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Dependent Background Risks and Asset Prices*

Yusuke Osaki[†]
Graduate School of Economics, Osaka University

Abstract

Dependent background risks which have functional forms are introduced into Lucas economies. This paper determines the conditions on preferences to guarantee the monotonicity of asset prices, when dependent background risks satisfy the monotonicity and the single crossing conditions.

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Keywords: Asset Price, Dependent Background Risk, Monotonicity, Single Crossing Condition.

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[†]Address: Graduate School of Economics, Osaka University, 1–7 Machikaneyama–Machi, Toyonaka, Osaka 560–0043, Japan. E–mail: `OsakiYusuke@srv.econ.osaka-u.ac.jp`. Phone Number: +81–6–6850–5234. Fax Number: +81–6–6850–5277

1 Introduction

In recent years, many studies concerning the economic theory of decision making under uncertainty have been analyzing situations with multiple sources of risk. In typical settings, decision makers face two sources of risk, controllable risks and uncontrollable risks, which are mutually independent. Uncontrollable risks are usually called background risks. However, independent background risks are controversial matters from the descriptive viewpoint. Surely, we do not face independent, but dependent background risks in most situations with uncertainty. Actually, the equity premium puzzle (Mehra and Prescott, 1985) cannot be explained by independent background risks.[1)] Therefore, it is necessary to analyze situations with dependent background risks. Yet, very few papers have investigated this topic. Motivated by the state of the art, we examine the effect of dependent background risks on asset prices in this paper. Of course, our analysis can be applied to other situations with dependent background risks. Since negatively dependent background risks have an insurance device for risk averse investors, we have the following intuition: positively (negatively) dependent background risks make asset prices lower (higher). We confirm that our intuition is correct at least for a class of preference justified from a descriptive viewpoint.

Since analyses of decision making under uncertainty with background risks are based on independence assumption, dealing with dependent background risks encounters some difficulties. To overcome these difficulties, we must restrict dependent background risks to have some specific forms. In this paper, dependent background risks have functional forms, that is, dependent background risks are represented as functions of marketable risks. An advantage of these forms is that situations with dependent background risks are changed to be ones with only one source of risks. The monotonicity of asset prices is examined when the difference between dependent background risks satisfy the motonocity and the single crossing conditions. The reasons for using the single crossing condition are the following: First, it is well known that the single crossing condition plays an important role in analyses of decision making under uncertainty, for example, the monotone comparative statics of decision making under uncertainty (Jewitt, 1987; Athey, 2002). Our paper can also seem to be an application of monotone comparative statics under uncertainty. Second, the analyses under other conditions satisfying the monotonicity condition are given by a direct application of that under the single crossing condition. Finally, the single crossing condition includes the case when dependent background risks have the same means. The same mean is a natural criterion for the comparison. By the covariance inequality (Wagener, 2003), the monotonicity condition can be ordered by its covariance with marketable risks. The purpose is achieved by using the method of comparative statics based on the property of the risk–neutral probability. Although this method is similar to work by Milgrom (1981), and Ohnishi and Osaki (2005), it is clear that our analysis is different from previous articles because these are not concerned with background risks.

The rest of this paper is organized as follows. In Section 2, we derive the equilibrium asset price in a static version of a Lucas (1978) economy with a dependent

[1)]See section 9.3 in Gollier (2001) and references therein for details.

background risk. In Section 3, we show the conditions on preferences to guarantee the monotonicity of asset prices, when the difference between dependent background risks satisfy the monotonicity and the single crossing conditions. Finally, we provide discussions on the results.

