Discussion Papers In Economics And Business

Utility Indifference Pricing in an Incomplete Market Model with Incomplete Information

Kazuhiro TAKINO

Discussion Paper 07-46

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Utility Indifference Pricing in an Incomplete Market Model
with Incomplete Information

Kazuhiro TAKINO

Discussion Paper 07-46

December 2007

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Utility Indifference Pricing in an Incomplete Market Model with Incomplete Information*

Kazuhiro TAKINO†

December, 2007

Abstract

In this article, we consider a derivative pricing model for the stochastic volatility model under an incomplete information. The incomplete information in our works, supposes that the true value of the drift for the stock price process is a random variable, investors only have an information of its distribution. This is more practical financial market than the situation with knowledge of the drift. There are many studies about the dynamic portfolio optimization problem under the incomplete information. In that situation, the corresponding problem becomes a easy to treat by *Separating Principle* and *Bayesian updating formula*. We apply these arguments to the utility indifference price approach, and present pricing method taken into account the incomplete information. On the other hand, Sircar and Zariphopoulou (2005) gives bounds and asymptotic approximations for the indifference prices in the stochastic volatility model. In them works, bounds include the drift parameter for the underlying price process. However, in practice, it is able to observe the drift parameter by estimation only. Therefore, it is meaningful to extended to the incomplete information. We derive bounds for the indifference prices using estimated drift, and the relationship between the buyer's indifference price and the seller's one.

JEL Classification. G11, G13, G14.

Key Words. Incomplete market, Incomplete information, Utility indifference price, Bayesian updating formula, Super/sub solution for PDEs.

*I would like to appreciate Dr. Masamitsu Ohnishi and Dr. Hiroshi Ishijima for helpful suggestions and comments throughout most of the paper and for their encouragement.

†Graduate School of Economics, Osaka University 1-7, Machikaneyama, Toyonaka, Osaka, 560-0043, Japan. E-mail: dg033tk@mail2.econ.osaka-u.ac.jp

1 Introduction

In this article, we consider a pricing model for an incomplete market model under an incomplete information. As typical example of the incomplete market model, we work the stochastic volatility model. The financial market in this paper, consists of the risk-free asset, one risky asset whose volatility drives stochastically over the time horizon and a derivative written on the risky asset. The incomplete information in our works, implies that investors don't have knowledge for a true value of the drift parameter of the risky-asset, and they only have an information of the distribution for that. Therefore, one can capture the value of the drift parameter only by estimation it. This situation is more close to the practical market than one with precise knowledge of the drift.

Under the incomplete information, there are many works for the decision of the dynamic optimal portfolio strategy based on the utility maximization up to now. ([Detemple, 1986], [Dothan and Feldman 1986], [Detemple and Sheldan 1991], [Gennote, 1986], [Ishijima, 1999]) The dynamic portfolio optimization problem under the incomplete information, requires that dynamic portfolio selection and estimation of the drift have to be done simultaneously. However, *Separating Principle* allows us to treat optimization and estimation separately. This changes from the optimization problem includes uncertain drift to the problem taken into account the estimation for it. On the other hand, the utility indifference pricing approach have been studied one of pricing methods in the incomplete market such the stochastic volatility model ([Grasselli and Hurd, 2005], [Sircar and Zariphopoulou, 2005]), the financial model with constraints ([Hodges and Neuberger, 1989]) and so on. The utility indifference price criterion is defined to determines the price to be equal maximized expected utilities for two different investment opportunities. The one is the problem without any claims, the another one is the problem with some claims. This pricing method is arrowed us to use various market models, although it requires us to solve expected maximization problems. So we apply studies of the portfolio optimization under the incomplete information to the indifference pricing approach, then we derive the price of derivatives under such circumstances.

Its procedure follows: At first, we set a financial market model with stochastic volatility under the incomplete market model. As mentioned above, we assume that the drift parameter of underlying asset of the derivative is not clear (i.e., a random variable), but investors have an information of the distribution of it. Next, we consider expected utility maximization problems for a portfolio consists of both the risk-free asset and such a risky-asset, and the derivative. In this time, we solve it using *Separating Principle*. The investor, under incomplete information, dynamically determine an optimal portfolio with estimating the drift of the stock price process. The separating principle allows the investor to treat them separately. Furthermore, Theorem 12.7 in Lipster and Shiryaev (2001) gives a dynamics of the estimator of the drift. This is called Bayesian updating formula. These makes the optimization problem with estimation of the drift easy to solve.

We apply Hamilton-Jacobi-Bellman (HJB) equation to solve optimization problems. In our works, partial differential equations obtained by related HJB equations, which are non-linear type. So we cannot explicitly represent solutions of optimization problems. Instead, we give bounds for solutions of these problems as Sircar and Zariphopoulou (2005), then we have a bound of utility indifference price in our financial market model. These bounds take into account the random drift term, that is, they include estimated drift. Furthermore we derive PDEs satisfied by the indifference prices, then we describe the relationship between the buyer's indifference price and seller's one by finding super/sub solution of these PDEs.

This paper is organized as follows: In the next section, we introduce a financial market model with stochastic volatility model with the incomplete information. In Section 3, we introduce estimation of random drift and derive the improvement of the estimation for the drift, i.e., Bayesian updating formula, with a change of measure. In Section 4, we examine expected utility maximization problems, and give solutions for these. In Section 5, we derive bounds of the utility indifference prices by using results of Section 4, and we show that bounds include the estimation of the drift instead of the true drift value. Furthermore, we provide the relationship between the buyer's indifference price and seller's one. As end of the paper, we give conclusions in our works.

2 Financial Market

In this section, we set a financial market model. We first define risk-free/risky asset price processes in a stochastic volatility environment, then we introduce an admissible portfolio strategies, and derive it's related wealth process.

2.1 Traded Assets

Let us consider the following financial market. There exist one risk free asset (typically the bank account) and one risky asset (typically stock). At first, as for the bank account, we assume that one dollar in the bank account at time 0 results in

$$B(s) = 1$$

at time $s \in [0, T]$ where $T > 0$ is the time horizon, that is, we assume that the risk free rate is equal to zero for simplicity without loss of generality, which implies that $\mathrm{d}B(s) = 0$.

