_{N-3})]. \end{aligned} \tag{51}$$

The sufficient conditions for non early exercise of the option is $A < B$. Using the Jensen's inequality,

$$\begin{aligned}
A &\geq D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+ \\
&+ D(T_{N-3},T_{N-1})\max\Big\{\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})], \\
&\mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})]\Big\} \\
&:= C. \qquad (52)
\end{aligned}$$

We derive the sufficient conditions under which $C < B$ is satisfied.These are the sufficient conditions under which the option holder does not exercise the option at T_{N-3}. We have two cases under which $C < B$ satisfies. The first case is the comparison between the payments of T_{N-2} and T_{N-3}, that is

$$\begin{aligned}
&\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})] \\
&< \mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})]. \qquad (53)
\end{aligned}$$

In this case we have

$$\begin{aligned}
C &= D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+ + D(T_{N-3},T_{N-1}) \\
&\mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})]. \qquad (54)
\end{aligned}$$

So in order to satisfy $C < B$, we need

$$\begin{aligned}
&D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+ \\
&< D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})]. \qquad (55)
\end{aligned}$$

Utilizing the relation

$$\begin{aligned}
&D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})] \\
&\geq D(T_{N-3},T_{N-2})\delta(\frac{L_{T_{N-2}}(T_{N-3})}{1+\delta L_{T_{N-2}}(T_{N-3})}-K)_+, \qquad (56)
\end{aligned}$$

we find that the sufficient condition for $C < B$ is

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-2}}(T_{N-3})}{1+\delta L_{T_{N-2}}(T_{N-3})}. \qquad (57)$$

The second case is comparison between the payments of T_{N-1} and T_{N-3}, that is

$$\begin{aligned}
&\mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})] \\
&< D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})]. \qquad (58)
\end{aligned}$$

In this case we have

$$\begin{aligned}
C &= D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+ + D(T_{N-3},T_{N-1}) \\
&\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})]. \qquad (59)
\end{aligned}$$

So in order to satisfy $C < B$ we need

$$\begin{aligned}
&D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+ \\
&< D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})] \\
&= D(T_{N-3},T_N)\mathbb{E}^{T_N}[\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})] \\
&= D(T_{N-3},T_N)\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-3})] \\
&= D(T_{N-3},T_{N-2})\frac{1}{1+2L(T_{N-3};T_{N-2},T_N)}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-3})] \\
&\geq D(T_{N-3},T_{N-2})\frac{1}{1+2L(T_{N-3};T_{N-2},T_N)}\delta(\mathbb{E}^{T_N}[L_{T_{N-1}}(T_{N-1})|L(T_{N-3})]-K)_+ \\
&\geq D(T_{N-3},T_{N-2})\delta(\frac{L_{T_{N-1}}(T_{N-3})}{1+2L(T_{N-3};T_{N-2},T_N)}-K)_+. \qquad (60)
\end{aligned}$$

So we can derive the following non early exercise condition in this case as

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-1}}(T_{N-3})}{1+2L(T_{N-3};T_{N-2},T_N)}. \qquad (61)$$

□

7.6 Proof of the Proposition 4.4

Proof. We consider the case of $2 \leq l < N-i$ at $t = T_i$. In the case of $t = T_{N-3}$, from the result of Proposition 5 we found that we do not exercise the option under the conditions (57) and (61).We suppose that we do not exercise the option under the conditions at $t = T_{i+1}$

$$L_{T_{i+1}}(T_{i+1}) < \frac{L_{T_{i+2}}(T_{i+1})}{1+\delta L_{T_{i+2}}(T_{i+1})}; \qquad (62)$$

$$L_{T_{i+1}}(T_{i+1}) < \frac{L_{T_{i+3}}(T_{i+1})}{1+2L(T_{i+1};T_{i+2},T_{i+4})}. \qquad (63)$$

That is,

$$W(T_{i+1},L_{T_{i+1}}(T_{i+1}),l) = \max\{H,I\} \qquad (64)$$

satisfies $H < I$, where

$$\begin{aligned}
H &= D(T_{i+1},T_{i+2})\delta(L_{T_{i+1}}(T_{i+1})-K)_+ + D(T_{i+1},T_{i+3}) \\
&\quad \mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2},L_{T_{i+2}}(T_{i+2}),l-1)}{D(T_{i+2},T_{i+3})}|L(T_{i+1})]; \qquad (65)
\end{aligned}$$

$$I = D(T_{i+1},T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2},L_{T_{i+2}}(T_{i+2}),l)}{D(T_{i+2},T_{i+3})}|L(T_{i+1})]. \qquad (66)$$