2 Asset Market

Let us consider a static version of a Lucas (1978) economy, that is, a two–date pure exchange economy with homogeneous investors. Every investor has an identical expected utility representation with a strictly increasing, strictly concave and sufficiently smooth von Neumann–Morgenstern utility function (utility function) u, which means that all of the required higher order derivatives are assumed to exist.[2)] One risk–free asset and one risky asset are traded in the asset market. The investor is endowed with w units of the risk–free asset, and one unit of the risky asset. Let us consider that the risk–free asset is the numeraire, and the gross risk–free rate is normalized to one without loss of any generality. The return on the risky asset at the final date is a random variable $\tilde{x}$ with a Cumulative Distribution Function (CDF) $F(x)$ defined over a support $[a, b]$. By limited liability, the risky asset return is typically defined over a positive reigion, $a \geq 0$. The CDF $F(x)$ is assumed to be differentiable, that is, the Probability Density Function (PDF) $F'(x) := f(x)$ exists, where primes denote derivatives. The investor also faces a non–marketable risk, which is dependent on the risky asset return, and is called the dependent background risk. The dependent relation between the marketable and dependent background risks is represented by a function $\epsilon(x)$, therefore the dependent background risk is given by the random variable $\tilde{\epsilon} := \epsilon(\tilde{x})$. The function $\epsilon(x)$ is assumed so that the final wealth in equilibrium $w+\epsilon(x)+x$ is an increasing function of x. This assumption is sufficient for $\epsilon(x)$ to be an increasing function of x. An economic interpretation is given as: By following the covariance inequality,[3)] this assumption means that the dependent background risk has positive covariance with the marketable risk. Since the economy satisfies the one–fund separation theorem, the risky asset can be considered as the market portfolio. The dependent background risk can be considered as the variation of gross domestic product (GDP). Therefore the assumption implies that the variation of GDP has positive covariance with the market portfolio. Considering an actual economy, we can consider that the assumption is a natural one.

The investor buys the portfolio $\boldsymbol{\alpha} = (\alpha_0, \alpha_1)$ to maximize her expected utility from final wealth, where α_0 and α_1 stands for investments in the risk–free asset and risky asset respectively. The asset prices are given by $\boldsymbol{q} = (1, q)$, where the risk–free asset price is equal to one because of normalization and the risky asset price is

[2)] We use the term "increasing" and "decreasing" in the weak sense.

[3)] The covariance inequality claims the following: if both g and h are increasing functions, then $\mathbb{E}[g(\tilde{x})h(\tilde{x})] \geq \mathbb{E}[g(\tilde{x})]\mathbb{E}[h(\tilde{x})]$ holds for all random variable $\tilde{x}$. See Wagener (2003) for the details and applications of the covariance inequality.

denoted as q. The investor's problem is given as follows:

$$\begin{aligned} \mathbf{P} \quad & \max_{\{\boldsymbol{\alpha}\}} \ \mathbb{E}[u(\alpha_0 + \epsilon(\tilde{x}) + \alpha_1 \tilde{x})] \\ & \text{s.t. } \alpha_0 + \alpha_1 q \leq w + q \end{aligned} \tag{1}$$

Let us define the Lagrangean $\mathcal{L}(\boldsymbol{\alpha}, \lambda) := \mathbb{E}[u(\alpha_0 + \epsilon(\tilde{x}) + \alpha_1 \tilde{x})] - \lambda(\alpha_0 + \alpha_1 q - w - q)$, where λ is the Lagrange multiplier. Since the objective function is concave and the constraint is linear, the first order conditions meet the necessary and sufficient conditions for the optimality. By the homogeneity of investors, a no–trade equilibrium occurs. Then the demands for assets in equilibrium are qual to the endowments: $\alpha_0 = w, \ \alpha_1 = 1$. The solutions of investor's problem in equilibrium are given as follows:

$$\frac{\partial \mathcal{L}}{\partial \alpha_0} = \mathbb{E}[u'(z(\tilde{x}))] - \lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha_1} = \mathbb{E}[\tilde{x} u'(z(\tilde{x}))] - \lambda q = 0, \tag{3}$$

where $z(x)$ is the final wealth in equilibrium defined by $z(x) := w + \epsilon(x) + x$, and is an increasing function of x. By Eqs. (2) and (3), the equilibrium asset price is given as follows:

$$q = \frac{\mathbb{E}[\tilde{x} u'(z(\tilde{x}))]}{\mathbb{E}[u'(z(\tilde{x}))]}. \tag{4}$$