Next, we set the stock price process with a stochastic volatility as follows.

$$\mathrm{d}S(s) = S(s)\{\mu \mathrm{d}s + \sigma(s, Y(s))\mathrm{d}W_1(s)\}, \quad S(t) = S, \tag{2.1}$$

where Y is a certain state variable driven by

$$\mathrm{d}Y(s) = a(s, Y(s))\mathrm{d}s + b(s, Y(s))\{\rho \mathrm{d}W_1(s) + \sqrt{1-\rho^2}\mathrm{d}W_2(s)\}, \quad Y(t) = Y, \tag{2.2}$$

for $(0 \leqslant)t \leqslant s \leqslant T$, where μ is a random variable on $(\Omega_\mu, \mathcal{F}_\mu, \nu)$ with $E\left[\mu^4\right] < \infty$, E is an expectation under P, P is defined later. Also, $\nu \circ \mu^{-1}$ is normally distributed with mean m_0 and variance Γ_0. And then, $W = (W_1, W_2)^\top$ is two-dimensional standard Brownian motion on a filtered probability space $(\Omega_W, \{\mathcal{F}_{W,s}; s \geqslant 0\}, \phi)$, where $\mathcal{F}_{W,s}$ is generated by informations of W up to s. In this time, we can construct an filtered probability space $(\Omega, \{\mathcal{F}_s; t \geqslant 0\}, P)$, where $\Omega = \Omega_\mu \times \Omega_W$, $\mathcal{F}_s = \mathcal{F}_\mu \otimes \mathcal{F}_{W,s}$, $P = \nu \otimes \phi$. Furthermore, we assume that investors only have informations obtained from the stock price up to now. That is, the information available for them is described by $\mathcal{G}_t = \sigma(S(u); 0 \leqslant u \leqslant t)$, with $\mathcal{G}_t \subset \mathcal{F}_t$.

We would price a European type claim already mentioned above, and it's payoff function is supposed to $g(T) := g(T, S(T), Y(T))$. In the next section we consider utility maximization problems. In order to ensure the existence of solutions for such problems, we set the following assumption.

Assmption 2.1.
- *The volatility function $\sigma(\cdot)$ and the diffusion coefficient $b(\cdot,\cdot)$ are smooth and, bounded above and below away from zero.*
- *The drift coefficient $a(\cdot,\cdot)$ in (2.2) is Lipschitz continuous on $[0, T] \times \mathbb{R}$.*
- *The payoff function $g(T)$ is a $\mathcal{G}_T$-measurable.*

2.2 Wealth Process

We denote by $\pi(s)$ the amount of money held in stock at time s. So that, if the investor have a wealth X at time s, then the amount of money invested in the bank account is $X - \pi(s)$. The negative value of $\pi(\cdot)$ usually means that he/she is short-selling the stock. It is assumed that $\pi(\cdot)$ is an adapted process, i.e., $\pi(s)$ is $\mathcal{G}_s$-measurable for any $s \in [t, T]$. This assumption means that the investor cannot foresee the future, so $\pi(s)$ is determined from the information up to time s. We also require another technical conditions. We give them in the following definition.

Definition 2.1 (Admissible). *We say that the portfolio strategy $\pi(s)$ is admissible, if it is $\mathcal{G}_s$-measurable, and satisfies the integrability condition $E\left[\int_t^T \sigma^2(s, Y(s))\pi^2(s)\mathrm{d}s\right] < +\infty$ and the self-financing condition. $\mathcal{A}$ denotes the set of all admissible policies. We also denote by X the wealth process (or portfolio process) corresponding to an admissible strategy $\pi \in \mathcal{A}$. The self-financing condition is defined by the following equation:*

$$\mathrm{d}X(s) = \frac{\pi(s)}{S(s)}\mathrm{d}S(s) + \frac{X(s) - \pi(s)}{B(s)}\mathrm{d}B(s) = \frac{\pi(s)}{S(s)}\mathrm{d}S(s), \tag{2.3}$$

for $t \leqslant s \leqslant T$.

Substituting (2.1) into (2.3), we have the wealth equation

$$\mathrm{d}X(s) = \pi(s)\{\mu \mathrm{d}s + \sigma(s, Y(s))\mathrm{d}W_1(s)\}. \tag{2.4}$$

3 Incomplete Information and Estimation for Drift

In this section, we consider the estimation of the drift μ. In order to use the indifference price approach, we examine utility maximization problems for two different investment opportunities respectively. The one is the problem without a claim, the another one is the problem with a claim. We suppose that the preference of the market participant is an exponential type, i.e., the utility function is $u(x) = -e^{-\gamma x}$ where γ is a risk-aversion parameter. Therefore, the former problem is presented by

$$U(t, X(t), Y(t)) = \sup_{\pi(t)\in\mathcal{A}} E_t\left[-e^{-\gamma X(T)}\right], \tag{3.1}$$

and the latter one is formulated as

$$V(t, S(t), X(t), Y(t); \eta) = \sup_{\pi(t)\in\mathcal{A}} E_t\left[-e^{-\gamma(X(T)+\eta g(T))}\right], \tag{3.2}$$

where $\eta = \pm 1$, where E_t denotes the expectation under P conditioned with informations up to t, and $\eta(= \pm 1)$ implies the position of the claim, namely $\eta = 1$ is a long position for the claim and $\eta = -1$ is short one.

We adopt *Separating Principle* [7] to solve these optimization problems. The separating principle allows us to estimate the drift μ and optimize the expected utility separately. Then this method stands in solving the above problems. In particularly, as shown latter, we utilize the continuous Bayesian updating formula of Lipster-Shiryaev [14].

As mentioned above, the investors cannot know a true value of the drift μ. Given an information $\mathcal{G}_t$, the investor estimates μ as

$$m(t) := E[\mu|\mathcal{G}_t], \tag{3.3}$$

$$\Gamma(t) := E\left[(\mu - m(t))^2 \middle| \mathcal{G}_t\right], \tag{3.4}$$

where $\Gamma(t)$ implies the estimation error. From Theorem 12.7 in [14], using infinitesimal observations $\mathrm{d}S(s)$, we can improve the estimation for μ and its error as

$$\mathrm{d}m(s) := -\frac{\Gamma(s)}{\sigma^2}\left\{\frac{\mathrm{d}S(s)}{S(s)} - m(s)\mathrm{d}s\right\}, \tag{3.5}$$

$$\mathrm{d}\Gamma(s) := -\left(\frac{\Gamma(s)}{\sigma}\right)^2 \mathrm{d}s \tag{3.6}$$

for $t \leqslant s \leqslant T$. Now we set a Doleans-Dade exponential as follows

$$\mathcal{E}(\theta,\zeta)(s) = \exp\left(-\frac{1}{2}\int_0^s \left\{\theta^2(u) + \zeta^2(u)\right\}\mathrm{d}u - \int_0^s \theta(u)\mathrm{d}W_1(u) - \int_0^s \zeta(u)\mathrm{d}W_2(u)\right), \tag{3.7}$$

for $t \leqslant s \leqslant T$, where $\theta(u) := \dfrac{\mu - m(u)}{\sigma(u, Y(u))}$ and $\zeta(u) := \dfrac{\rho\mu}{\sqrt{1-\rho^2}\sigma(u, Y(u))}$.

