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Abstract

Using the standard neoclassical growth model with two types of agents, we examine how the presence of heterogenous agents affects the stabilization role of progressive income taxation. We first show that if the marginal tax payment of each agent increases with her relative income, the steady state satisfies local saddlepoint stability so that the equilibrium is determinate. However, unlike the representative agent models with progressive taxation, our model with heterogeneous agents may have the possibility of equilibrium indeterminacy. The indeterminacy conditions depend not only on the property of tax functions but also on production and preference structures.

Keywords: heterogeneous agents, progressive taxation, wealth distribution, aggregate stability

JEL Classification Code: E52, O42

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1 Introduction

It has been widely acknowledged that progressive income taxation under the balanced-budget rule is one of the most effective tools for establishing macroeconomic stability. In fact, Guo and Lansing (1998) demonstrate that progressive income taxation may eliminate sunspot-driven economic fluctuations caused by equilibrium indeterminacy even in the presence of strong degree of external increasing returns.[1] Guo and Harrison (2004) also claim that equilibrium indeterminacy in a model with regressive income taxation under a fixed government spending shown by Schmitt-Grohé and Uribe (1997) does not hold if the government spending is adjusted to keep a fixed level of income tax.

Although those findings are intuitively plausible, they are obtained in the context of representative agent models. The purpose of this paper is to reconsider the stabilization effect of progressive income taxation in a model with heterogeneous agents. For analytical clarity, we use a simple neoclassical growth model with fixed labor supply in which there are only two types of agents. Each group consists of infinitely-lived agents who have an identical time discount rate. Each group of agents, however, may have different utility functions and hold different level of initial wealth. Our main concern is to investigate how the presence of heterogeneous agents affects the stabilization effect of progressive income taxation under the balanced-budget discipline. We first examine the case in which the same rate of tax applies to both labor and capital incomes. Following Guo and Lansing (1998), we assume that the rate of tax is assumed to increase with the private income relative to the average income in the economy at large. We then consider the model with factor specific income taxation: different rates of tax apply to labor and capital incomes, respectively. Given each taxation scheme, we characterize the steady state equilibrium and explore its local stability.

We obtain three main results. First, if the marginal rate of tax is a monotonic function of the relative income, the economy has a unique steady state equilibrium where all the agents hold an identical amount of capital. Second, if the marginal tax payment of each agent increases with her relative income, then the steady state

[1] See also Guo (1999) and Dormel and Pintus (2007).

satisfies local saddlepoint stability so that the equilibrium is determinate. Third, if the marginal tax payment decreases with the relative income, then the steady state equilibrium is either unstable or locally indeterminate. In the latter, there may exist a continuum of converging paths around the steady state. It is also shown that indeterminacy of equilibrium tends to emerge when the elasticity of intertemporal substitution in consumption of each types of agents is sufficiently different from each other.

The present study is closely related to some of the existing investigations on wealth distribution in the neoclassical growth model with heterogenous agents and non-linear income taxation. Sarte (1997) first demonstrates that introducing progressive income taxation may yield a unique interior steady state even though every agent's time discount rate is different from each other. As is well known, in the standard neoclassical growth model with heterogenous households, the agent who has the lowest time discount rate ultimately owns the entire stock of capital: see Becker (1980),.Chatterjee (1994) and Sorger (2000). The presence of non-linear income taxation avoids yielding such an extreme conclusion.[2] Carroll and Young (2007) analyze stationary wealth distribution under progressive taxation when each agent's labor supply is heterogenous. While Sarte (1997) and Carroll and Young (2007) focus on the wealth distribution in the steady state equilibrium, Sorger (2002) re-examines Sarte's model and presents numerical examples of dynamic analysis in which converging equilibrium path is indeterminate around the steady state. Since Sorger (2002) assumes that the time discount rate of each agent is not identical, the wealth of each agent may not be equalized in the steady state equilibrium. Such an asymmetry in the steady state could be a source of complex behavior of the model economy. In contrast, we assume that the income tax depends on the relative level of income and the time discount rate is identical for all agents, so that the steady-state level of wealth is completely equalized. This assumption enables us to inspect the relationship between tax functions, preference structure and the dynamic behavior of the economy near the steady state in the absence of asymmetric wealth distribution.