We would like to show that, under the conditions

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)}; \qquad (67)$$

$$L_{T_i}(T_i) < \frac{L_{T_{i+2}}(T_i)}{1+2L(T_i;T_{i+1},T_{i+3})}, \tag{68}$$

we do not exercise the option at $t = T_i$. The optimality equation is

$$W(T_i, L_{T_i}(T_i), l) = \max\{J, K\}, \tag{69}$$

where

$$\begin{aligned} J &= D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \\ &+ D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)}{D(T_{i+1}, T_{i+2})}|L(T_i)]; \end{aligned} \tag{70}$$

$$K = D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, L_{T_i+1}(T_{i+1}), l)}{D(T_{i+1}, T_{i+2})}|L(T_i)]. \tag{71}$$

From the hypothesis, $H < I$, substituting H for K we obtain

$$\begin{aligned} K &> D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{1}{D(T_{i+1}, T_{i+2})}\Big\{D(T_{i+1}, T_{i+2})\delta(L_{T_{i+1}}(T_{i+1}) - K)_+ \\ &+ D(T_{i+1}, T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})]\Big\}|L(T_i)] \\ &= D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)] + D(T_i, T_{i+2}) \\ &\mathbb{E}^{T_{i+2}}[\frac{D(T_{i+1}, T_{i+3})}{D(T_{i+1}, T_{i+2})}\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})]|L(T_i)]. \end{aligned} \tag{72}$$

Utilizing the relation

$$\begin{aligned} &D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)] \\ &\geq D(T_i, T_{i+1})\delta(\frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)} - K)_+, \end{aligned} \tag{73}$$

and comparing the first terms of J and (72), we obtain that the sufficient condition of non early exercise condition is

$$D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \leq D(T_i, T_{i+1})\delta(\frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)} - K)_+. \tag{74}$$

Then the sufficient condition of non early exercise is

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)}. \tag{75}$$

This condition is satisfied from the hypothesis, (67). In order to compare the second terms of J and (72), we use the corollary 2. From the hypothesis, $H < I$ with the exercise opportunity l, we know that the option holder does not exercise the option with $l-1$ at the same state and T_{i+1}. So we have

$$\begin{aligned} &W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1) = D(T_{i+1}, T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})] \\ &> D(T_{i+1}, T_{i+2})\delta(L_{T_{i+1}}(T_{i+1}) - K)_+ \\ &+ D(T_{i+1}, T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-2)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})] \end{aligned} \tag{76}$$

We substitute $W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)$ for the second term of J. We obtain

$$\begin{aligned} & D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)}{D(T_{i+1}, T_{i+2})}|L(T_i)] \\ & = D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{D(T_{i+1}, T_{i+3})}{D(T_{i+1}, T_{i+2})}\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})]|L(T_i)]. \quad (77) \end{aligned}$$

So, both of the second terms of J and (72) are equal. Then under the conditions (67) and (68) we do not exercise the option at $t = T_i$. □

7.7 Proof of the Proposition 5.1

Proof. We prove that the option holder exercises the i-caplet at higher state than a state at which s/he exercises the option.

7.7.1 The case $l = 1$ at T_{N-2}

In this subsection we consider the case l=1 at T_{N-2}. We define

$$f(L_{T_{N-1}}(T_{N-1})) := D(T_{N-2}, T_N)\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})]; \quad (78)$$

$$g(L_{T_{N-2}}(T_{N-2}) := D(T_{N-2}, T_{N-1})\delta(L_{T_{N-2}}(T_{N-2}) - K)_+. \quad (79)$$

The optimality equation is

$$v(L_{T_{N-1}}(T_{N-1}), L_{T_{N-2}}(T_{N-2}) = \max\{g(L_{T_{N-2}}(T_{N-2}), f(L_{T_{N-1}}(T_{N-1}))\}. \quad (80)$$