Definition 3.1. *We assume that (3.7) is a P-martingale. Then we define an equivalent martingale measure (EMM) $\tilde{P}$ by*

$$\left.\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P}\right|_{\mathcal{G}_s} = \mathcal{E}(\theta,\zeta)(s). \tag{3.8}$$

From Girsanov theorem,

$$\tilde{W}(s) := \begin{pmatrix} \tilde{W}_1(s) \\ \tilde{W}_2(s) \end{pmatrix} := \begin{pmatrix} \tilde{W}_1(s) + \int_0^s \theta(u)\mathrm{d}u \\ \tilde{W}_2(s) + \int_0^s \zeta(u)\mathrm{d}u \end{pmatrix} \tag{3.9}$$

is a two-dimensional standard Brownian motion under $\tilde{P}$. Therefore, under measure $\tilde{P}$, the stock price process, it's variance process and the wealth process introduced in the above, are rewritten the form taken into account of Bayes updating. That is

$$\frac{\mathrm{d}S(s)}{S(s)} = m(s)\mathrm{d}s + \sigma(s, Y(s))\mathrm{d}\tilde{W}_1(s), \tag{3.10}$$

$$\mathrm{d}Y(s) = \left[a(s, Y(s)) + \frac{\rho b(s, Y(s))m(s)}{\sigma(s, Y(s))}\right]\mathrm{d}s + b(s, Y(s))\left(\rho\mathrm{d}\tilde{W}_1(s) + \sqrt{1-\rho^2}\mathrm{d}\tilde{W}_2(s)\right), \tag{3.11}$$

$$\mathrm{d}X(s) = \pi(s)\left\{m(s)\mathrm{d}s + \sigma(s, Y(s))\mathrm{d}\tilde{W}_1(s)\right\}. \tag{3.12}$$

Also, m satisfies

$$\mathrm{d}m(s) = -\frac{\Gamma(s)}{\sigma(s, Y(s))}\mathrm{d}\tilde{W}_1(s).$$

In the rest of the paper, we discuss based on these dynamics. And also, the value function (3.1) and (3.2) are written as

$$U\left(t, m(t), X(t)\right) = \sup_{\pi(t)\in\mathcal{A}} \tilde{E}_t\left[-e^{-\gamma X(T)}\right], \tag{3.13}$$

$$V\left(t, m(t), S(t), X(t); \eta\right) = \sup_{\pi(t)\in\mathcal{A}} \tilde{E}_t\left[-e^{-\gamma(X(T)+\eta g(T))}\right], \tag{3.14}$$

where $\tilde{E}_t$ denotes the expectation under $\tilde{P}$ conditioned with informations up to t.

4 Optimization Problems

4.1 Operators

To facilitate the presentation, we define the following operators: for any smooth function f,

$$\begin{aligned}
\mathcal{A}^{\langle m,Y\rangle} f :=& \left(a(t,Y(t)) + \frac{\rho b(t,Y(t))m(t)}{\sigma(t,Y(t))}\right) f_Y \\
&+ \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2 f_{mm} + \frac{1}{2} b^2(t,Y(T)) f_{YY} - \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))} f_{mY},
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^{\langle m,S,Y\rangle} f :=& m(t)S(t)f_S + \left(a(t,Y(t)) + \frac{\rho b(t,Y(t))m(t)}{\sigma(t,Y(t))}\right) f_Y \\
&+ \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2 f_{mm} + \frac{1}{2}\sigma^2(t,Y(t))S^2 f_{SS} + \frac{1}{2} b^2(t,Y(T)) f_{YY} \\
&- \Gamma(t)S(t)f_{mS} - \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))} f_{mY} + \rho\sigma(t,Y(t))b(t,Y(t))S(t)f_{SY},
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^{\langle m,Y\rangle}(f_X, f_{XX}, f_{mX}, f_{XY}) :=& \max_{\pi(t)\in\mathcal{A}} \Big[\pi(t)m(t)f_X + \frac{1}{2}\left(\pi(t)\right)^2 \sigma^2(t,Y(t)) f_{XX} \\
&- \pi(t)\Gamma(t)f_{mX} + \pi(t)\rho\sigma(t,Y(t))b(t,Y(t))f_{XY}\Big],
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^{\langle m,S,Y\rangle}(f_X, f_{XX}, f_{mX}, f_{SX}, f_{XY}) :=& \max_{\pi(t)\in\mathcal{A}} \Big[\pi(t)m(t)f_X + \frac{1}{2}\left(\pi(t)\right)^2 \sigma^2(t,Y(t)) f_{XX} \\
&- \pi(t)\Gamma(t)f_{mX} + \pi(t)\sigma^2(t,Y(t))Sf_{SX} + \pi(t)\rho\sigma(t,Y(t))b(t,Y(t))f_{XY}\Big],
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}^{\langle m,Y\rangle}(f_X, f_{XX}, f_{mX}, f_{XY}) :=& -\frac{1}{2}\left(\frac{m(t)}{\sigma(t,Y(t))}\right)^2 \frac{f_X^2}{f_{XX}} - \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2 \frac{f_{mX}^2}{f_{XX}} - \frac{1}{2}\rho^2 b^2(t,Y(t)) \frac{f_{XY}^2}{f_{XX}} \\
&+ \frac{\Gamma(t)m(t)}{\sigma(t,Y(t))^2}\frac{f_{mX}f_X}{f_{XX}} - \frac{\rho b(t,Y(t))m(t)}{\sigma(t,Y(t))}\frac{f_{XY}f_X}{f_{XX}} + \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))}\frac{f_{mX}f_{XY}}{f_{XX}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}^{\langle m,S,Y\rangle}(f_X, f_{XX}, f_{mX}, f_{SX}, f_{XY}) :=& -\frac{1}{2}\left(\frac{m(t)}{\sigma(t,Y(t))}\right)^2 \frac{f_X^2}{f_{XX}} - \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2 \frac{f_{mX}^2}{f_{XX}} \\
&- \frac{1}{2}\sigma^2(t,Y(t))S^2\frac{f_{SX}^2}{f_{XX}} - \frac{1}{2}\rho^2 b^2(t,Y(t))\frac{f_{XY}^2}{f_{XX}} + \frac{\Gamma(t)m(t)}{\sigma(t,Y(t))^2}\frac{f_{mX}f_X}{f_{XX}} - m(t)S(t)\frac{f_X f_{SX}}{f_{XX}} \\
&- \frac{\rho b(t,Y(t))m(t)}{\sigma(t,Y(t))}\frac{f_{XY}f_X}{f_{XX}} + \Gamma(t)S(t)\frac{f_{mX}f_{SX}}{f_{XX}} + \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))}\frac{f_{mX}f_{XY}}{f_{XX}} \\
&- \rho\sigma(t,Y(t))b(t,Y(t))S(t)\frac{f_{SX}f_{XY}}{f_{XX}},
\end{aligned}$$

