Discussion Papers In Economics And Business

Endogenous Political Instability

Ryo Arawatari and Kazuo Mino

Discussion Paper 07-32-Rev.

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Endogenous Political Instability

Ryo Arawatari and Kazuo Mino

Discussion Paper 07-32-Rev.

November 2007

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Endogenous Political Instability*

Ryo Arawatari† and Kazuo Mino‡

November 2007

Abstract

In this paper, we construct a simple dynamic two-party electoral competition model in which the degree of political instability is endogenously determined. We consider the campaign contributions as stock variable which is gradually accumulated by both party's direct investment and induced the Markov-perfect Nash equilibrium. We then examine the stability of the symmetric steady state and find that it may be either totally stable or unstable depending on the parameter values involved in the model. We also found that under certain conditions, at least near the symmetric steady state, there exists indeterminacy of equilibrium path: there exist both stable and unstable paths, that is, under given levels of political assets, both high instability political system and low instability political system can emerge depending on expectations of political parties.

Keywords: Political assets; Dynamic political economy; Differential game; Markov-perfect Nash equilibrium; Two-party model

JEL Classification: C73; D72; D78

---

*We are very grateful to Junichi Itaya and Akira Yakita for their invaluable comments. We also acknowledge the helpful comments by Takeo Hori, Shingo Ishiguro, Koichi Kawamoto and Keiichi Morimoto. Arawatari acknowledges financial support from the Japan Society for the Promotion of Science (JSPS) Research Fellowship for Young Scientists. Mino acknowledges financial support from Grant-in-Aid for Scientific Research (No. 18530136).

†Corresponding Author. Graduate School of Economics, Osaka University, 1-7 Machikaneyama, Toyonaka, Osaka, 560-0043, Japan. E-mail address: ege001ar@mail2.econ.osaka-u.ac.jp

‡Graduate School of Economics, Osaka University, 1-7 Machikaneyama, Toyonaka, Osaka, 560-0043, Japan. E-mail address: mino@econ.osaka-u.ac.jp

1 Introduction

It has been argued that there is a close connection between political instability and economic performance. Casual observation suggests that the countries with poor economic performances are often associated with unstable political systems. Several authors such as Devereux and Wen (1998), Persson and Tabellini (1998) and Darby et al. (2004) present the analytical frameworks that establish the negative relationship between political instability and economic growth. Such a theoretical prediction fits well to the empirical findings by Alesina and Perotti (1996), Barro and Sala-i-Martin (1995), Perotti (1996) and Alesina et al. (1996)[2].

In view of the theoretical and empirical studies mentioned above, the relationship between political instability and economic performance seem to be ascertained. However, the problem with these studies is that since they all treat the political instability as an exogenously given factor, they cannot explain what is the determinant factor of the degree of political instability. Actually, the degree of political instability differs vastly among countries. This paper examines a theoretical model in which the degree of political instability is endogenously determined. The main purpose of our study is to present a step towards a general equilibrium consideration on the relationship between political instability and economic performance.

We construct a model of electoral competition between two political parties. Following Austen-Smith (1987), Snyder (1990), Barron (1994) and Grossman and Helpman (1996), we assume that each party tries to increase its probability of winning by the political campaign. The key difference between the foregoing investigations and the present paper is that in our setting the relevant determinant of the probability of winning depends on the accumulated stock of past spending for campaign, while the existing studies assume that current level of campaign contributions alone determines the probability of winning of each party. Our discussion, therefore, explores the political instability in a dynamic framework, as oppose to the foregoing studies that treat static models.

[2]See also Fredriksson and Svensson (2003).

More specifically, we set up a dynamic-game model of electoral competition between two parties. The state variables of our dynamic game is the stocks of accumulated campaign spending ('political assets') and the strategy of each party is the level of campaign spending. The probability of winning is determined by the stocks of political assets as well as by the exogenous political shocks. Each party is assumed to maximize a discounted sum of expected profits given by the expected rent revenue minus costs for campaign. Assuming that the political shocks are uniformly distributed, we formulate the electoral competition in terms of a linear-quadratic differential game and characterize the Markov-perfect Nash equilibrium of the game.

In our model, the degree of political instability is represented by the probability of winning. We consider that the two-party system is highly unstable when the probability of winning of each party is close to 50%. In contrast, the political system is highly stable if the probability of winning of one party is close to 100%. We first show that there is a symmetric steady state where both parties accumulate the same level of political asset and the probability of winning of both parties are 50%. We then examine the stability of the symmetric steady sate and find that it may be either totally stable or unstable depending on the parameter values involved in the model. It is shown that whether the political instability prevails or not depends on the initial differences in the political assets of both parties. It is also revealed that under certain conditions the symmetric steady state has a saddle point property. If this is the case, there exist a stable as well as an unstable paths, that is, both stable and unstable political systems can emerge. In this case, realization of one particular path is determined by the expectations of political parties. Such a situation is particularly interesting, because it may explain why the similar types of economies sometimes exhibit substantially different degree of political instability.

The rest of the paper is organized as follows. Section 2 sets up the analytical framework. Section 3 characterizes the Markov-perfect Nash equilibrium of the electoral game. This section presents our main findings and their implications. Section 4 concludes the paper.

2 The Model

2.1 Citizens

The society consists of a continuum of citizens with a unit mass who have identical preferences over private consumption and public services. Each citizen receives the same amount of endowment in each period. The instantaneous utility of a citizen is specified as

$$W(c(t), g(t)) = hc(t) + H(g(t)), \quad h > 0,$$

where $c(t)$ and $g(t)$ are private consumption and the public spending per capita, respectively. Function $H(g)$ is monotonically increasing, strictly concave and satisfies the Inada conditions: $\lim_{g \to 0} H'(g) = +\infty$ and $\lim_{g \to +\infty} H'(g) = 0$. We assume that citizens do not hold any financial asset, so that the private consumption is determined by their disposable income:

$$c(t) = (1 - \tau(t))y + y_0,$$

where $\tau(t) \in [0, 1]$ denotes the rate of income tax, y (> 0) is a given, taxable endowment and y_0 (> 0) expresses a given, untaxable endowment. Thus even if $\tau(t) = 1$, each consumer can maintain a positive level of consumption, y_0.

2.2 Electoral Competition

There are two political parties, indexed by $j = A$ and B. If party j wins an election and holds the office, it can appropriate a part of tax revenue as a rent. Letting $r_j(t)$ be rent appropriation, the budget constraint for the government office is given by

$$\tau_j(t)y = g_j(t) + r_j(t), \quad j = A, B, \tag{1}$$

where $\tau_j(t)$ is the rate of income tax set by party j.

Using the government's budget constraint, we rewrite each voter's utility function under party j's policy as follows:

$$W(g_j(t), r_j(t)) = h(y - g_j(t) - r_j(t)) + hy_0 + H(g_j(t)). \tag{2}$$

At the time of the election, the voters base their voting decision not only on the relative welfare gain under alternative offices but also on the campaign contributions by the parties as well as on exogenous political shocks and individual-specific ideological bias. Specifically, voter i prefers the candidate representing party A, if the following holds:

$$W(g_A(t), r_A(t)) \geq W(g_B(t), r_B(t)) + \phi_i + \Theta(t).$$

The above condition assumes that voter i's preference are affected by two additional factors, ϕ_i and $\Theta(t)$.

First, $\phi_i \in \mathbb{R}$ expresses an individual-specific parameter that measures voter i's individual ideological bias toward party B. A positive value of ϕ_i implies that voter i has a bias in favor of party B's political stance, whereas the voter prefers party A's ideological position if $\phi_i < 0$. The voters with $\phi_i = 0$ are ideologically neutral, that is, they do not have individual ideological bias. We assume that ϕ_i is distributed according to a cumulative distribution function $F(\cdot)$, and the distribution is symmetric, i.e., $F(-\phi) = 1 - F(\phi)$ $\forall \phi \in \mathbb{R}$ and $F(0) = 1/2$.