By the Black–Scholes formula,

$$f(L_{T_{N-1}}(T_{N-1})) = \delta D(T_{N-2}, T_N)[L_{T_{N-1}}(T_{N-2})\Phi(d) - K\Phi(d-\psi)], \quad (81)$$

where

$$d = \frac{\log(\frac{L_{T_{N-1}}(T_{N-2})}{K})}{\psi} + \frac{\psi}{2}; \quad (82)$$

$$\frac{\mathrm{d}L_{T_{N-1}}(t)}{L_{T_{N-1}}(t)} = \sigma_{N-1}(t)\mathrm{d}W^{T_N}(t), \quad 0 \le t \le T_i; \quad (83)$$

$$\psi^2 = \int_{T_{N-2}}^{T_{N-1}} \{\sigma_{N-1}(s)\}^2 ds, \quad (84)$$

$W^{T_N}(t)$ is the Brownian motion under the forward neutral probability $\mathbb{P}^{T_N}$. We define

$$\begin{aligned} \overline{f}(L_{T_{N-1}}(T_{N-2})) &:= \frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N-1})}\mathbb{E}^{T_N}[(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})] \\ &= \frac{1}{1+\delta L_{T_{N-1}}(T_{N-2})}[L_{T_{N-1}}(T_{N-2})\Phi(d) - K\Phi(d-\psi)]. \quad (85) \end{aligned}$$

We also define

$$
\begin{aligned}
&u(L_{T_{N-1}}(T_{N-1}), L_{T_{N-1}}(T_{N-2})) := g(L_{T_{N-1}}(T_{N-1})) - f(L_{T_{N-1}}(T_{N-2})) \\
&= \delta D(T_{N-2}, T_{N-1})[(L_{T_{N-2}}(T_{N-2}) - K)_+ \\
&- \frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N-1})}\mathbb{E}^{T_N}[(L_{T_{N-1}}(T_{N-1}) - K)_+ | L(T_{N-2})]] \\
&= \delta D(T_{N-2}, T_{N-1})[(L_{T_{N-2}}(T_{N-2}) - K)_+ - \overline{f}(L_{T_{N-1}}(T_{N-2}))]. \qquad (86)
\end{aligned}
$$

We look into features of the function $\overline{f}(L_{T_{N-1}}(T_{N-2}))$.

$$
\begin{aligned}
&\lim_{L_{T_{N-1}}(T_{N-2})\uparrow\infty} \overline{f}(L_{T_{N-1}}(T_{N-2})) = \\
&\frac{L_{T_{N-1}}(T_{N-2})}{1+\delta L_{T_{N-1}}(T_{N-2})}\Phi(d) - \frac{K}{1+\delta L_{T_{N-1}}(T_{N-2})}\Phi(d-\psi) = 1 \qquad (87)
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{L_{T_{N-1}}(T_{N-2})\downarrow 0} \overline{f}(L_{T_{N-1}}(T_{N-2})) = \\
&\frac{L_{T_{N-1}}(T_{N-2})}{1+\delta L_{T_{N-1}}(T_{N-2})}\Phi(d) - \frac{K}{1+\delta L_{T_{N-1}}(T_{N-2})}\Phi(d-\psi) = 0 \qquad (88)
\end{aligned}
$$

We define

$$
h(L_{T_{N-1}}(T_{N-2})) := L_{T_{N-1}}(T_{N-2})\Phi(d) - K\Phi(d-\psi). \qquad (89)
$$

Using the following relation

$$
\begin{aligned}
\frac{\mathrm{d}\Phi(d-\psi)}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} &= \frac{1}{\sqrt{2\pi}}\exp(-\frac{d^2}{2})\exp(d\psi - \frac{\psi^2}{2})\frac{1}{\psi L_{T_{N-1}}(T_{N-2})} \\
&= \frac{\mathrm{d}\Phi(d)}{\mathrm{d}d}\frac{1}{\psi L_{T_{N-1}}(T_{N-2})}, \qquad (90)
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{\mathrm{d}h(L_{T_{N-1}}(T_{N-2}))}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} &= \Phi(d) + L_{T_{N-1}}(T_{N-2})\frac{\mathrm{d}\Phi(d)}{\mathrm{d}d}\frac{1}{\psi L_{T_{N-1}}(T_{N-2})} \\
&- K\frac{\mathrm{d}\Phi(d-\psi)}{\mathrm{d}d}\frac{1}{\psi L_{T_{N-1}}(T_{N-2})} \\
&= \Phi(d). \qquad (91)
\end{aligned}
$$