$$\mathcal{L}^{\langle m,Y\rangle} f := \mathcal{A}^{\langle m,Y\rangle} f + \frac{\Gamma(t)m(t)}{\sigma^2(t,Y(T))} f_m - \frac{\rho b(t,Y(t))m(t)}{\sigma(t,Y(t))} f_Y,$$

$$\mathcal{L}^{\langle m,S,Y\rangle} f := \mathcal{A}^{\langle m,S,Y\rangle} f - m(t)S(t)f_S + \frac{\Gamma(t)m(t)}{\sigma^2(t,Y(T))} f_m - \frac{\rho b(t,Y(t))m(t)}{\sigma(t,Y(t))} f_Y,$$

$$\mathcal{M}^{\langle m,Y\rangle}(f_m,f_Y,f) := \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2\frac{f_m^2}{f} - \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))}\frac{f_m f_Y}{f} + \frac{1}{2}\rho^2 b^2(t,Y(t))\frac{f_Y^2}{f},$$

$$\begin{aligned}\mathcal{M}^{\langle m,S,Y\rangle}(f_m,f_S,f_Y,f) &:= \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2\frac{f_m^2}{f} + \frac{1}{2}\sigma^2(t,Y(t))S^2(t)\frac{f_S^2}{f} + \frac{1}{2}\rho^2 b^2(t,Y(t))\frac{f_Y^2}{f} \\ &\quad - \Gamma(t)S(t)\frac{f_m f_S}{f} - \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))}\frac{f_m f_Y}{f} + \rho\sigma(t,Y(t))b(t,Y(t))S(t)\frac{f_S f_Y}{f}.\end{aligned}$$

4.2 Optimizations

Recall that, the utility maximization problem without any claims is

$$U\left(t,m(t),X(t),Y(t)\right) = \sup_{\pi\in\mathcal{A}}\tilde{E}_t\left[-e^{-\gamma X(T)}\right]. \tag{4.1}$$

The HJB equation of U is

$$\begin{cases} U_t + \mathcal{A}^{\langle m,Y\rangle}U + \mathcal{H}^{\langle m,Y\rangle}(U_X,U_{XX},U_{mX},U_{XY}) = 0, \\ U\left(T,m(T),X(T),Y(T)\right) = -e^{-\gamma X(T)}. \end{cases} \tag{4.2}$$

The maximum in (4.2) is achieved at

$$\pi^* = -\frac{m(t)}{\sigma^2(t,Y(t))}\frac{U_X}{U_{XX}} + \frac{\Gamma(t)}{\sigma^2(t,Y(t))}\frac{U_{mX}}{U_{XX}} - \frac{\rho b(t,Y(T))}{\sigma(t,Y(t))}\frac{U_{XY}}{U_{XX}}. \tag{4.3}$$

Plugging (4.3) into (4.2) gives us the following non-linear partial differential equation (PDE):

$$\begin{cases} U_t + \mathcal{A}^{\langle m,Y\rangle}U + \mathcal{J}^{\langle m,Y\rangle}(U_X,U_{XX},U_{mX},U_{XY}) = 0, \\ U\left(T,m(T),X(T),Y(T)\right) = -e^{-\gamma X(T)}. \end{cases} \tag{4.4}$$

From Section 3 in Pham [16] or Theorem 2.5 in Sircar and Zariphopoulou [17], U is given by

$$U\left(t,m(t),X(t),Y(t)\right) = -e^{-\gamma X(t)}F\left(t,m(t),Y(t)\right) \tag{4.5}$$

where $F\in\mathbf{C}^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ is the unique bounded solution of the quasi linear partial differential equation

$$\begin{cases} F_t + \mathcal{L}^{\langle m,Y\rangle}F = \frac{1}{2}\left(\frac{m(t)}{\sigma(t,Y(t))}\right)^2 + \mathcal{M}^{\langle m,Y\rangle}(F_m,F_Y,F), \\ F\left(T,m(T),Y(T)\right) = 1. \end{cases} \tag{4.6}$$

Note that (4.6) is obtained by substituting (4.5) into (4.4), and F is a non-negative because U is an indirect utility function of u.

Next we consider the optimization problem with a claim. This problem is formulated as

$$V\left(t,m(t),S(t),X(t),Y(t);\eta\right) = \sup_{\pi\in\mathcal{A}}\tilde{E}_t\left[-e^{-\gamma(X(T)+\eta g(T))}\right], \tag{4.7}$$

where $g(T) := g(T, S(T), Y(T))$, and recall that $\eta(= \pm 1)$ implies the position of the claim. The HJB equation of (4.7) is

$$\begin{cases} V_t + \mathcal{A}^{\langle m,S,Y\rangle} U + \mathcal{H}^{\langle m,S,Y\rangle}(V_X, V_{XX}, V_{mX}, V_{SX}, V_{XY}) = 0, \\ V(T, m(T), S(T), X(T), Y(T); \eta) = -e^{-\gamma(X(T)+\eta g(T))}. \end{cases} \tag{4.8}$$

The maximum in (4.8) is achieved at

$$\pi^* = -\frac{m(t)}{\sigma^2(t, Y(t))}\frac{V_X}{V_{XX}} + \frac{\Gamma(t)}{\sigma^2(t, Y(t))}\frac{V_{mX}}{V_{XX}} - S\frac{V_{SX}}{V_{XX}} - \frac{\rho b(t, Y(T))}{\sigma(t, Y(t))}\frac{V_{XY}}{V_{XX}}. \tag{4.9}$$