Second, $\Theta(t)$ represents the average (relative) popularity of party B in period t and it is defined as:

$$\Theta(t) \equiv \sigma_B(t) - \sigma_A(t) - S. \tag{3}$$

Here, $\sigma_j \in (-\infty, +\infty)$ denotes the 'political asset' accumulated by party j through its campaign and S denotes an exogenous shock that disturbs the voters' decision. The behavior of σ_i is discussed in the next subsection. For analytical simplicity, we assume that S is a random variable uniformly distributed on $[-\bar{s}, \bar{s}]$ where $\bar{s}$ is a positive constant. Note that in our formulation, ϕ_i is exogenously specified by voter's preference, while $\Theta(t)$

is an endogenous variable determined by accumulation of political assets of both parties.

2.3 Strategic Behaviors of the Political Parties

The above specification basically follows the modelling by Austen-Smith (1987), Snyder (1990), Baron (1994), and Grossman and Helpman (1996) who emphasize the campaign contribution as an important determinant of implemented policies. The existing studies, however, assume that the relative popularity is determined by the instantaneous campaign contributions and thus σ_j is a flow variable. Additionally, the previous literature usually assumes that campaign activities are conducted by interest groups. In contrast to the foregoing formulations, we consider that σ_j is an intangible stock variable which is gradually accumulated by party j's direct investment. Letting $d_j(t)$ be party j's investment level, σ_j changes according to

$$\dot{\sigma}_j(t) = d_j(t) - \delta\sigma_j(t), \tag{4}$$

where δ denotes the depreciation rate of σ_j. We assume that the rate of depreciation is the same for both parties. In the above, $d_j(t)$ represents party j's effort (or labor) level to increase its political reputation. It is assumed that one unit of effort raises one unit of the political asset. Since voting habits and preferences over alternative parties usually change over time, our formulation may capture the voting criteria of citizens in a more plausible way than the existing studies. Now define

$$\sigma(t) \equiv \sigma_B(t) - \sigma_A(t).$$

Then we may summarize two equations in (4) as:

$$\dot{\sigma}(t) = d_B(t) - d_A(t) - \delta\sigma(t). \tag{5}$$

Given the assumptions displayed above, citizens vote for the party whose platform,

$\{\tau_j, g_j, r_j\}$, provides them with the highest utility. Recall that the distribution of ϕ_i is symmetric and $F(0) = 1/2$. Therefore, party A wins an election if the following holds:

$$F\left(W(g_A(t), r_A(t)) - W(g_B(t), r_B(t)) - \sigma(t) + S\right) > \frac{1}{2}$$
$$\Leftrightarrow \; W(g_A(t), r_A(t)) - W(g_B(t), r_B(t)) - \sigma(t) + S > 0.$$

As a result, under our distributional assumption on S, each party's probability of winning,

$$P_A(t) = \Pr\left(S > W(g_B(t), r_B(t)) - W(g_A(t), r_A(t)) + \sigma(t)\right),$$

is given by

$$P_A(t) = \begin{cases} 1 & \text{if } W(g_B, r_B) - W(g_A, r_A) + \sigma \leq -\bar{s}, \\ 1 - \frac{W(g_B, r_B) - W(g_A, r_A) + \sigma + \bar{s}}{2\bar{s}} & \text{if } W(g_B, r_B) - W(g_A, r_A) + \sigma \in (-\bar{s}, \bar{s}), \\ 0 & \text{if } W(g_B, r_B) - W(g_A, r_A) + \sigma \geq \bar{s}. \end{cases} \tag{6}$$

Namely, if the divergence in political assets, $\sigma(= \sigma_B - \sigma_A)$, is small (resp. large) enough to satisfy $W(g_B, r_B) - W(g_A, r_A) + \sigma \leq -\bar{s}$ (resp. $W(g_B, r_B) - W(g_A, r_A) + \sigma \geq \bar{s}$), then party A (resp. party B) always wins, regardless of the magnitude of political disturbance, $S \in [-\bar{s}, \bar{s}]$. As a result, $P_B(= 1 - P_A)$ is given by

$$P_B(t) = \begin{cases} 0 & \text{if } W(g_B, r_B) - W(g_A, r_A) + \sigma \leq -\bar{s}, \\ \frac{W(g_B, r_B) - W(g_A, r_A) + \sigma + \bar{s}}{2\bar{s}} & \text{if } W(g_B, r_B) - W(g_A, r_A) + \sigma \in (-\bar{s}, \bar{s}), \\ 1 & \text{if } W(g_B, r_B) - W(g_A, r_A) + \sigma \geq \bar{s}. \end{cases} \tag{7}$$

We may consider that political system is stable if P_A is close either to 1 or 0, while it is unstable if P_A is close to 1/2.

Party j's instantaneous benefit is defined as the expected utility of rent appropriation, $\theta P_j(t) r_j(t)$, minus the investment cost. We assume that the investment cost function is given by $(a/2)\, d_j^2(t)$ $(a > 0)$. Here, θ and a are positive scale parameters that respectively transform rent appropriation, r_i, and effort, d_j, to utilities. We set the objective function for party j as

$$U_j = \int_0^{\infty} e^{-\rho t} \left(\theta P_j(t) r_j(t) - \frac{a}{2} \cdot d_j(t)^2 \right) dt, \quad j = A, B,$$

where $\rho > 0$ is discount rate. Each party maximizes U_j subject to (1) and a given initial level of $\sigma(0) (= \sigma_B(0) - \sigma_A(0))$. The timing of events is as follows. First, both parties announce their policy platforms, $\{\tau_A, g_A, r_A\}$ and $\{\tau_B, g_B, r_B\}$, simultaneously. Then an election is held, in which voters choose between the two parties. The elected party implements the announced policy platform. We assume that the each party commits to their policy platform in each moment. Formally seaking, our model is a differential game between two parties in which each player's control variables are τ_j, g_j and d_j, and the state variable is $\sigma(t) (\equiv \sigma_B(t) - \sigma_A(t))$. In what follows, we characterize the Markov-perfect Nash (feedback Nash) equilibrium of this dynamic game.

3 Political Instability

3.1 Probability of Winning

In order to derive the Markov-perfect equilibrium, we first define the value function for party j's dynamic optimization problem in such a way that

$$V_j(\sigma(t)) \equiv \max \left\{ \int_t^{\infty} e^{-\rho(s-t)} \left(\theta P_j(s) r_j(s) - \frac{a}{2} \cdot d_j(s)^2 \right) ds \ : \ \text{subject to (4)} \right\}.$$

Then the Hamilton-Jacobi-Bellman (HJB) equation for party i's optimization problem is given by

$$\rho V_j(\sigma(t)) = \max_{r_j, d_j, g_j} \left\{ \theta P_j(t) r_j(t) - \frac{a}{2} \cdot d_j(t)^2 + V_j'(\sigma(t))[d_B(t) - d_A(t) - \delta\sigma(t)] \right\}, \quad j = A, B, \tag{8}$$

The above equation should holds for all $t \geq 0$. In the following, we abbreviate t for notational simplicity.

Before examining the HJB equations given above, we should determine the probability of winning P_j. More specifically, we must know the relationship between P_j and the state variable, σ. First, the resource constraint and feasibility conditions require that

$$(r_j, g_j) \in \mathfrak{g} \equiv \{(r_j, g_j) : \ r_j \geq 0, \ g_j \geq 0, \ r_j + g_j \leq y\}, \quad j = A, B. \tag{9}$$

As a result, function $W(g_j, r_j)$ is expressed as

$$W(g_j, r_j) \in [0, hy + H(\min(g^*, y)) - h \cdot \min(g^*, y)], \quad j = A, B,$$

where g^* satisfies the first-order condition for an interior solution that maximizes $W_j(g_j, r_j)$ under a given r_j, that is, $h = H'(g^*)$. To ensure the presence of an interior solution satisfying $g^* < y$, we make the following assumption:

Assumption 1: $H'^{-1}(h) < y$.

Additionally, keeping in mind that

$$W(g_B, r_B) - W(g_A, r_A) = h(g_A - g_B + r_A - r_B) + H(g_B) - H(g_A), \tag{10}$$

we see that the divergence between $W(g_A, r_A)$ and $W(g_B, r_B)$ fulfills

$$W(g_B, r_B) - W(g_A, r_A) \in [-\psi, \psi], \tag{11}$$

where $\psi \equiv hy + H(g^*) - hg^* > 0$. In what follows, we assume:

Assumption 2: $\bar{s} \geq \psi/2$.