So using (91) we have

$$
\begin{aligned}
\frac{\mathrm{d}\overline{f}(L_{T_{N-1}}(T_{N-2}))}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} &= \frac{\Phi(d)(1+\delta L_{T_{N-1}}(T_{N-2})) - \delta[L_{T_{N-1}}(T_{N-2})\Phi(d) - K\Phi(d-\psi)]}{(1+\delta L_{T_{N-1})^2}} \\
&= \frac{\Phi(d) + \delta K\Phi(d-\psi)}{(1+\delta L_{T_{N-1})^2}} > 0. \qquad (92)
\end{aligned}
$$

Hence, we find

$$
\lim_{L_{T_{N-1}}(T_{N-2})\uparrow\infty} \frac{\mathrm{d}\overline{f}(L_{T_{N-1}}(T_{N-2}))}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} = 0; \qquad (93)
$$

$$\lim_{L_{T_{N-1}}(T_{N-2})\downarrow 0} \frac{\mathrm{d}\overline{f}(L_{T_{N-1}}(T_{N-2}))}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} = 0. \tag{94}$$

On the other hand, we can show that we have only one solution to the following equation for $L_{T_{N-1}}(T_{N-2}) > K$.

$$\overline{f}(L_{T_{N-1}}(T_{N-2})) = L_{T_{N-1}}(T_{N-2}) - K, \tag{95}$$

that is

$$\frac{1}{1+\delta L_{T_{N-1}}(T_{N-2})}[L_{T_{N-1}}(T_{N-2})\Phi(d) - K\Phi(d-\psi)] = L_{T_{N-1}}(T_{N-2}) - K. \tag{96}$$

$$L_{T_{N-1}}(T_{N-2})\Phi(d) - K\Phi(d-\psi) = \{1+\delta L_{T_{N-1}}(T_{N-2})\}\{L_{T_{N-1}}(T_{N-2}) - K\} \tag{97}$$

We investigate the function $h(L_{T_{N-1}}(T_{N-2}))$, LHS of (97).

$$\begin{aligned}\lim_{L_{T_{N-1}}(T_{N-2})\uparrow\infty} h(L_{T_{N-1}}(T_{N-2})) &= L_{T_{N-1}}(T_{N-2})\Phi(d) - K\Phi(d-\psi) \\ &= L_{T_{N-1}}(T_{N-2}) - K \end{aligned} \tag{98}$$

Using (91) we have

$$\lim_{L_{T_{N-1}}(T_{N-2})\uparrow\infty} \frac{\mathrm{d}h(L_{T_{N-1}}(T_{N-2}))}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} = 1; \tag{99}$$

$$\lim_{L_{T_{N-1}}(T_{N-2})\downarrow 0} h(L_{T_{N-1}}(T_{N-2})) = 0; \tag{100}$$

$$\lim_{L_{T_{N-1}}(T_{N-2})\downarrow 0} \frac{\mathrm{d}h(L_{T_{N-1}}(T_{N-2}))}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} = 0; \tag{101}$$

$$\begin{aligned}\frac{\mathrm{d}^2 h(L_{T_{N-1}}(T_{N-2}))}{\mathrm{d}L_{T_{N-1}}(T_{N-2})^2} &= \frac{\mathrm{d}\Phi(d)}{\mathrm{d}L_{T_{N-1}}(T_{N-2})} \\ &= \frac{\mathrm{d}\Phi(d)}{\mathrm{d}d}\frac{1}{\psi L_{T_{N-1}}(T_{N-2})} \\ &= \frac{1}{\sqrt{2\pi}}\exp(-\frac{d^2}{2})\frac{1}{\psi L_{T_{N-1}}(T_{N-2})} > 0. \end{aligned} \tag{102}$$

On the other hand, we have

$$RHSof(97) = \{L_{T_{N-1}}(T_{N-2}) + \frac{1-\delta K}{2\delta}\}^2 - \frac{4\delta K + (1-\delta K)^2}{4\delta^2}. \tag{103}$$

Hence we can draw the function $h(L_{T_{N-1}}(T_{N-2}))$ and RHS of (97) as Figure 1. Then the figures of $\overline{f}(L_{T_{N-1}}(T_{N-2}))$ and $\{L_{T_{N-1}}(T_{N-2}) - K\}_+$ cross only one time. We can draw the function $\overline{f}(L_{T_{N-1}}(T_{N-2}))$ and $\{L_{T_{N-1}}(T_{N-2}) - K\}_+$ as Figure 2. We have the figure $u(L_{T_{N-1}}(T_{N-1}), L_{T_{N-1}}(T_{N-2}))$ as Figure 3. As the result we find that the option holder exercises the $N-2$-caplet at higher state than a state at which s/he exercises the option.