[2]Wealth distribution may not degenerate in the steady state in an overlapping generations economy:as well see Hendiriks (2007).

Finally, it is to be pointed out that Li and Sarte (2004) consider an endogenous growth model with heterogenous agents in which the taxation rule is assumed to be the same as that used in our study. Due to the assumption of Ak technology, the model economy in Li and Sarte (2004) always stays at the balanced growth path. Thus equilibrium dynamics out of the steady state is not discussed in their study.[3]

The next section constructs an analytical framework. Section 3 characterizes the steady state equilibrium and investigates equilibrium dynamics under the uniform income tax, while Section 4 discusses the model with factor-specific taxation. Section 5 presents numerical examples. Concluding remarks are given in Section 6.

2 The Base Model

2.1 Households

There are two groups of infinitely-lived agents who have the same time discount rate. Two types of agent have different levels of initial wealth and their utility functions could be different from each other. For simplicity, population in the economy is assumed to be constant over time so that the number of agents in each group will not change. The economy is closed and the government does not issue interest bearing bonds. Thus the stock of capital is the only asset held by the households. The representative agent in group i ($i = 1, 2$) supplies one unit of labor in each moment and maximizes a discounted sum of utility

$$U_i = \int_0^\infty e^{-\rho t} u_i(c_i) dt, \quad \rho > 0, \quad i = 1, 2, \tag{1}$$

over an infinite time horizon. The flow budget constraint for each agent is

$$\dot{k}_i = \hat{r}_i k_i + \hat{w}_i - c_i + T_i, \quad i = 1, 2. \tag{2}$$

Here, k_i is capital stock owned by an agent in group i, c_i consumption, $\hat{r}_i$ after-tax rate of return to capital, $\hat{w}_i$ the after-tax real wage rate and T_i expresses a transfer

[3] García-Peñalosa and Turnovsky (2006 and 2007) study equilibrium dynamics of neoclassical as well as endogenous growth models with heterogenous agents. They, however, treat models in which all agents have an identical quasi-homothetic utility function so that wealth distribution does not affect aggregate dynamics.

from the government. The initial holding of capital, $k_i(0)$, is given. The instantaneous utility function of each type of agent, $u_i(c_i)$, is monotonically increasing, strictly concave in c_i and satisfies the Inada conditions.

2.2 Production

The representative firm produces a single good according to a constant-returns-to-scale technology given by

$$\bar{Y} = F\left(\bar{K}, N\right),$$

where $\bar{Y}$, $\bar{K}$ and N denote the total output, capital and labor, respectively. Using the homogeneity assumption, in what follows we write the production function in such a way that

$$Y = f(K),$$

where $Y \equiv \bar{Y}/N$ and $K \equiv \bar{K}/N$. The productivity function, $f(K)$, is assumed to be monotonically increasing and strictly concave in the capital-labor ratio, K, and fulfills the Inada conditions. The commodity market is competitive so that the before-tax rate of return to capital and real wage are respectively determined by

$$r = f'(K), \quad w = f(K) - Kf'(K). \tag{3}$$

For simplicity, we assume that capital does not depreciate.

If we denote the number of agents in group i by N_i $(i = 1, 2)$, then the full-employment condition for labor and capital are as follows:

$$N_1 + N_2 = N,$$

$$N_1 k_1 + N_2 k_2 = \bar{K}.$$

Letting $\theta_i = N_i/N$, we may express the full-employment conditions in the following manner:

$$K = \theta_1 k_1 + \theta_2 k_2, \quad 0 < \theta_i < 1, \quad \theta_1 + \theta_2 = 1. \tag{4}$$

For notational simplicity, we normalize the total population, N, to one. Thus θ_i represents the mass of agents of type i as well as the population share of that type.