This assumption means that the variance of the stochastic term is high enough. Given Assumptions 1 and 2, we obtain the following lemma that plays a key role in the subsequent discussion:

Lemma 1: *Under Assumptions 1 and 2, the following results are established:*

(i) *If* $\sigma \le -3\bar{s}$, then $P_A = 1$ so that Party A always wins. The optimal levels of r and g are: $r_A = y$, $g_A = 0$, $(r_B, g_B) \in \{(r_B, g_B) : r_B \ge 0,\ g_B \ge 0,\ r_B + g_B \le y\}$.

(ii) *If* $\sigma \in (-3\bar{s}, 3\bar{s})$, it holds that $P_A \in (0,1)$, $r_A(\sigma) = (3\bar{s} - \sigma)/3h$, $r_B(\sigma) = (3\bar{s} + \sigma)/3h$, $g_A = g_B = g^*$.

(iii) *If* $\sigma \ge 3\bar{s}$, then $P_A = 0$ so that party B always wins. The optimal levels of r and g are: $(r_A, g_A) \in \{(r_A, g_A) : r_A \ge 0,\ g_A \ge 0,\ r_A + g_A \le y\}$, $r_B = y$ and $g_B = 0$.

Proof. (i) Suppose that $\sigma \le -3\bar{s}$ holds. Using (10), (11) and Assumption 2, we obtain:

$$W(g_B, r_B) - W(g_A, r_A) + \sigma \le \psi + \sigma \le \psi - 3\bar{s} \le -\bar{s}.$$

By (6), $P_A = 1$ holds if $\sigma \le -3\bar{s}$. When $P_A = 1$, HJB equations in (8) are rewritten as:

$$\rho V_A(\sigma) = \max_{r_A, d_A, g_A} \left\{ \theta r_A - \frac{a}{2} \cdot d_A^2 + V_A'(\sigma)[d_B - d_A - \delta\sigma] \right\},$$
$$\rho V_B(\sigma) = \max_{r_B, d_B, g_B} \left\{ -\frac{a}{2} \cdot d_B^2 + V_B'(\sigma)[d_B - d_A - \delta\sigma] \right\}.$$

From the resource constraint in (9), it is obvious that each party finds it optimal to set $r_A = y$, $g_A = 0$ and $(r_B, g_B) \in \{(r_B, g_B) : r_B \ge 0,\ g_B \ge 0,\ r_B + g_B \le y\}$.

(ii) Suppose that $\sigma \in (-3\bar{s}, 3\bar{s})$ holds. Now let us guess that $P_A \in (0,1)$ in equilibrium. Then, using (6), (7) and (10), we write the HJB equations for each party in the following manner:

$$\rho V_A(\sigma) = \max_{r_A, d_A, g_A} \left\{ \theta \left[1 - \frac{h(g_A - g_B) + h(r_A - r_B) - H(g_A) + H(g_B) + \sigma + \bar{s}}{2\bar{s}} \right] \cdot r_A \right.$$
$$\left. - \frac{a}{2} \cdot d_A^2 + V_A'(\sigma)[d_B - d_A - \delta\sigma] \right\}, \tag{12}$$
$$\rho V_B(\sigma) = \max_{r_B, d_B, g_B} \left\{ \theta \left[\frac{h(g_A - g_B) + h(r_A - r_B) - H(g_A) + H(g_B) + \sigma + \bar{s}}{2\bar{s}} \right] \cdot r_B \right.$$
$$\left. - \frac{a}{2} \cdot d_B^2 + V_B'(\sigma)[d_B - d_A - \delta\sigma] \right\}. \tag{13}$$

By use of the first-order conditions for selecting g_j in the left-hand sides of the HJB equations, we obtain[3]:

$$g_A = g_B = g^* \equiv H'^{-1}(h). \tag{14}$$

Therefore, each party chooses same level of public spending in this case. Note that under the Assumption 1, it should hold that $g^* < y$. Substituting (14) into the (12) and (13), we may rewrite the HJB equations as follows:

$$\rho V_A(\sigma) = \max_{r_A, d_A} \left\{ R_A - \frac{a}{2} \cdot d_A^2 + V_A'(\sigma)[d_B - d_A - \delta\sigma] \right\}, \tag{15}$$

$$\rho V_B(\sigma) = \max_{r_B, d_B} \left\{ R_B - \frac{a}{2} \cdot d_B^2 + V_B'(\sigma)[d_B - d_A - \delta\sigma] \right\}, \tag{16}$$

where

$$R_A \equiv \theta \left[1 - \frac{h(r_A - r_B) + \sigma + \bar{s}}{2\bar{s}} \right] \cdot r_A, \tag{17}$$

$$R_B \equiv \theta \left[\frac{h(r_A - r_B) + \sigma + \bar{s}}{2\bar{s}} \right] \cdot r_B. \tag{18}$$

Here, R_j $(j = A, B)$ denotes the expected rent obtained by party j. Solving the first-order conditions about r_A and r_B, we obtain the following policy functions:

$$r_A(\sigma) = \frac{1}{3h} \cdot (3\bar{s} - \sigma), \qquad r_B(\sigma) = \frac{1}{3h} \cdot (3\bar{s} + \sigma). \tag{19}$$

Now, we verify that $P_A \in (0, 1)$ holds under above optimal policy strategies. Using (10) and the condition $\sigma \in (-3\bar{s}, 3\bar{s})$, we obtain:

$$W(g_B, r_B) - W(g_A, r_A) + \sigma = \frac{1}{3} \cdot \sigma \in (-\bar{s}, \bar{s}).$$

Therefore, from equation (6), $P_A \in (0, 1)$ holds.

(iii) The proof of Lemma 1(iii) is similar to that of Lemma 1(i) and is hence omitted. □

[3]Note that $r_A \neq 0$ and $r_B \neq 0$ in equilibrium. If $r_j = 0$ the value of felicity function must be zero. Therefore, rational parties do not choose $r_j = 0$.

This lemma demonstrates that if the state variable, σ, is out of the interval, $(-3\bar{s}, 3\bar{s})$, then one party always wins. In words, the political system is completely stable, when the gap between the two party's political assets exceeds certain limits. Conversely speaking, when the divergence between σ_A and σ_B is not too large, each party can win the election and thus political instability remains.

3.2 Optimal Investment

In this subsection, we determine the optimal investment of each party, d_j. We first consider the region, $\sigma \in (-3\bar{s}, 3\bar{s})$, in which each party has a positive probability of winning. By use of (17), (18) and (ii) in Lemma 1, the expected rent function of party j defined on $\sigma \in (-3\bar{s}, 3\bar{s})$ is respectively represented by

$$R_A(\sigma) = \frac{\theta}{3h}\left[1 - \frac{\sigma/3 + \bar{s}}{2\bar{s}}\right](3\bar{s} - \sigma),$$
$$R_B(\sigma) = \frac{\theta}{3h}\left[\frac{\sigma/3 + \bar{s}}{2\bar{s}}\right](3\bar{s} + \sigma).$$

As (19) shows, in our differential game each playe is not symmetric. Therefore, even though both parties have the identical preferences and the same cost functions, the policy function of each party is not identical.

In view of the first-order condition for selecting the optimal d_A and d_B, we find

$$ad_A = -V_A'(\sigma), \quad ad_B = V_B'(\sigma).$$

Equations (15) and (16) mean that the model in this case is a linear-quadratic game with an infinite time horizon. Although it has been known that a linear-quadratic differential game with an infinite horizon may sustain nonlinear strategies, the following discussion focuses on the case where each political party selects a linear strategy.[4] Therefore, we set

[4]For the presence of nonlinear strategies in the infinite horizon, linear quadratic differential games, see Tsutsui and Mino (1990). As for the recent development in this issue, see, for example, Rowat (2007).

the optimal policy functions of each party in the following manner:

$$d_j = \alpha_j \sigma + \beta_j, \quad j = A, B.$$

Restricting our attention to the linear strategies reduces the generality of our analytical consideration. We will, however, show that even if we only examine the linear policy functions, we may have a rich set of equilibrium outcomes of the game, and hence our central message can be conveyed in the simple case of linear strategies.