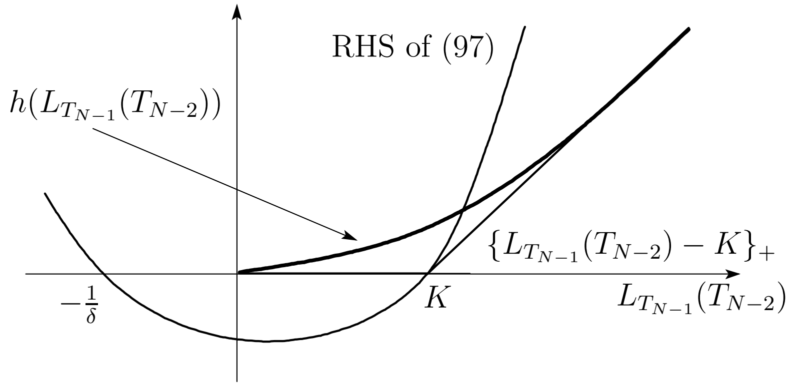

Figure 1: The function of $h(L_{T_{N-1}}(T_{N-2}))$, RHS of (97) and $\{L_{T_{N-1}}(T_{N-2}) - K\}_+$.

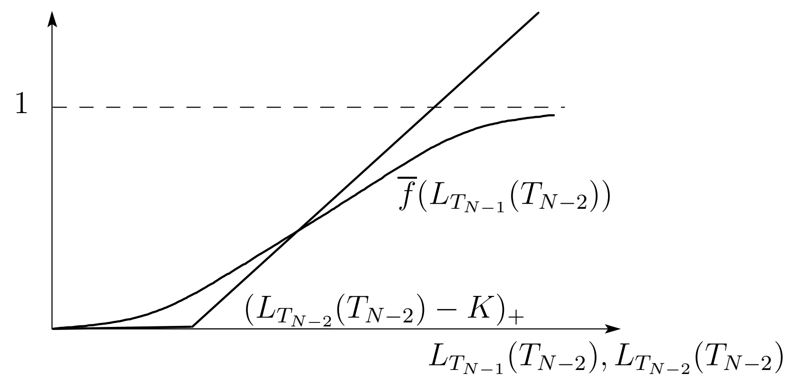

Figure 2: The function of $\overline{f}(L_{T_{N-1}}(T_{N-2}))$ and $\{L_{T_{N-1}}(T_{N-1}) - K\}_+$.

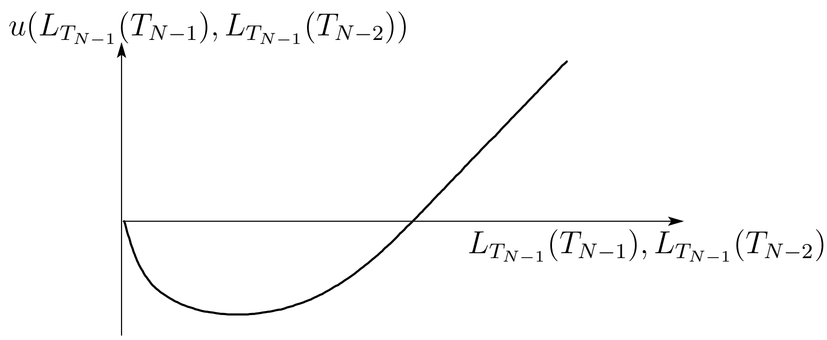

Figure 3: The function of $u(L_{T_{N-1}}(T_{N-1}), L_{T_{N-1}}(T_{N-2}))$.