Plugging (4.9) into (4.8) gives us the following PDE:

$$\begin{cases} V_t + \mathcal{A}^{\langle m,S,Y\rangle} V + \mathcal{J}^{\langle m,S,Y\rangle}(V_X, V_{XX}, V_{mX}, V_{SX}, V_{XY}) = 0, \\ V(T, m(T), S(T), X(T), Y(T); \eta) = -e^{-\gamma(X(T)+\eta g(T))}. \end{cases} \tag{4.10}$$

From Pham [16], V is given by

$$V(t, m(t), S(t), X(t), Y(t); \eta) = -e^{-\gamma X(t)} G(t, m(t), S(t), Y(t); \eta) \tag{4.11}$$

where $G \in \mathbf{C}^{1,2,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ is the unique bounded solution of the quasi linear partial differential equation

$$\begin{cases} G_t + \mathcal{L}^{\langle m,S,Y\rangle} G = \frac{1}{2}\left(\frac{m(t)}{\sigma(t, Y(t))}\right)^2 + \mathcal{M}^{\langle m,S,Y\rangle}(G_m, G_S, G_Y, G), \\ G(T, m(T), S(T), Y(T); \eta) = e^{-\gamma \eta g(T)}. \end{cases} \tag{4.12}$$

PDE (4.12) is also obtained by substituting (4.11) into (4.10), and G must be a non-negative if V maintain a property of the indirect utility function for u.

5 Indifference Prices and Bounds

5.1 Bounds for the Utility Indifference Prices

In this section, we provide utility indifference prices and their bounds. We derive bounds of indifference prices using the method of super/sub solution of non-linear PDE as Sircar and Zariphopoulou [17]. At first we define the utility indifference price.

Definition 5.1 (Utility Indifference Pirce). *The buyer's utility indifference price* $p_b(t) := p_b(t, m(t), S(t), Y(t))$ *at time* t *is defined as the solution of*

$$U(t, m(t), X(t), Y(t)) = V(t, m(t), S(t), X(t) - p_b(t), Y(t); 1).$$

While the seller's utility indifference price $p_s(t) := p_s(t, m(t), S(t), Y(t))$ *at time* t *is defined as the solution of*

$$U(t, m(t), X(t), Y(t)) = V(t, m(t), S(t), X(t) + p_s(t), Y(t); -1).$$

In the previous subsection, we derived solutions for utility maximization problems with or without a claim and corresponding PDEs. By consider a super/sub solutions for these PDEs, we introduce bounds of the buyer's/seller's indifference price respectively. Before that, we additionally construct an EMM. Define

$$\tilde{\mathcal{E}}(\lambda,0)(s) = \exp\left(-\frac{1}{2}\int_0^s \lambda^2(u)\mathrm{d}u - \int_0^s \lambda(u)\mathrm{d}\tilde{W}_1(u)\right) \tag{5.1}$$

where $\lambda(s) := \dfrac{m(s)}{\sigma(s,Y(s))}$. From Assumption 2.1, this is a $\tilde{P}$-martingale.

Definition 5.2. *We define an equivalent measure Q according to*

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}\tilde{P}}\right|_{\mathcal{G}_s} = \tilde{\mathcal{E}}\left(0,\lambda\right)(s).$$

In this time,

$$M(s) := \begin{pmatrix} M_1(s) \\ M_2(s) \end{pmatrix} := \begin{pmatrix} \tilde{W}_1(s) + \int_0^s \lambda(u)\mathrm{d}u \\ \tilde{W}_2(s) \end{pmatrix} \tag{5.2}$$

is a two-dimensional standard Brownian motion under Q. Under this, dynamics become

$$\frac{\mathrm{d}S(s)}{S(s)} = \sigma(s,Y(s))\mathrm{d}M_1(s), \tag{5.3}$$

$$\mathrm{d}Y(s) = a(s,Y(s))\mathrm{d}s + b(s,Y(s))\left(\rho\mathrm{d}M_1(s) + \sqrt{1-\rho^2}\mathrm{d}M_2(s)\right), \tag{5.4}$$

$$\mathrm{d}m(s) = \frac{\Gamma(s)m(s)}{\sigma^2(s,Y(s)} - \frac{\Gamma(s)}{\sigma(s,Y(s))}\mathrm{d}M_1(s). \tag{5.5}$$

Proposition 5.1. *The solution F of (4.6) has a bound. That is*

$$\bar{F}\left(t,m(t),Y(t)\right) < F\left(t,m(t),Y(t)\right) < \hat{F}\left(t,m(t),Y(t)\right) \tag{5.6}$$

where $\bar{F}$ and $\hat{F}$ are given by

$$\begin{aligned} \bar{F}\left(t,m(t),Y(t)\right) &= \exp\left(E_t^Q\left[-\frac{1}{2}\int_t^T\left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2\mathrm{d}s\right]\right), \\ \hat{F}\left(t,m(t),Y(t)\right) &= E_t^Q\left[\exp\left(-\frac{1}{2}\int_t^T\left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2\mathrm{d}s\right)\right], \end{aligned} \tag{5.7}$$

where E_t^Q denotes the expectation under Q conditioned with informations up to time t.

Proof. Let us consider a lower bound. By $F>0$ and $\rho^2<1$, it holds that

$$\begin{aligned} F_t + \mathcal{L}^{\langle m,Y\rangle}F <& \frac{1}{2}\left(\frac{m(t)}{\sigma(t,Y(t))}\right)^2 F + \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2\frac{F_m^2}{F} + \frac{1}{2}b^2(t,Y(t))\frac{F_Y^2}{F} \\ &- \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))}\frac{F_mF_Y}{F}. \end{aligned} \tag{5.8}$$