The above discussion yields:

$$V_A'(\sigma) = -a\alpha_A \sigma - a\beta_A, \qquad V_B'(\sigma) = a\alpha_B \sigma + a\beta_B.$$

Using the above relations and the HJB equations, we can expresses α_j and β_j by the given parameter values. The following lemma presents our findings:

Lemma 2: *Suppose that the policy functions of d_A and d_B are linear in σ, i.e., $d_j = \alpha_j \sigma + \beta_j$, $j = A, B$. Then, there exists two sets of equilibrium policy functions whose coefficients are respectively given by*

$$\alpha_A = \alpha_A^1 \equiv \frac{-(\rho + 2\delta) + \sqrt{(\rho + 2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6}, \quad \alpha_B = \alpha_B^1 \equiv \frac{(\rho + 2\delta) - \sqrt{(\rho + 2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6},$$
$$\beta_A = \beta_B = \beta^1 \equiv \frac{\theta}{3ah(\rho + \delta - \alpha_B^1)},$$

and

$$\alpha_A = \alpha_A^2 \equiv \frac{-(\rho + 2\delta) - \sqrt{(\rho + 2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6}, \quad \alpha_B = \alpha_B^2 \equiv \frac{(\rho + 2\delta) + \sqrt{(\rho + 2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6},$$
$$\beta_A = \beta_B = \beta^2 \equiv \frac{\theta}{3ah(\rho + \delta - \alpha_B^2)}.$$

Proof. See Appendix. □

At this moment, it is indeterminate which set of parameter values is selected. We discuss this issue in the next subsection. In the following, we impose two restrictions on the parameter values. First, α_A and α_B given above should be real numbers, and hence we assume:

Assumption 3: $\bar{s} > \frac{4\theta}{3ah(\rho+2\delta)^2}$.

This assumption implies that the variance of the stochastic term should be high enough to ensures that the α_A and α_B must be real numbers.

Second, when we obtain a set of interior solutions of $d_j(t)$, $\sigma(t)$ should be in some interval. More specifically, we find that $d_j(t) = \alpha_j^i \sigma(t) + \beta_j^i \geq 0$ for any set of coefficients, $\left(\alpha_j^i, \beta_j^i\right)$ $(j = A, B$ and $i = 1, 2)$, if $\sigma(t)$ satisfies

$$\sigma(t) \in [-\sigma_0, \sigma_0],$$

where

$$\sigma_0 = \frac{12}{ah\left[(\rho+2\delta)(5\rho+4\delta) + 2(2\rho+\delta)\sqrt{D} - D\right]},$$
$$D \equiv (\rho+2\delta)^2 - \frac{4\theta}{3ah\bar{s}}.$$

Therefore, if $-3\bar{s} < -\sigma_0 < \sigma_0 < 3\bar{s}$, at least one of the parties obtains a corner solution $(d_j(t) = 0)$, even though both party have a positive probability of winning. To avoid analytical complexity, we make the following assumption:

Assumption 4: $\bar{s} \leq \frac{(3-\theta)^2}{9ah(\delta+2\rho)[\rho(1-\theta)+\delta(2-\theta)]}$.

This assumption ensures that $\sigma_0 > 3\bar{s}$. Given this restriction, each parties' investment level is strictly positive for all $\sigma(t) \in (-3\bar{s}, 3\bar{s})$ in which $P_j > 0$ $(j = A, B)$. We now consider the cases of $\sigma \leq -3\bar{s}$ and $\sigma \geq 3\bar{s}$. The optimal investment in those intervals is summarized by the following lemma:

Lemma 3: *Suppose that each party takes the equilibrium policy, $\{r_j, g_j\}$, $j = A, B$, characterized in Lemma 1.*

(i) *If* $\sigma \leq -3\bar{s}$, *then the optimal levels of investment are* $d_A = d_B = 0$.

(ii) *If* $\sigma \geq 3\bar{s}$, *them the optimal levels of investment are* $d_A = d_B = 0$.

Proof. See Appendix. □

Figure 1 displays the both parties' optimal investment levels. When $\sigma \in (-3\bar{s}, 3\bar{s})$, there exist two sets of equilibrium policy functions. If $\sigma \notin (-3\bar{s}, 3\bar{s})$, both parties cease to invest. This reduces the divergence between the political asset of each party and weakens the situation in which one party dominates the other in the electoral competition.

[Figure 1 is around here.]

In the strategic equilibrium with $\alpha_j = \alpha_j^1$ $(j = A, B)$, both parties' investment choices are less sensitive to a change in σ than the case where they select the strategies with $\alpha_j = \alpha_j^2$ $(j = A, B)$. Suppose that $\sigma \in (-3\bar{s}, 3\bar{s})$ and that σ increases. Then, in response to the rise in σ, party A lowers its investment spending, while party B raises it (see Fig.1). If party B selects the insensitive strategy, $\alpha_B = \alpha_B^1$, the increase in party B's investment is small so that party A's incentive to reduce its investment level is relatively weak. As a result, party A also chooses the less sensitive strategy, $\alpha_A = \alpha_A^1$. Conversely, suppose that σ increases and party A takes insensitive strategy, $\alpha_A = \alpha_A^1$. In this case a rise in σ yields a relatively small decline in party B's investment. Hence, responding to the small change in the opposite party's investment, party B selects the insensitive strategy, $\alpha_B = \alpha_B^1$, as well. Consequently, $\alpha_A = \alpha_A^1$ and $\alpha_B = \alpha_B^1$ constitute one of the Nash equilibrium. In the rest of this paper, we refer to the Nash equilibrium with $\alpha_j = \alpha_j^1$ $(j = A, B)$ as the *equilibrium with insensitive strategies.*

Alternatively, in the strategic equilibrium with $\alpha_j = \alpha_j^2$ $(j = A, B)$, the optimal investment of both parties are more sensitive than in the case of $\alpha_j = \alpha_j^1$ $(j = A, B)$. Intuitive implication of this Nash equilibrium is similar to the case of insensitive strategies. We will refer to the Nash equilibrium with $\alpha_j = \alpha_j^2$ $(j = A, B)$ as the *equilibrium with sensitive strategies.*

3.3 Patterns of Dynamics

We can now examine equilibrium dynamics generated by our political game. Lemma 2 and 3 mean that in the Markov-perfect Nash equilibrium the dynamic equation (5) is expressed as

$$\dot{\sigma}(t) = \begin{cases} -\delta\sigma(t) & \text{if } \sigma(t) \leq -3\bar{s}, \\ (\alpha_B - \alpha_A - \delta)\sigma(t) & \text{if } -3\bar{s} < \sigma(t) < 3\bar{s}, \\ -\delta\sigma(t) & \text{if } \sigma(t) \geq 3\bar{s}, \end{cases},$$

where $\alpha_A = \alpha_A^1,\ \alpha_B = \alpha_B^1$ or $\alpha_A = \alpha_A^2,\ \alpha_B = \alpha_B^2$ given in Lemma 2.

It is obvious that when the initial value of σ is either less than $-3\bar{s}$ or larger than $3\bar{s}$, the dynamic behavior of σ is stable. For example, if the initial level of σ is less than $-3\bar{s}$ so that $\sigma_A(0) > \sigma_B(0) + 3\bar{s}$, then $\sigma(t)$ continues increasing. Since in this situation both parties cease to invest, the divergence between σ_A and σ_B continues reducing at a rate of depreciation, δ. In this sense, if the initial value of σ is out of $(-3\bar{s}, 3\bar{s})$, the political system is stable.