Let us define the function

$$\hat{f}(x; \epsilon) := \frac{u'(z(x)) f(x)}{\mathbb{E}[u'(z(\tilde{x}))]}, \quad x \in [a, b], \tag{5}$$

where ϵ represents the parameter of $\hat{f}$. As $\hat{f}(x; \epsilon) \geq 0$ for all $x \in [a, b]$ and $\int_a^b \hat{f}(t; \epsilon) dt = 1$, $\hat{f}(x; \epsilon)$ can be regarded as a PDF defined over the support $[a, b]$. By taking the expectation with respect to the PDF $\hat{f}$, we can rewrite the equilibrium asset price as

$$q = \hat{\mathbb{E}}[\tilde{x}], \tag{6}$$

where $\hat{\mathbb{E}}$ denotes the expectation operator with respect to the PDF $\hat{f}$. Since equilibrium asset prices become equal to the expected values of their returns under the expectation with respect to the probability $\hat{F}(x; \epsilon) = \int_a^x \hat{f}(t; \epsilon) dt$, the probability $\hat{F}(\epsilon)$ is called the risk–neutral probability.

3 Main Results

In this section, we examine the effects of dependent background risks on asset prices. Let us consider two different economies with dependent background risk $\epsilon_1(x)$ and $\epsilon_2(x)$, say economy 1 and 2 respectively. First, we assume the monotonicity condition:

$$\epsilon_1(x) - \epsilon_2(x) \text{ is an increasing function of } x. \tag{7}$$

Applying the covariance inequality, the monotonicity condition (7) means that

$$\mathrm{cov}(\epsilon_1(\tilde{x}) - \epsilon_2(\tilde{x}), \tilde{x}) = \mathrm{cov}(\epsilon_1(\tilde{x}), \tilde{x}) - \mathrm{cov}(\epsilon_2(\tilde{x}), \tilde{x}) \geq 0. \tag{8}$$

A condition satisfying the monotonicty condition (7) is the following:

$$\begin{aligned} &\epsilon_1(x) - \epsilon_2(x) \leq 0 \ \forall x \in [a, \underline{x}_0); \\ &\epsilon_1(x) - \epsilon_2(x) = 0 \ \forall x \in [\underline{x}_0, \overline{x}_0]; \\ &\epsilon_1(x) - \epsilon_2(x) \geq 0 \ \forall x \in (\overline{x}_0, b]. \end{aligned} \tag{9}$$

The above condition is called the single crossing condition. We determine the condition on prferences to guarantee the monotonicity of asset prices under the monotonicity condition (7) and the single crossing condition (9). There are two reasons why we are interested in the case of the single crossing condition. The first is that analyses under two other conditions satisfying the monotonicty condition are just dirict applications for that under the single crossing condition. The other conditions are: $\epsilon_1(x) - \epsilon_2(x) \leq 0, \ \forall x \in [a, b]$, and $\epsilon_1(x) - \epsilon_2(x) \geq 0, \ \forall x \in [a, b]$. The second reason is that the single crossing condition includes the interesting case, $\mathbb{E}[\epsilon_1(\tilde{x})] = \mathbb{E}[\epsilon_2(\tilde{x})]$. The same mean can be considered to be a natural criterion for the comparison. Preparing for the analysis, we give some definitions and properties for the risk aversion and stochastic dominance.

Definition 3.1. Absolute risk aversion of u at wealth x is defined by $\mathcal{A}(x) := -u''(x)/u'(x)$. We call absolute risk aversion as simply risk aversion.

We call that utility functions u exhibit Increasing, Constant, and Decreasing Absolute Risk Aversion (IARA, CARA, and DARA) respectively, if risk aversion is increasing, constant, and decreasing functions of x. Note that IARA and DARA include CARA as a special case.

Definition 3.2.

- $F(2)$ dominates $F(1)$ in the sense of Monotone Likelihood Ratio Dominance (MLRD) if $f(y;2)/f(y;1) \geq f(x;2)/f(x;1)$ holds for all $x, y \in [a, b]$ such that $x \leq y$; We denote this as $F(2) \geq_{\mathrm{MLRD}} F(1)$.
- $F(2)$ dominates $F(1)$ in the sense of First–order Stochastic Dominance (FSD) if $F(x;1) \geq F(x;2)$ holds for all $x \in [a, b]$. We denote this as $F(2) \geq_{\mathrm{FSD}} F(1)$.