So F is a super solution of $\bar{F}$ which solves to

$$\begin{aligned}\bar{F}_t + \mathcal{L}^{\langle m,Y\rangle}\bar{F} =& \frac{1}{2}\left(\frac{m(t)}{\sigma(t,Y(t))}\right)^2\bar{F} + \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2\frac{\bar{F}_m^2}{\bar{F}} + \frac{1}{2}b^2(t,Y(t))\frac{\bar{F}_Y^2}{\bar{F}}\\ &- \frac{\rho\Gamma(t)b(t,Y(t))}{\sigma(t,Y(t))}\frac{\bar{F}_m\bar{F}_Y}{\bar{F}},\end{aligned} \tag{5.9}$$

and $\bar{F}(T, m(T), Y(T)) = 1$. That is

$$F(t, m(t), Y(t)) > \bar{F}(t, m(t), Y(t)). \tag{5.10}$$

We transform $\bar{F}$ as

$$\bar{F}(t, m(t), Y(t)) = e^{\phi^F(t,m(t),Y(t))}, \tag{5.11}$$

where $\phi^F \in \mathbf{C}^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}_+)$ and $\phi^F(T, m(T), Y(T)) = 0$. Then ϕ^F solves to

$$\begin{cases} \phi_t^F + \mathcal{L}^{\langle m,Y\rangle}\phi^F = \dfrac{1}{2}\left(\dfrac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2, \\ \\ \phi^F(T, m(T), Y(T)). \end{cases} \tag{5.12}$$

Therefore, by Feynman-Kac formula and under Q, we have

$$\phi^F(t, m(t), Y(t)) = E_t^Q\left[-\frac{1}{2}\int_t^T\left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right]. \tag{5.13}$$

The lower bound is obtained from (5.10), (5.11) and (5.13).

We turn to derive an upper bound of F. From (4.6) it holds that

$$F_t + \mathcal{L}^{\langle m,Y\rangle}F > \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2 F.$$

It is following that F is a sub solution of $\hat{F}$, that is, $F < \hat{F}$. Where $\hat{F}$ is a solution of

$$\begin{cases} \hat{F}_t + \mathcal{L}^{\langle m,Y\rangle}\hat{F} = \dfrac{1}{2}\left(\dfrac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2\hat{F}, \\ \\ \hat{F}(T, m(T), Y(T)) = 1. \end{cases} \tag{5.14}$$

Again, by using Feynman-Kac formula to (5.14), we obtain the upper bound $\hat{F}$ as

$$\hat{F}(t, m(t), Y(t)) = E_t^Q\left[\exp\left(-\frac{1}{2}\int_t^T\left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right)\right].$$

□

On the other hand, the bound of G is given by the following proposition.

Proposition 5.2. *The solution G of (4.12) has a bound. That is*

$$\bar{G}\left(t, m(t), S(t), Y(t); \eta\right) < G\left(t, m(t), S(t), Y(t); \eta\right) < \hat{G}\left(t, m(t), S(t), Y(t); \eta\right) \tag{5.15}$$

where $\bar{G}$ and $\hat{G}$ are given by

$$\begin{aligned}\bar{G}\left(t, m(t), S(t), Y(t); \eta\right) &= \exp\left(E_t^Q\left[-\gamma\eta g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s, Y(s))}\right)^2 \mathrm{d}s\right]\right),\\ \hat{G}\left(t, m(t), S(t), Y(t); \eta\right) &= E_t^Q\left[\exp\left(-\gamma\eta g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s, Y(s))}\right)^2 \mathrm{d}s\right)\right],\end{aligned} \tag{5.16}$$

where $\eta = \pm 1$.

Proof. Let us consider a lower bound. By $G > 0$ and $\rho^2 < 1$, it holds that

$$\begin{aligned}G_t + \mathcal{L}^{\langle m,S,Y\rangle} G &< \frac{1}{2}\left(\frac{m(t)}{\sigma(t, Y(t))}\right)^2 G + \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t, Y(t))}\right)^2 \frac{G_m^2}{G} + \frac{1}{2}\sigma^2(t, Y(t))S^2(t)\frac{G_S^2}{G}\\ &+ \frac{1}{2}b^2(t, Y(t))\frac{G_Y^2}{G} - \Gamma(t)S(t)\frac{G_m G_S}{G} - \frac{\rho\Gamma(t)b(t, Y(t))}{\sigma(t, Y(t))}\frac{G_m G_Y}{G} - \rho\sigma(t, Y(t))b(t, Y(t))S(t)\frac{G_S G_Y}{G}.\end{aligned} \tag{5.17}$$

So G is a super solution of $\bar{G}$ which solves to

$$\begin{aligned}\bar{G}_t + \mathcal{L}^{\langle m,Y\rangle} \bar{G} &= \frac{1}{2}\left(\frac{m(t)}{\sigma(t, Y(t))}\right)^2 \bar{G} + \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t, Y(t))}\right)^2 \frac{\bar{G}_m^2}{\bar{G}} + \frac{1}{2}\sigma^2(t, Y(t))S^2(t)\frac{\bar{G}_S^2}{\bar{G}}\\ &+ \frac{1}{2}b^2(t, Y(t))\frac{\bar{G}_Y^2}{\bar{G}} - \Gamma(t)S(t)\frac{\bar{G}_m \bar{G}_S}{\bar{G}} - \frac{\rho\Gamma(t)b(t, Y(t))}{\sigma(t, Y(t))}\frac{\bar{G}_m \bar{G}_Y}{\bar{G}} - \rho\sigma(t, Y(t))b(t, Y(t))S(t)\frac{\bar{G}_S \bar{G}_Y}{\bar{G}},\end{aligned} \tag{5.18}$$

and $\bar{G}\left(T, m(T), S(T), Y(T); \eta\right) = e^{-\gamma\eta g(T)}$. That is

$$G\left(t, m(t), S(t), Y(t); \eta\right) > \bar{G}\left(t, m(t), S(t), Y(t); \eta\right). \tag{5.19}$$

We transform $\bar{G}$ as

$$\bar{G}\left(t, m(t), S(t), Y(t); \eta\right) = e^{\phi^G\left(t, m(t), S(t), Y(t); \eta\right)}, \tag{5.20}$$

where $\phi^G \in \mathbf{C}^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+)$ and $\phi^G\left(T, m(T), Y(T); \eta\right) = -\gamma\eta g(T)$. Then ϕ^G solves to

$$\begin{cases} \phi_t^G + \mathcal{L}^{\langle m,S,Y\rangle}\phi^G = \dfrac{1}{2}\left(\dfrac{\Gamma(t)}{\sigma(t, Y(t))}\right)^2, \\ \phi^G\left(T, m(T), Y(T); \eta\right) = -\gamma\eta g(T). \end{cases} \tag{5.21}$$

Therefore, by Feynman-Kac formula and under Q, we have

$$\phi^G\left(t, m(t), S(t), Y(t); \eta\right) = E_t^Q\left[-\gamma\eta g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s, Y(s))}\right)^2 \mathrm{d}s\right]. \tag{5.22}$$

The lower bound is obtained from (5.19), (5.20) and (5.22).