As for the case of $\sigma \in (-3\bar{s}, 3\bar{s})$, the dynamic system has a symmetric steady state where $\sigma = 0$. Lemma 2 shows that there are two dynamic equations: $\dot{\sigma}(t) = (\alpha_B - \alpha_A - \delta)\sigma(t)$ is expressed as either

$$\dot{\sigma}(t) = \frac{\sigma(t)}{3}\left[(\rho - \delta) - \sqrt{(\rho + 2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}\right] \tag{20}$$

or

$$\dot{\sigma}(t) = \frac{\sigma(t)}{3}\left[(\rho - \delta) + \sqrt{(\rho + 2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}\right]. \tag{21}$$

Now suppose that $\rho < \delta$. Then, we can obtain $[(\rho - \delta) - \sqrt{(\rho + 2\delta)^2 - 4\theta/3ah\bar{s}}]/3 < 0$ and

$$\frac{1}{3}\left[(\rho - \delta) + \sqrt{(\rho + 2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}\right] \begin{cases} < 0 & \text{if } \bar{s} < 4\theta/9ah\delta(\delta + 2\rho), \\ > 0 & \text{if } \bar{s} > 4\theta/9ah\delta(\delta + 2\rho). \end{cases}.$$

Conversely, if $\rho > \delta$, it holds that $[(\rho-\delta)+\sqrt{(\rho+2\delta)^2-4\theta/3ah\bar{s}}]/3>0$ and

$$\frac{1}{3}\left[(\rho-\delta)-\sqrt{(\rho+2\delta)^2-\frac{4\theta}{3ah\bar{s}}}\right]\begin{cases}>0 & \text{if } \bar{s}<4\theta/9ah\delta(\delta+2\rho),\\ <0 & \text{if } \bar{s}>4\theta/9ah\delta(\delta+2\rho).\end{cases}.$$

Therefore, dynamic equations of (21) and (20) yield the following proposition, which is the main finding of this paper:

Proposition 1: *Suppose that the state variable, σ, is in the range of $\sigma \in (-3\bar{s}, 3\bar{s})$ and Assumption 2, 3 and 4 hold. Then we obtain three patterns of dynamics listed below:*

(i) *If* $\bar{s} < 4\theta/9ah\delta(\delta+2\rho)$ and $\rho < \delta$, *then regardless of the equilibrium strategies taken by the parties,* $\sigma(t)$ *converges to zero.*

(ii) *If* $\bar{s} < 4\theta/9ah\delta(\delta+2\rho)$ and $\rho > \delta$, *then regardless of the equilibrium strategies taken by the parties,* $\sigma(t)$ *diverges from zero.*

(iii) *If* $\bar{s} > 4\theta/9ah\delta(\delta+2\rho)$, *there are two possible trajectories of* $\sigma(t)$, *one of which converges to the symmetric steady state, while the other diverges from it.*

Before discussing Proposition 1, let us make the following assumption.

Assumption 5: $\theta < 1$.

This assumption ensures that we can have all the cases in Proposition 1 under the restrictions on parameter values given by Assumptions 3 and 4.[5]

Now, we discuss each case of Proposition 1. First, consider case (i). In this case, even though we have restricted our attention to the linear policy functions, there are dual equilibrium strategies under which σ converges to zero: see Figure 2(a). Which equilibrium path is actually taken depends on the expectations coordination among both parties

[5]More precisely, $\theta < 1$ ensures that the graph of $\bar{s} = (3-\theta)^2/9ah(\delta+2\rho)\,[\rho(1-\theta)+\delta(2-\theta)]$ is above the graph of $\bar{s} = 4\theta/3ah(\rho+2\delta)^2$ at least for $\rho = 0$ and $\rho = \delta$.

(and the public). Since we have not introduced a specific mechanism of expectations coordination, the actual trajectory may fluctuate between the two paths in the presence of extrinsic uncertainty that disturbs expectations of agents. However, regardless of the selection of equilibrium path, the absolute value of σ continues converging to zero. Notice that in the symmetric steady state where $\sigma = 0$, it holds that $P_A = P_B = 1/2$, i.e., we have the highest degree of political instability. Thus case (i) always produces the highest political instability in the steady-state equilibrium.

[Figure 2 is around here.]

In Proposition 1 (ii), there is no converging path towards the highest political instability $(\sigma = 0)$. Regardless of the selection of strategies, the unstable behavior of σ leads the dominance of one party unless the initial value of σ happens to equal zero. Figure 2 (b) depicts case (ii). Proposition 1 (iii) means that dynamic equation (21) is unstable, while (20) is stable. Namely, the symmetric steady state is stable for $\alpha_A = \alpha_A^1,\ \alpha_B = \alpha_B^1$, and it is unstable for $\alpha_A = \alpha_A^2,\ \alpha_B = \alpha_B^2$ (see Fig.2(c)). If both parties select the second set of strategies, then σ diverges from zero. On this equilibrium path, the destiny of the political system is that party A dominates if the initial value of σ is negative $(\sigma_B(0) < \sigma_A(0))$. The result is opposite if the initial value of σ is positive $(\sigma_B(0) > \sigma_A(0))$. In those situations, the political system maintains complete stability. In contrast, if the first set of strategies is selected, then the political instability continues rising. Again, if expectations change, the temporary equilibrium may fluctuate between the stable and unstable trajectories.

If the initial value of $\sigma(t)$ is out of $(-3\bar{s}, 3\bar{s})$, the dynamic behavior of the model is extremely simple. Since in this region $d_j(t) = 0$, $\sigma(t)$ changes according to $\dot{\sigma}(t) = -\delta\sigma(t)$. Hence, if $\sigma(0) > 3\bar{s}$ (resp. $\sigma(0) < -3\bar{s}$), then $\sigma(t)$ reaches $3\bar{s}$ (resp. $-3s$) in a finite time.

3.4 Discussion

Figure 3 depicts the set of parameters $(\bar{s}, \rho)$ classified according to the each case of Proposition 1. This figure clearly shows that if $\rho < \delta$, then case (i) holds under relatively small levels of $\bar{s}$ and ρ. Namely, the symmetric steady state ($\sigma = 0$) is stable under both sensitive and insensitive strategies. Conversely, as stated in case (ii) of Proposition 1, if $\rho > \delta$, then the system is totally unstable under a smaller level of $\bar{s}$ and ρ. If $\bar{s}$ and ρ are large enough to satisfy $\bar{s} > 4\theta/9ah\delta(\delta+2\rho)$, the sensitive strategies $\left(\alpha_j = \alpha_j^2\right)$ yield an unstable path diverging from the symmetric steady state, while the path with insensitive strategies $\left(\alpha_j = \alpha_j^1\right)$ is stable. This last case corresponds to (iii) in Proposition 1.

[Figure 3 is around here.]

In order to obtain intuitive implications of the graphical analysis shown above, let us inspect the investment behavior of each party out of the symmetric steady state. First, consider the case of $\sigma(t) \in (-3\bar{s}, 3\bar{s})$ in which both party conduct positive investment. In this situation, the stability of $\sigma(t)$ simply depends on the sign of

$$\frac{d\dot{\sigma}(t)}{d\sigma(t)} = \alpha_B^i - \alpha_A^i - \delta \quad (i = 1, 2). \tag{22}$$

By use of Lemma 2, we can show the following:

$$\frac{\partial(\alpha_B^1 - \alpha_A^1 - \delta)}{\partial\rho} < 0, \quad \frac{\partial(\alpha_B^2 - \alpha_A^2 - \delta)}{\partial\rho} > 0, \tag{23}$$

$$\frac{\partial(\alpha_B^1 - \alpha_A^1 - \delta)}{\partial\bar{s}} < 0, \quad \frac{\partial(\alpha_B^2 - \alpha_A^2 - \delta)}{\partial\bar{s}} > 0, \tag{24}$$

where $\alpha_A^i < 0$ and $\alpha_B^i > 0$ $(i = 1, 2)$. Inequalities in (23) mean that a rise in discount rate, ρ, makes the insensitive strategies (i.e., $\alpha_j = \alpha_j^1 : \ j = A, B$) less sensitive, while it makes the sensitive strategies ($\alpha_j = \alpha_j^2 : \ j = A, B$) more sensitive. An increase in the time discount rate, ρ, means that both parties become more myopic, implying that they have weaker incentive to invest. As a consequence, investment behaviors of both parties become less sensitive to change in σ. On the other hand, the myopic parties have deeper concern with the probability of winning and the level utility in the near future, which makes them more sensitive to change in σ. As shown by (23), the first effect dominates the second in the equilibrium with insensitive strategies ($\alpha_j = \alpha_j^1 \ j = A, B$) . In contrast, the second effect dominates in the equilibrium with sensitive strategies ($\alpha_j = \alpha_j^2 \ j = A, B$).