The following properties are well known in the theory of stochastic dominance. Hence we give the following theorem without proofs. The reader may refer to Gollier (2001) and Shaked and Shanthikumar (1994) for more details.

Theorem 3.1.

- If $F(2) \geq_{\mathrm{MLRD}} F(1)$, then $F(2) \geq_{\mathrm{FSD}} F(1)$;
- $F(2) \geq_{\mathrm{FSD}} F(1)$, if and only if $\mathbb{E}[g(\tilde{x}(2))] \geq \mathbb{E}[g(\tilde{x}(1))]$ for every increasing function g.

For the first step, we show the following lemma.

Lemma 3.1. If utility function u displays DARA for $x \le \underline{x}_0$ and IARA for $\overline{x}_0 \le x$, then $\hat{F}(\epsilon_1) \le_{\text{MLRD}} \hat{F}(\epsilon_2)$.

Proof. We have that

$$\frac{\hat{f}(x_t;\epsilon_2)}{\hat{f}(x_t;\epsilon_1)} \ge \frac{\hat{f}(x_s;\epsilon_2)}{\hat{f}(x_s;\epsilon_1)} \Leftrightarrow \frac{u'(z_2(x_t))}{u'(z_1(x_t))} \ge \frac{u'(z_2(x_s))}{u'(z_1(x_s))}. \tag{10}$$

Hence, we can replace the condition $\hat{F}(\epsilon_1) \le_{\text{MLRD}} \hat{F}(\epsilon_2)$ with the right–hand side of Eq. (10) for all $x_s, x_t \in [a.b]$ such that $x_s \le x_t$. To shorten the proof, we consider the case $\underline{x}_0 = \overline{x}_0 = x_0$.

(i) $x_s < x_t < x_0$:
It follows by the straightforward calculation that

$$\text{sgn}\left\{\left(\frac{u'(z_2(x))}{u'(z_1(x))}\right)'\right\} = \text{sgn}\{\mathcal{A}(z_1(x))z_1'(x) - \mathcal{A}(z_2(x))z_2'(x)\}. \tag{11}$$

Since $z_1(x) \le z_2(x)$ and u displays DARA for all $x \le x_0$, we have that

$$\mathcal{A}(z_1(x)) \ge \mathcal{A}(z_2(x)). \tag{12}$$

Since $\epsilon_1(x) - \epsilon_2(x)$ is an increasing function of x, we have that

$$\epsilon_1'(x) \ge \epsilon_2'(x) \;\Leftrightarrow\; z_1'(x) \ge z_2'(x). \tag{13}$$

Combining Eqs. (12) and (13), we obtain that $\mathcal{A}(z_1(x))z_1'(x) - \mathcal{A}(z_2(x))z_2'(x) \ge 0$, that is $u'(z_2(x))/u'(z_1(x))$ is an increasing function. Hence,

$$\frac{u'(z_2(x_t))}{u'(z_1(x_t))} \ge \frac{u'(z_2(x_s))}{u'(z_1(x_s))}, \text{ for all } x_s \le x_t \le x_0. \tag{14}$$

(ii) $x_s < x_0 < x_t$:
Since $u'(x)$ is a decreasing function of x, $z_1(x_s) \le z_2(x_s)$ means that

$$u'(z_1(x_s)) \ge u'(z_2(x_t)) \;\Leftrightarrow\; 1 \ge \frac{u'(z_2(x_t))}{u'(z_1(x_t))}, \tag{15}$$

and $z_1(x_t) \ge z_2(x_t)$ means that

$$u'(z_1(x_s)) \le u'(z_2(x_t)) \;\Leftrightarrow\; 1 \le \frac{u'(z_2(x_t))}{u'(z_1(x_t))}. \tag{16}$$

Combining Eqs. (15) with (16), we obtain that

$$\frac{u'(z_2(x_s))}{u'(z_1(x_s))} \le \frac{u'(z_2(x_t))}{u'(z_1(x_t))}, \text{ for all } x_s \le x_0 \le x_t. \tag{17}$$

(iii) $x_0 < x_s < x_t$:
We can obtain a similar proof in the manner as case (i) except for changing the sign, hence we omit the proof.