We turn to derive an upper bound of G. From (4.12) it holds that

$$G_t + \mathcal{L}^{\langle m,S,Y\rangle} G > \frac{1}{2}\left(\frac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2 G.$$

It is following that G is a sub solution of $\hat{G}$, that is, $G < \hat{G}$. Where $\hat{G}$ is a solution of

$$\begin{cases} \hat{G}_t + \mathcal{L}^{\langle m,S,Y\rangle} \hat{G} = \dfrac{1}{2}\left(\dfrac{\Gamma(t)}{\sigma(t,Y(t))}\right)^2 \hat{G}, \\ \\ \hat{G}\left(T, m(T), Y(T)\right) = 1. \end{cases} \tag{5.23}$$

Again using Feynman-Kac formula to (5.23), we obtain the upper bound $\hat{G}$ as

$$\hat{G}\left(t, m(t), S(t), Y(t); \eta\right) = E_t^Q\left[\exp\left(-\gamma\eta g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right)\right].$$

□

From the above calculations, we derive bounds for indifference prices. Recall that $p^b\left(t, m(t), S(t), Y(t)\right)$ and $p^s\left(t, m(t), S(t), Y(t)\right)$ respectively denote the buyer's indifference price and the seller's one at time t. By (5.1), (5.6) and (5.15), it holds that

$$\begin{aligned} \frac{1}{\gamma}\ln\frac{\bar{F}(t)}{\hat{G}(t;1)} < p^b < \frac{1}{\gamma}\ln\frac{\hat{F}(t)}{\bar{G}(t;1)}, \\ \frac{1}{\gamma}\ln\frac{\bar{G}(t;-1)}{\hat{F}(t)} < p^s < \frac{1}{\gamma}\ln\frac{\hat{G}(t;-1)}{\bar{F}(t)}, \end{aligned} \tag{5.24}$$

where $F(t) \equiv F\left(t, m(t), Y(t)\right)$ and $G(t;\eta) \equiv G\left(t, m(t), S(t), Y(t); \eta\right)$. Substituting (5.7) and (5.16) into both of (5.24), we obtain the following theorem.

Theorem 5.1. *(i) The buyer's indifference price of the claim $g(T)$, has a bound as follows.*

$$\begin{aligned} &\frac{1}{\gamma}\left(E_t^Q\left[-\frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right]\right. \\ &\qquad \left. - \ln E_t^Q\left[\exp\left(-\gamma g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right)\right]\right) \\ &\qquad < p^b(t) < \\ &\frac{1}{\gamma}\left(\ln E_t^Q\left[\exp\left(-\frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right)\right]\right. \\ &\qquad \left. - E_t^Q\left[-\gamma g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right]\right) \end{aligned} \tag{5.25}$$

(ii) The seller's indifference price lies in a bound as follows.

$$\begin{aligned}
&\frac{1}{\gamma}\Bigg(E_t^Q\left[\gamma g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right] \\
&\quad - \ln E_t^Q\left[\exp\left(-\frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right)\right]\Bigg) \\
&\quad < p^s(t) < \\
&\frac{1}{\gamma}\Bigg(\ln E_t^Q\left[\exp\left(\gamma g(T) - \frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right)\right] \\
&\quad - E_t^Q\left[-\frac{1}{2}\int_t^T \left(\frac{m(s)}{\sigma(s,Y(s))}\right)^2 \mathrm{d}s\right]\Bigg)
\end{aligned} \tag{5.26}$$

From Theorem 5.1, under the incomplete information, we observe that bounds of utility indifference prices include estimated drift term. Compared to Theorem 3.1 in [17], it is replaced μ with $m(s) = E_s[\mu]$.

5.2 Utility Indifference Pricing Equation

It is not clear the relationship of the buyer's indifference price and the seller's one from (5.25) and (5.26) under the assumption that the buyer of the derivative and seller have same preference (i.e., the risk-aversion parameters for the buyer and seller are equal). In this section, we derive PDEs satisfied by the indifference prices[1], then we describe the relationship between the buyer's indifference price and seller's one by finding super/sub solution of these PDEs.

Assmption 5.1. *We assume that F_Y/F is smooth and bounded, where F is given by (4.5).*

Proposition 5.3. *(i) The buyer's indifference price p^b is a unique and bounded solution of the pricing equation*

$$p_t^b + \mathcal{L}^{\langle m,S,Y\rangle} p^b + (1-\rho^2) b^2 \frac{F_Y}{F} p_Y^b = \frac{1}{2} b^2 \gamma (1-\rho^2)(p_Y^b)^2 \tag{5.27}$$

with $p^b(T) = g(T)$.

(ii) The seller's indifference price p^s is a unique and bounded solution of the pricing equation

$$p_t^s + \mathcal{L}^{\langle m,S,Y\rangle} p^s + (1-\rho^2) b^2 \frac{F_Y}{F} p_Y^s = -\frac{1}{2} b^2 \gamma (1-\rho^2)(p_Y^s)^2 \tag{5.28}$$

with $p^s(T) = g(T)$.

Proof. We only show (i). The second statement is derived by the same way. From Definition 5.1, (4.5) and (4.11), we have

$$G(t,m,S,y) = e^{-\gamma p^b} F(t,m,y). \tag{5.29}$$

Substituting (5.29) into (4.12) together with (4.6), it holds that (5.27). The uniqueness and boundedness follow by proof of Theorem 2.6 in [17]. □

[1] Sircar and Zariphopoulou [17] call this PDE the indifference pricing equation.

From Assumption 2.1 and 5.1, $\kappa(s) := -\sqrt{1-\rho^2}b(s,Y(s))\dfrac{F_Y}{F}$ is smooth and bounded. Thus, it holds that

$$\mathcal{E}^M(0,\kappa)(s) := \exp\left(-\frac{1}{2}\int_0^s \kappa^2(u)\mathrm{d}s - \int_0^s \kappa(u)\mathrm{d}M_2(u)\right) \tag{5.30}$$

is a Q-martingale.

Definition 5.3. *We define an equivalent measure $\hat{Q}$ according to*

$$\left.\frac{\mathrm{d}\hat{Q}}{\mathrm{d}Q}\right|_{\mathcal{G}_s} = \mathcal{E}^M(0,\kappa)(s)$$

for $t \leqslant s \leqslant T$.