7.7.2 The case $l = 1$ at T_i

We consider the case $l = 1$ at T_i. We will prove that if $W(T_i, j, 1; E)$ is satisfied then $W(T_i, j', 1; E)$, where j' is upper state than j, by the induction. We already proved the case $l = 1$ at T_{N-2} in Subsection 7.7.1. We assume

$$W(T_{i+1}, j, 1; E) = D(T_{i+1}, T_{i+2})\delta(L_{T_{i+1}}(T_{i+1}) - K)_+ \\ > D(T_{i+1}, T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, j, 1)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})]. \tag{104}$$

We also assume

$$W(T_i, j, 1; E) = D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \\ > D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1}, j, 1)}{D(T_{i+1}, T_{i+2})}|L(T_i)]. \tag{105}$$

Substituting (104) for RHS of (105), we have

$$RHSof(105) = D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)] \\ = D(T_i, T_{i+1})\frac{1}{1 + \delta L_{T_{i+1}}(T_i)}\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)]. \tag{106}$$

With the same reasoning as Subsection 7.7.1, we can conclude that the option holder exercises the i-caplet at higher state than a state at which s/he exercises the option if s/he exercises the $i+1$-caplet at the same state on $t = T_{i+1}$.

7.7.3 The case $l = 2$ at T_{N-3}

We consider the case $l = 2$ at T_{N-3}. We assume $W(T_{N-3}, j, 2; E)$, that is

$$D(T_{N-3}, T_{N-2})\delta(L_{T_{N-3}}(T_{N-3}) - K)_+ \\ + D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{W(T_{N-2}, L_{T_{N-2}}(T_{N-2}), 1)}{D(T_{N-2}, T_{N-1})}|L(T_{N-3})] \\ > D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{W(T_{N-2}, L_{N-2}(T_{N-2}), 2)}{D(T_{N-2}, T_{N-1})}|L(T_{N-3})]. \tag{107}$$

This equation is rewritten as

$$D(T_{N-3}, T_{N-2})\delta(L_{T_{N-3}}(T_{N-3}) - K)_+ + D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\max\Big\{ \\ \delta(L_{T_{N-2}}(T_{N-2}) - K)_+, \frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})]\Big\}|L(T_{N-3})] \\ > D(T_{N-3}, T_{N-1})\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2}) - K)_+|L(T_{N-3})] + D(T_{N-3}, T_{N-1}) \\ \mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})]|L(T_{N-3})]. \tag{108}$$

We consider two cases. The first case is

$$\delta(L_{T_{N-2}}(T_{N-2}) - K)_+ \\ > \frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1}) - K)_+|L(T_{N-2})]|L(T_{N-3})]. \tag{109}$$

In this case the (108) is simplified as

$$\begin{aligned}
&D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+\\
&> D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\\
&\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})]\\
&= D(T_{N-3},T_{N-1})\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-3})]\\
&= D(T_{N-3},T_{N-2})\frac{1}{1+\delta L_{T_{N-2}}(T_{N-3})}\frac{1}{1+\delta L_{T_{N-1}}(T_{N-3})}\\
&\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-3})]. \qquad (110)
\end{aligned}$$

Considering $L_{T_{N-2}}(T_{N-3})$ as constant, we can show that the option holder exercises the $N-3$–caplet at higher state than a state at which s/he exercises the option with the same reasoning as Subsection 7.7.1. The second case is

$$\begin{aligned}
&\delta(L_{T_{N-2}}(T_{N-2})-K)_+\\
&< \frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}[\delta(L_{T_{N-1}}(T_{N-1})-K)_+|L(T_{N-2})]|L(T_{N-3})]. \qquad (111)
\end{aligned}$$

In this case the (108) is simplified as

$$\begin{aligned}
&D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+\\
&> D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})]\\
&= D(T_{N-3},T_{N-2})\frac{1}{1+\delta L_{T_{N-2}}(T_{N-3})}\mathbb{E}^{T_{N-1}}[\delta(L_{T_{N-2}}(T_{N-2})-K)_+|L(T_{N-3})]. \qquad (112)
\end{aligned}$$

We can show that the option holder exercises the $N-3$–caplet at higher state than a state at which s/he exercises the option with the same reasoning as Subsection 7.7.1.