Through the above discussion, we obtain the right–hand side of Eq. (10) for all $x_s, x_t \in [a.b]$ such that $x_s \le x_t$. Hence, we complete the proof. □

Recalling that MLRD is a sufficient condition for FSD, we can obtain

$$q_2 = \hat{\mathbb{E}}_2[\tilde{x}] \geq \hat{\mathbb{E}}_1[\tilde{x}] = q_1, \tag{18}$$

if utility function u displays DARA for $x \leq \underline{x}_0$ and IARA for $\overline{x} \leq x$. We summarize the discussion with the following proposition:

Proposition 3.1. Let us consider economy 1 and 2 with dependent background risk ϵ_1 and ϵ_2, and denote asset price by q_1 and q_2. Suppose that $\epsilon_1(x) - \epsilon_2(x)$ satisfies the monotonicty condition (7) and the single crossing condition (9). If preferences display DARA for $x \leq \underline{x}_0$ and IARA for $\overline{x}_0 \leq x$, then $q_1 \leq q_2$.

Let us consider the case where there does not exist a single crossing point. By the above analysis, we have the following corollary.

Corollary 3.1. Let us consider economy 1 and 2 with dependent background risk ϵ_1 and ϵ_2, and denote asset price by q_1 and q_2. Suppose that $\epsilon_1(x) - \epsilon_2(x)$ satisfies the monotonicty condition (7) and the condition, $\epsilon_1(x) - \epsilon_2(x) \leq (\geq)\ 0,\ \forall x \in [a, b]$. If preferences display DARA (IARA), then $q_1 \leq q_2$.

4 Discussion

We provide two discussions: the first concerns the validity of the conditions on preferences, and the second is connected to the equity premium puzzle. The conditions on preferences in Prop. 3.1 are that risk aversion is a decreasing function for low wealth and an increasing function for high wealth. This condition has the validity from a descriptive viewpoint as suggested by Pratt (1964).[4)] Moreover, our results are consistent with the following intuition for asset prices: 'Risk averse investors have desirability for consumption–smoothing. They decrease (increase) demands for assets with higher (lower) positively dependent background risks. As a result, higher (lower) positively dependent background risks decrease (increase) asset prices.' On the other hand, DARA is the least demanding condition for most problems with independent background risks. Clearly, our conditions on preferences do not satisfy DARA in the global sense. Through the above two opposite discussions, the validity of the conditions on preferences remains an open question. However, this is a difficult question because we have no consensus for the conditions on preferences from empirical and/or experimental observations.

Second, an important implication of the result concernes the equity premium puzzle (Mehra and Prescott, 1985). The equity premium puzzle means that the equity premium in a Lucas economy (1978) is too small compared to the observed one. As a theoretical resolution, Weil (1992) modified the economy by adding an independent background risk to a standard Lucas economy (1978) and concluded that this tends to solve the puzzle under standard risk aversion which means that both risk aversion and prudence are decreasing functions. Here, (absolute) prudence

[4)] Pratt (1964, pp.123) *"And consideration of the yield and riskiness per investment dollar of investor's portfolios may suggest, at least in some context, description by utility functions for which $r^*(x)$ is first decreasing and then increasing."*

is defined by $\mathcal{P} := -u'''/u''$. Through empirical studies, however, Lucas (1994) and Telmer (1993) observed that this effect is too small to explain the puzzle. Our results imply that the equity premium in an economy with both positively dependent and independent background risks is higher than the premium in the economy considered by Weil (1992) with an additional restriction on preferences that is CARA for high wealth.[5)] Here, the positively dependent background risk means an increasing function $\epsilon(x)$ with single crossing points at the x–axis. In other words, the puzzle can be explained by the introduction of the positively dependent background risk. Recall that background risk and risky asset can be considered as the variation of GDP and the market portfolio respectively in an actual economy. Hence the positively dependent background risk is a natural requirement from an empirical viewpoint.

[5)] Recall that CARA is the negative exponential preference that means constant absolute prudence (CAP), and the DARA and decreasing absolute prudence include both CARA and CAP.

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