In this time,

$$\hat{M}(s) := \begin{pmatrix} \hat{M}_1(s) \\ \hat{M}_2(s) \end{pmatrix} := \begin{pmatrix} M_1(s) \\ M_2(s) + \int_0^s \kappa(u)\mathrm{d}u \end{pmatrix} \tag{5.31}$$

is a two-dimensional standard Brownian motion under $\hat{Q}$. Under this measure, the dynamics of Y become

$$\mathrm{d}Y(s) = \left[a(s,Y(s)) + (1-\rho^2)b^2(s,Y(s))\frac{F_Y}{F}\right]\mathrm{d}s + b(s,Y(s))\left(\rho\mathrm{d}\hat{M}_1(s) + \sqrt{1-\rho^2}\mathrm{d}\hat{M}_2(s)\right).$$

Proposition 5.4. *The buyer's indifference price and seller's price have the following relationship:*

$$p^b(t) < p^s(t)$$

for $0 \leqslant t < T$.

Proof. We observe that

$$p_t^b + \mathcal{L}^{\langle m,S,Y\rangle}p^b + (1-\rho^2)b^2\frac{F_Y}{F}p_Y^b = \frac{1}{2}b^2\gamma(1-\rho^2)(p_Y^b)^2 > 0.$$

So p^b is a sub solution of q, i.e., $p^b(t,m,S,y) < q(t,m,S,y)$ where q is a solution of

$$q_t + \mathcal{L}^{\langle m,S,Y\rangle}q + (1-\rho^2)b^2\frac{F_Y}{F}q_Y = 0$$

with $q(T) = g(T)$. Under $\hat{Q}$-measure, Feynman-Kac formula gives the following representation:

$$q(t,m,S,y) = E_t^{\hat{Q}}[g(T)].$$

On the other hand, since it holds that

$$p_t^s + \mathcal{L}^{\langle m,S,Y\rangle}p^s + (1-\rho^2)b^2\frac{F_Y}{F}p_Y^s = -\frac{1}{2}b^2\gamma(1-\rho^2)(p_Y^s)^2 < 0,$$

hence p^s is a super solution of q, i.e., $p^s(t,m,S,y) > q(t,m,S,y)$. Therefore, we have

$$p^b(t,m,S,y) < E_t^{\hat{Q}}[g(T)] < p^s(t,m,S,y)$$

as required. □

Remark 5.1. *The result of Proposition 5.4 is not restricted to the incomplete information model. It might hold same relationship for other incomplete market models with respect to appropriate EMMs.*

6 Summary

In this paper, we considered the utility indifference pricing for the derivative written on the stock with stochastic volatility under incomplete information, i.e., the drift parameter of the stock is assumed to be a normally distributed random variable. In order to derive the indifference price, we should solve expected utility maximization problems, although Separating principle and Bayesian updating formula make them easy to treat. By using HJB equation for optimization problems, we obtained non-linear PDEs. Since it is so hard to find an explicit solution for such PDEs, we provide bounds of indifference prices by solve super/sub solutions of corresponding PDEs. Furthermore, under the incomplete information, bounds of utility indifference prices include estimated drift term instead the true drift parameter μ. This is the main result in this paper. Also we found the relationship between the buyer's indifference price and the seller's one by the indifference pricing equations.

References

[1] Cvitanić, J., Pham, H., & Touzi, N. (1999). Super-Replication in Stochastic Volatility Models under Portfolio Constraints. Journal of Applied Probability, 36, 523-545.

[2] Cvitanić, J., & Zapatero, F. (2004). Introduction to the Economics and Mathematics of Financial Markets. Massachusetts: MIT press.

[3] Davis, M.H.A. (1997). Option Pricing in Incomplete Markets. Mathematics of Derivative Securities (pp. 216-227), edited by Dempster, M.A.H., & Pliska, S.R.. Cambridge: Cambridge University Press.

[4] Detemple, J.B. (1986). Asset pricing in an economy with incomplete information. Journal of Finance, 61, 383-392.

[5] Detemple, J., and Selden, L. (1991). A general equilibrium analysis of option and stock market interactions. International Economic Review, 32, 279-304.

[6] Dothan, M.U. & Feldman, D. (1986). Equilibrium interest rates and multiperiod bonds in a partially observable economy. Journal of Finance, 41, 369-382.

[7] Fleming, W.H., & Rishel, R.W. (1975). Deterministic and Stochastic Optimal Control. New York: Springer-Verlag.

[8] Gennote, G. (1986). Optimal portfolio choice under information. Journal of Finance, 41, 733-746.

[9] Grasselli, M.R., & Hurd, T.R. (2005). Indifference pricing and hedging in stochastic volatility models. Technical report, McMaster University.

[10] Heston, S. (1993). A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. Review of Financial Studies, 6, 327-343.

[11] Henderson, V., & Hobson, D. (2004). Utility Indifference Pricing - An Overview. Working paper. Available at; `http://www2.warwick.ac.uk/fac/soc/wbs/subjects/finance/finance_faculty/vicky_henders/papers/Indifference_survey.pdf`

[12] Hodges, S.D., & Neuberger, A. (1989). Optimal replication of contingent claim under transaction costs. Review of Futures Markets, 8, 222-239.

[13] Ishijima, H. (1999). The Dynamic Portfolio Management under Incomplete Information. Doctoral Dissertation, Tokyo Institute of Technology.

[14] Lipster, R.S., & Shiryaev, A.N. (2001). Statistics of Random Processes II: Applications (Applied Mathematics). Berlin: Springer-Verlag.

[15] Musiela, M., & Zariphopoulou, T. (2004). An example of indifference prices under exponential preferences. Finance and Stochastics, 8, 229-239.

[16] Pham, H. (2002). Smooth Solutions to Optimal Investment Models with Stochastic Volatilities and Portfolio Constraints. Appl. Math. & Optim., 46, 55-78.

[17] Sircar, R., & Zariphopoulou, T. (2005). Bounds and Asymptotic Approximations for Utility Prices When Volatility is Random. SIAM J. Control Optim., 43, 1328-1353.

[18] Young, V.R. (2004). Pricing in an incomplete market with an affine term structure. Mathematical Finance, 14, 359-381.

[19] Young, V.R., & Zariphopoulou, T. (2002). Pricing dynamic insurance risks using the principle of equivalent utility. Scandinavian Actuarial Journal, 4, 246-279.

[20] Zariphopoulou, T. (2001). A solution approach to valuation with unhedgeable risks. Finance and Stochastics, 5, 61-82.