In a similar vein, inequalities in (24) reveal that a higher variance of political shock, $\bar{s}$, makes the insensitive strategies (i.e., $\alpha_j = \alpha_j^1 : \ j = A, B$) less sensitive, while it makes the sensitive strategies ($\alpha_j = \alpha_j^2 : \ j = A, B$) more sensitive. Since a rise in $\bar{s}$ implies that both parties are more likely to be hit by bigger political shocks, it reduces the expected gain of investment. As a consequence, the investment behaviors of both parties become less sensitive to a change in σ. At the same time, a higher $\bar{s}$ increases each party's incentive to accumulate a larger stock of political asset in order to keep an advantageous position over the opposite party. This effect makes investment of both parties more sensitive to a change in σ. Inequalities in (23) show that in the equilibrium with insensitive strategies ($\alpha_j = \alpha_j^1 \ j = A, B$), the first effect dominates the second. The opposite result holds in the equilibrium with sensitive strategies ($\alpha_j = \alpha_j^2 \ j = A, B$).

Finally, combining Lemma 1 with Proposition 1, we can examine the global dynamics of our model. If we assume that $\bar{s} < 4\theta/9ah\delta(\delta + 2\rho)$ and $\rho < \delta$, the interior solutions are stable. Therefore, $\sigma(t)$ ultimately converges to the symmetric steady state where $\sigma = 0$. In this case, the dynamic system satisfies global stability, which implies that the destiny of political system is totally unstable because $P_j = 1/2$ in the steady state. In contrast, if $\bar{s} < 4\theta/9ah\delta(\delta + 2\rho)$ and $\rho > \delta$, then the interior solutions are unstable. Hence, $\sigma(t)$ converges to $3\bar{s}$ (resp. $-3\bar{s}$), if the initial value of σ is higher (resp. lower) than zero. In this situation, the destiny of the political system depends on the history (the initial value of the state variable) alone. In the steady state, political system becomes stable in the sense that one party always holds the office. If $\bar{s} > 4\theta/9ah\delta(\delta + 2\rho)$, there are one stable and one unstable path for $\sigma(t) \in (-3\bar{s}, 3\bar{s})$. Thus the ultimate state to which the political system converges depends on expectations of agents: if both parties select α_j^1, the symmetric steady state is globally stable. In contrast, if the parties choose a_j^2, then the realized steady-state value of σ will be either $3\bar{s}$ or $-3\bar{s}$, depending on the initial level of $\sigma(t)$. This case is particularly interesting, because it may explain why the similar types of economies sometimes exhibit substantially different degree of political instability.

4 Conclusion

The existing literature on the relationship between political instability and economic performances have assumed that the degree of political instability is exogenously given. In this paper, we have constructed a simple, dynamic two-party electoral competition model in which the degree of political instability is endogenously determined. The key assumption of our model is that the campaign contributions can be accumulated as a political asset which is gradually increased by the party's investment. We formulate the electoral competition as a differential game and characterized the Markov-perfect Nash equilibrium. By examining the stability of the symmetric steady-state, we have found that it may be either totally stable or unstable depending on the parameter values involved in the model. We have also revealed that under certain conditions the symmetric steady state exhibits a saddle point property: there exist both stable and unstable paths. In this case, the path the political system actually follows depends on expectations formation of agents.

To focus on endogenous determination of political instability, we have assumed away the production side of the economy. When discussing the effects of political instability on economic growth and development, we should integrate our model with an appropriate framework of growth economics. Our next task is to conduct such an extension of our study.

References

lesina, A., Perotti, R.: Political instability, income distribution and investment. European Economic Review. **40**, 1203–28 (1996)

Alesina, A., Özler, S., Roubini, N., Swagel, P.: Political Instability and Economic Growth. Journal of Economic Growth. **1**, 189–211 (1996)

Austen-Smith, D.: Interest Groups, Campaign Contributions and Probabilistic Voting. Public Choice. **54**, 123–39 (1996)

Baron, D.: Electoral Competition with Informed and Uninformed Voters. American Political Science Review. **88**, 33–47 (1994)

Barro, R., Sala-i-Martin, X.: Economic Growth. McGraw Hill, New York (1995)

Darby, J., Li, C., Muscatelli, V.: Political uncertainty, public expenditure and growth. European Journal of Political Economy. **20**, 153–79 (2004)

Devereux, M., Wen, J.: Political instability, capital taxation, and growth. European Economics Review. **42**, 1635–51 (1998)

Fredriksson, P., Svensson, J.: Political instability, corruption and policy formation: the case of environmental policy. Journal of Public Economics. **87**, 1383–1405 (2003)

Grossman, G., Helpman, E.: Electoral Competition and Special Interest Politics. Review of Economic Studies. **63**, 265–86 (1996)

Perotti, R.: Growth, Income Distribution, and Democracy: What the Data Say. Journal of Economic Growth. **1**, 149–87 (1996)

Persson, T., Tabellini, G.: Political economics and macroeconomic policy. In: Taylor, J., Woodford, M. Handbook of of Macroeconomics. North-Holland, Amsterdam, 1397–1482 (1998)

Rowat, C. Non-linear strategies in a linear quadratic differential game, Journal of Economic Dynamics and Control, **31**, 31, 79-3202 (2007).

Snyder, J.: Campaign Contributions as Investment: The US House of Representatives 1980-86. Journal of Political Economy. **98**, 1125–1227 (1990).

Tsutsui, S. and Mino, K., Nonlinear strategies in dynamic duopolistic competition with sticky prices, Journal of Economic Theory 52, 136-161 (1990)

Proof of Lemma 2.

Substituting (19) into (17) and (18), we find that

$$R_A(\sigma) = \theta \left[\frac{1}{18h\bar{s}}\sigma^2 - \frac{1}{3h}\sigma + \frac{\bar{s}}{2h} \right], \tag{A.1}$$

$$R_B(\sigma) = \theta \left[\frac{1}{18h\bar{s}}\sigma^2 + \frac{1}{3h}\sigma + \frac{\bar{s}}{2h} \right]. \tag{A.2}$$

Using (A.1), (A.2), (12) and (13), we obtain following envelope conditions:

$$\rho V_A'(\sigma) = \theta \left[\frac{1}{9h\bar{s}}\sigma - \frac{1}{3h} \right] + V_A''(\sigma)\left[d_B(\sigma) - d_A(\sigma) - \delta\sigma\right] + V_A'(\sigma)\left[d_B'(\sigma) - \delta\right], \tag{A.3}$$

$$\rho V_B'(\sigma) = \theta \left[\frac{1}{9h\bar{s}}\sigma + \frac{1}{3h} \right] + V_B''(\sigma)\left[d_B(\sigma) - d_A(\sigma) - \delta\sigma\right] + V_B'(\sigma)\left[-d_A'(\sigma) - \delta\right]. \tag{A.4}$$

Suppose that the policy functions are linear in $\sigma : d_j = \alpha_j\sigma + \beta_j,\ j = A, B$. Then it holds that

$$V_A'(\sigma) = -a\alpha_A\sigma - a\beta_A, \quad V_B'(\sigma) = a\alpha_B\sigma + a\beta_B,$$
$$V_A''(\sigma) = -a\alpha_A, \quad V_B''(\sigma) = a\alpha_B.$$

Substituting these expressions into (A.3) and (A.4) gives

$$\left[-\rho a\alpha_A - \frac{\theta}{9h\bar{s}} + a\alpha_A(\alpha_B - \alpha_A - \delta) + a\alpha_A(\alpha_B - \delta)\right]\sigma + \left[\frac{\theta}{3h} - \rho a\beta_A + a\alpha_A(\beta_B - \beta_A) + a\beta_A(\alpha_B - \delta)\right] = 0, \tag{A.5}$$

$$\left[\rho a\alpha_B - \frac{\theta}{9h\bar{s}} - a\alpha_B(\alpha_B - \alpha_A - \delta) - a\alpha_B(-\alpha_A - \delta)\right]\sigma + \left[\rho a\beta_B - \frac{\theta}{3h} - a\alpha_B(\beta_B - \beta_A) - a\beta_B(-\alpha_A - \delta)\right] = 0. \tag{A.6}$$