7.7.4 The case $2 \le l < N-i$ at T_i

We consider the case $2 \le l < N-i$ at T_i. We assume $W(T_{i+1},j,l;E)$, that is

$$\begin{aligned}
&D(T_{i+1},T_{i+2})\delta(L_{T_{i+1}}(T_{i+1})-K)_+ + D(T_{i+1},T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2},j,l-1)}{D(T_{i+2},T_{i+3})}|L(T_{i+1})]\\
&> D(T_{i+1},T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2},j,l)}{D(T_{i+2},T_{i+3})}|L(T_{i+1})]. \qquad (113)
\end{aligned}$$

We also assume $W(T_i,j,l;E)$, that is

$$\begin{aligned}
&D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ + D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l-1)}{D(T_{i+1},T_{i+2})}|L(T_i)]\\
&> D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{W(T_{i+1},j,l)}{D(T_{i+1},T_{i+2})}|L(T_i)]\\
&= D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1})-K)_+\\
&+ \frac{D(T_{i+1},T_{i+3})}{D(T_{i+1},T_{i+2})}\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2},j,l)}{D(T_{i+2},T_{i+3})}|L(T_{i+1})]|L(T_i)]. \qquad (114)
\end{aligned}$$

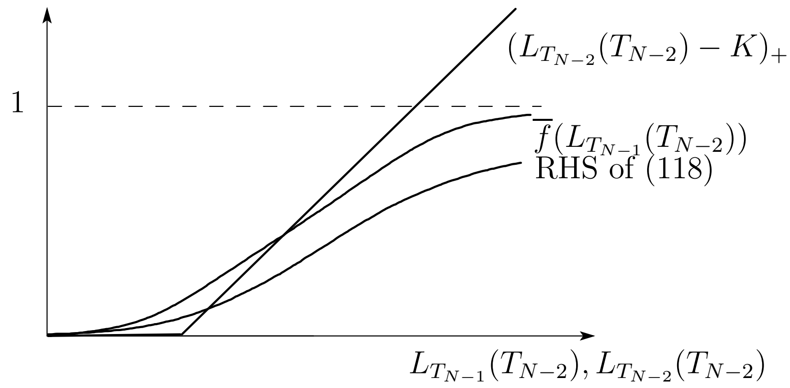

Figure 4: The function of $\overline{f}(L_{T_{N-1}}(T_{N-2}))$, RHS of (118) and $\{L_{T_{N-1}}(T_{N-1})-K\}_+$.

We define

$$P := W(T_{i+1}, j, l-1) - D(T_{i+1}, T_{i+3})\mathbb{E}^{T_{i+3}}[\frac{W(T_{i+2}, j, l-1)}{D(T_{i+2}, T_{i+3})}|L(T_{i+1})]. \tag{115}$$

The first term is the chooser flexible cap price with the exercise opportunity $l-1$ at T_{i+1} and the second term represents the price at T_{i+1} of the chooser flexible cap of T_{i+2} with $l-1$. So P is considered as the premium for the right to exercise $i+1$–caplet. It is clear that $P > 0$ and the premium P gets bigger as the state is bigger because the payoff from exercising $i+1$–caplet gets bigger as the state is bigger. On the other hand, the premium is not bigger than the payoff value, that is

$$D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)] > D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{P}{D(T_{i+1}, T_{i+2})}|L(T_i)]. \tag{116}$$

The premium increases with the same move as the change of the payoff from the $i+1$–caplet if the state is higher. (114) can be rewritten as follows.

$$\begin{aligned} &D(T_i, T_{i+1})\delta(L_{T_i}(T_i) - K)_+ \\ &> D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)] \\ &- D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}[\frac{P}{D(T_{i+1}, T_{i+2})}|L(T_i)] \end{aligned} \tag{117}$$

$$\begin{aligned} &(L_{T_i}(T_i) - K)_+ \\ &> \frac{1}{1+\delta L_{T_{i+1}}(T_i)}\mathbb{E}^{T_{i+2}}[\delta(L_{T_{i+1}}(T_{i+1}) - K)_+|L(T_i)] \\ &- \frac{1}{1+\delta L_{T_{i+1}}(T_i)}\mathbb{E}^{T_{i+2}}[\frac{P}{D(T_{i+1}, T_{i+2})}|L(T_i)] \end{aligned} \tag{118}$$

From the above consideration, the figure of each side of (118) is shown as Figure 4. With the same reasoning as Subsection 7.7.1, we can conclude that the option holder exercises the i-caplet at higher state than a state at which s/he exercises the option if s/he exercises the $i+1$-caplet at the same state on $t = T_{i+1}$. □