If both (A.5) and (A.6) hold for any feasible σ, we should have

$$-\rho a\alpha_A - \frac{\theta}{9h\bar{s}} + a\alpha_A(\alpha_B - \alpha_A - \delta) + a\alpha_A(\alpha_B - \delta) = 0, \tag{A.7}$$

$$\frac{\theta}{3h} - \rho a\beta_A + a\alpha_A(\beta_B - \beta_A) + a\beta_A(\alpha_B - \delta) = 0, \tag{A.8}$$

$$\rho a\alpha_B - \frac{\theta}{9h\bar{s}} - a\alpha_B(\alpha_B - \alpha_A - \delta) - a\alpha_B(-\alpha_A - \delta) = 0, \tag{A.9}$$

$$\rho a\beta_B - \frac{\theta}{3h} - a\alpha_B(\beta_B - \beta_A) - a\beta_B(-\alpha_A - \delta) = 0. \tag{A.10}$$

Combining (A.7) and (A.9), we obtain:

$$(\rho + 2\delta)(-\alpha_A) - (-\alpha_A)^2 = (\rho + 2\delta)\alpha_B - \alpha_B^2.$$

The above means that

$$\alpha_A = -\alpha_B, \tag{A.11}$$

In view of (A.11), (A.7) and (A.9) present two sets of solutions:

$$\alpha_A = \frac{-(\rho+2\delta) + \sqrt{(\rho+2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6}, \ \alpha_B = \frac{(\rho+2\delta) - \sqrt{(\rho+2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6},$$

and

$$\alpha_A = \frac{-(\rho+2\delta) - \sqrt{(\rho+2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6}, \ \alpha_B = \frac{(\rho+2\delta) + \sqrt{(\rho+2\delta)^2 - \frac{4\theta}{3ah\bar{s}}}}{6}.$$

Next, consider β_A and β_B. Combining (A.8) and (A.10) and using (A.11), we obtain:

$$\frac{\rho + \sqrt{D}}{2} \cdot (\beta_B - \beta_A) = 0,$$

which implies $\beta_A = \beta_B$. As a result, using (A.8) and (A.10), β_A and β_B are given by:

$$\beta_A = \beta_B = \frac{\theta}{3ah(\rho + \delta - \alpha_B)}.$$

$\square$

Proof of Lemma 3.

(i) Suppose that $\sigma \leq -3\bar{s}$. Using Lemma 1, HJB equations of each party is given by:

$$\rho V_A(\sigma) = \max_{d_A} \left\{ \theta y - \frac{a}{2} \cdot d_A^2 + V_A'(\sigma)[d_B - d_A - \delta\sigma] \right\},$$
$$\rho V_B(\sigma) = \max_{d_B} \left\{ -\frac{a}{2} \cdot d_B^2 + V_B'(\sigma)[d_B - d_A - \delta\sigma] \right\}.$$

Kuhn-Tucker conditions are given by:

$$-ad_A - V_A'(\sigma) \leq 0, \quad d_A \geq 0, \tag{B.1}$$
$$-ad_B + V_B'(\sigma) \leq 0, \quad d_B \geq 0. \tag{B.2}$$

Now let us assume that

$$d_A = d_B = 0, \tag{B.3}$$

are optimal policy functions. Substituting (B.3) into HJB equations, we obtain following differential equations:

$$V_A(\sigma) = \frac{\theta y}{\rho} - \frac{\delta V_A'(\sigma)}{\rho} \cdot \sigma,$$
$$V_B(\sigma) = -\frac{\delta V_B'(\sigma)}{\rho} \cdot \sigma.$$

Solving these differential equations, we obtain:

$$V_A(\sigma) = \frac{\theta y}{\rho}, \tag{B.4}$$
$$V_B(\sigma) = 0. \tag{B.5}$$

Then we obtain:

$$V_A'(\sigma) = V_B'(\sigma) = 0.$$

Therefore, the strategies given by (B.3) fulfill all of Kuhn-Tucker conditions of (B.1) and (B.2). As a result, we conclude that the strategies given by (B.3) are optimal policy.

(ii) The proof of Lemma 3(ii) is similar to that of Lemma 3(i) and is hence omitted.

$\square$

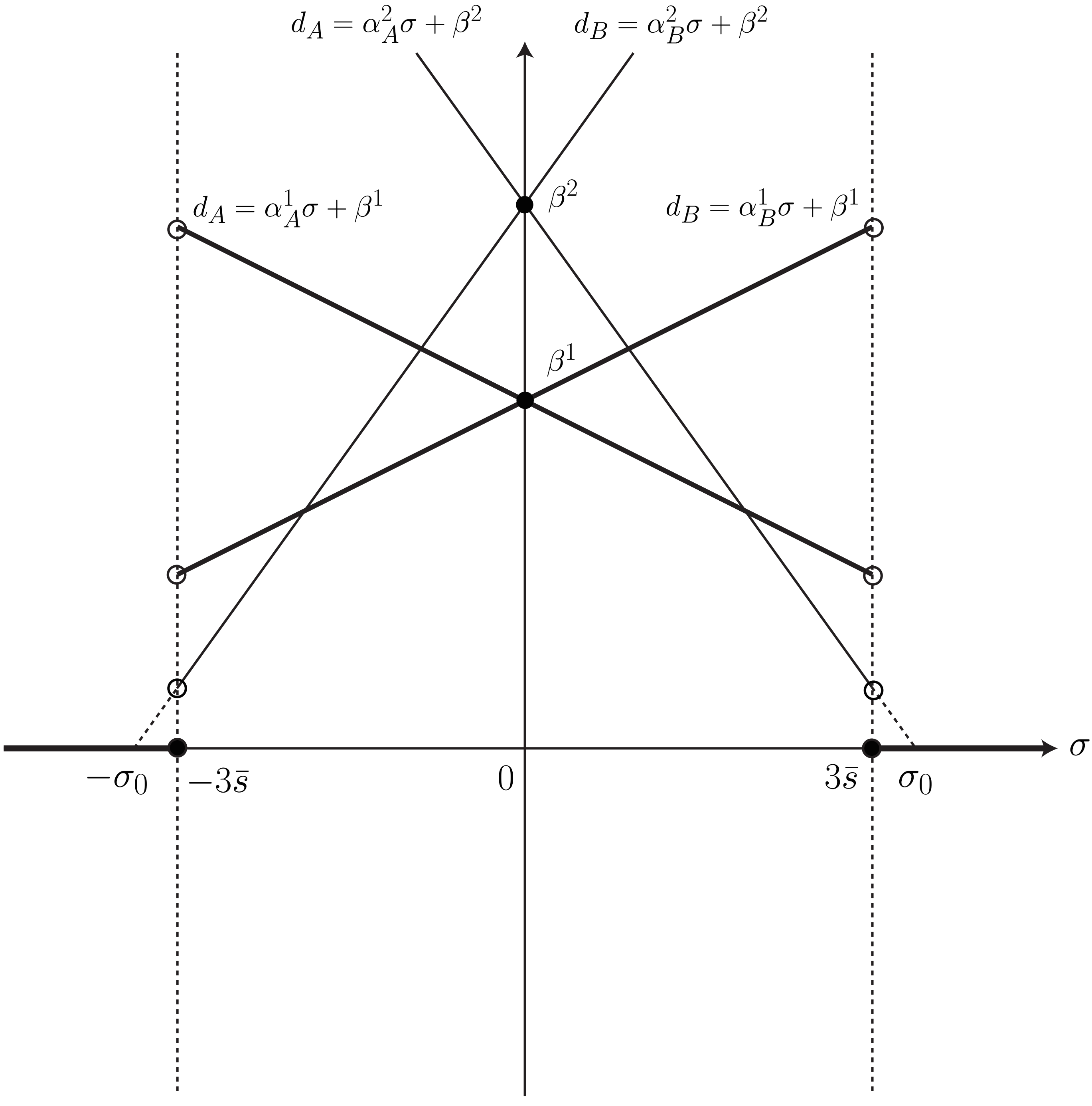
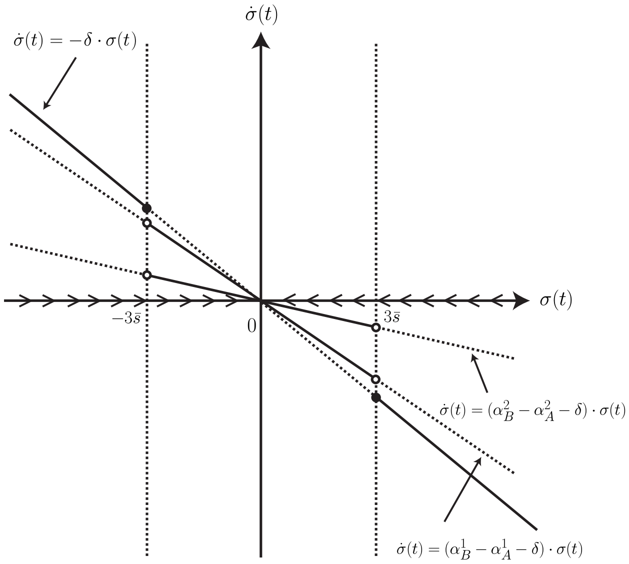
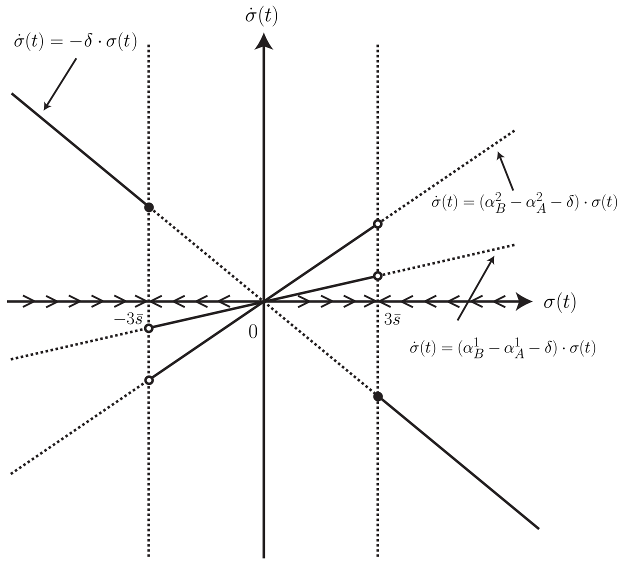


Figure 1: The figures displays the both parties' investment level.

(a) The case of $\bar{s} < \frac{4\theta}{9ah\delta(\delta+2\rho)}$ and $\rho < \delta$



(b) The case of $\bar{s} < \frac{4\theta}{9ah\delta(\delta+2\rho)}$ and $\rho > \delta$



(c) The case of $\bar{s} > \frac{4\theta}{9ah\delta(\delta+2\rho)}$

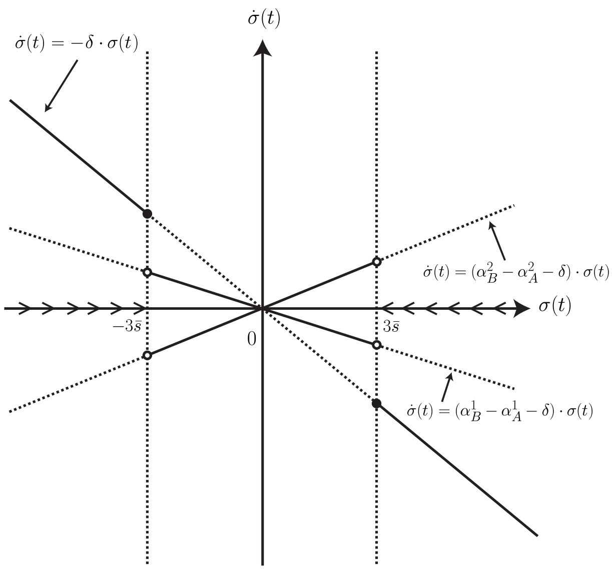


Figure 2: The figures display the dynamics of σ. The panel (a) illustrates the case of $\rho < \delta$ and $\bar{s} < 4\theta/9ah\delta(\delta + 2\rho)$; the panel (b) illustrates the case of $\rho > \delta$ and $\bar{s} < 4\theta/9ah\delta(\delta + 2\rho)$; the panel (c) illustrates the case of $\bar{s} > 4\theta/9ah\delta(\delta + 2\rho)$.

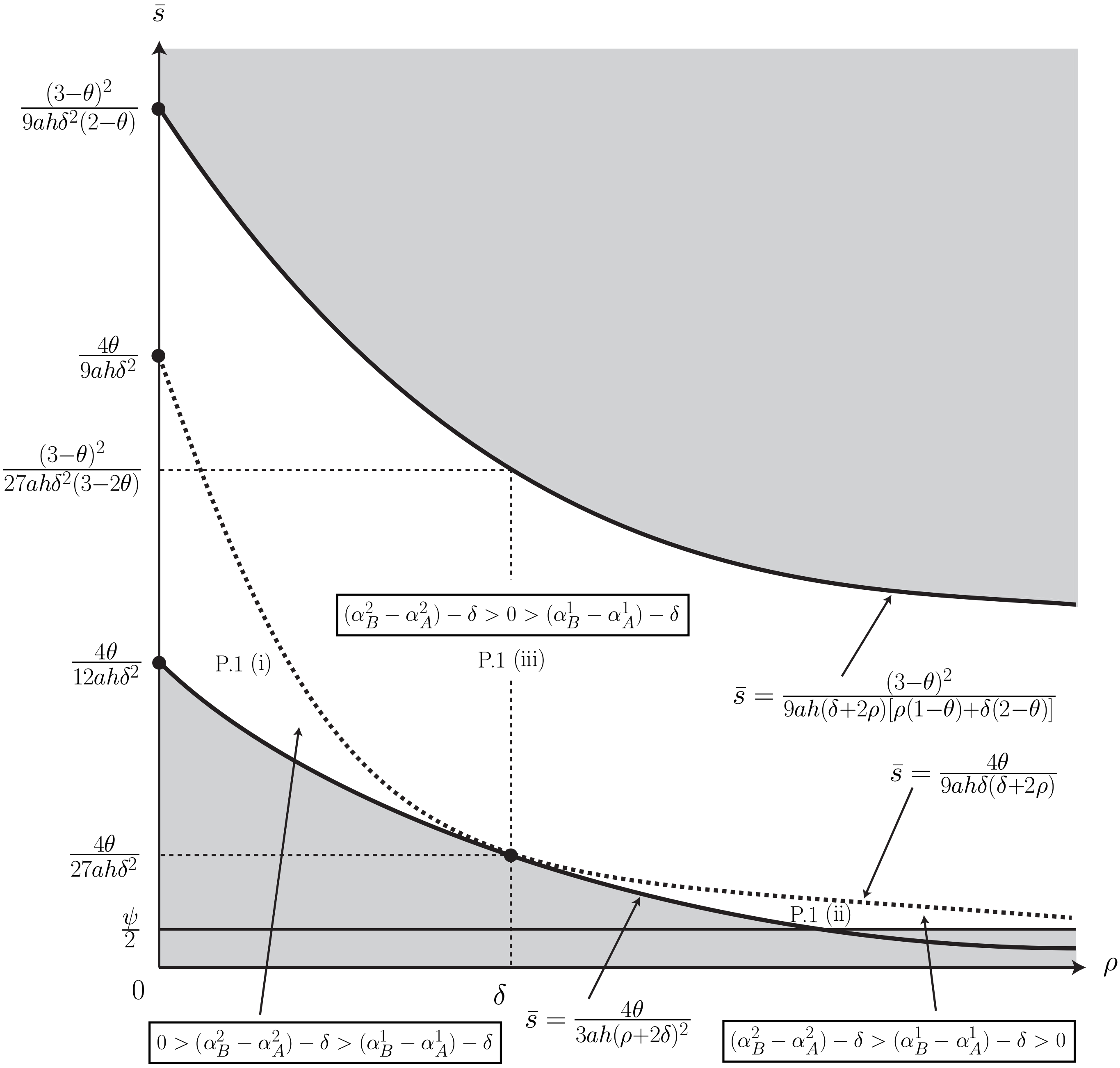


Figure 3: The figures display the set of parameters $(\bar{s}, \rho)$ classified according to the each case of Proposition 1.