2.3 Fiscal Rules

The government levies discretionary income taxes and distributes back its tax revenue as a transfer to each agent. In the main part of the paper, we assume that the same rate of tax applies to both capital and labor incomes. The rate of tax applies to income of an agent in group i is

$$\tau_i = \tau\left(\frac{y_i}{Y}\right), \qquad i = 1, 2,$$

where τ_i is the rate of tax and y_i $(= rk_i + w_i)$ denotes the total income of an agent in group i. Namely, the tax rate applied to each agent depends only on its standing in the economy.[4] The tax function $\tau(y_i/Y)$: $\Re_+ \to \Re_+$ is continuous, monotonically increasing, a twice differentiable function and satisfies $0 < \tau(y_i/Y) < 1$.

Denoting the amount of tax payment by $T(y_i, Y) = \tau\left(\frac{y_i}{Y}\right) y_i$, the average rate of tax is $T(y_i, Y)/y_i = \tau(y_i/Y)$ and the marginal tax payment is

$$\frac{\partial T(y_i, Y)}{\partial y_i} = \tau\left(\frac{y_i}{Y}\right) + \tau'\left(\frac{y_i}{Y}\right)\frac{y_i}{Y} \equiv T_m\left(\frac{y_i}{Y}\right).$$

Note that the ratio of marginal and average tax payments expresses the degree of progressiveness of taxation. When this measure is higher (resp. lower) than one, taxation is progressive (resp. regressive). In our formulation, progressiveness of taxation is represented by

$$\frac{T_m(y_i/Y)}{\tau(y_i/Y)} = 1 + \frac{\tau'(y_i/Y)}{\tau(y_i/Y)}\frac{y_i}{Y} > 1, \tag{5}$$

implying that taxation is progressive. Since the marginal tax payment depends on the relative income y_i/Y, we obtain Notice that the 'marginal progressiveness' of taxation is

$$T_m'\left(\frac{y_i}{Y}\right) = 2\tau'\left(\frac{y_i}{Y}\right) + \tau''\left(\frac{y_i}{Y}\right)\frac{y_i}{Y}. \tag{6}$$

If the above has a positive value, the marginal tax payment increases with the relative income. In contrast, if $T_m'(y_i/Y) < 0$ (so $\tau''(y_i/Y) < 0$), then the marginal tax payment decreases with the relative income. In what follows, we see that the sign of (6) may play a pivotal role in determining macroeconomic stability of the economy.

[4] This formulation is used by Guo and Lansing (1998) and Li and Sarte (2004).

The after-tax rate of return and real wage received by type i agents are respectively written as

$$\hat{r}_i = \left[1 - \tau\left(\frac{y_i}{Y}\right)\right] r, \qquad \hat{w}_i = \left[1 - \tau\left(\frac{y_i}{Y}\right)\right] w, \qquad i = 1, 2.$$

As a result, the flow budget constraint for the household (2) is rewritten as

$$\dot{k}_i = \left[1 - \tau\left(\frac{y_i}{Y}\right)\right] y_i - c_i + T_i, \qquad i = 1, 2.$$

We assume that the government follows the balanced-budget rule and, therefore, its flow budget constraint (in per-capita term) is

$$\theta_1 T_1 + \theta_2 T_2 = \theta_1 \tau\left(\frac{y_1}{Y}\right) y_1 + \theta_2 \tau\left(\frac{y_2}{Y}\right) y_2.$$

In addition, if we assume that the government pays back an identical amount of transfer to each agent, the lump-sum transfers of the group 1 and the group 2 are given by

$$T_1 = T_2 = \theta_1 \tau\left(\frac{y_1}{Y}\right) y_1 + \theta_2 \tau\left(\frac{y_2}{Y}\right) y_2. \tag{9}$$

2.4 Consumption and Capital Formation

Under the fiscal rules shown above, the type i agent's flow budget constraint is expressed as

$$\dot{k}_i = \left[1 - \tau\left(\frac{y_i}{Y}\right)\right] (r k_i + w) - c_i + T_i, \quad i = 1, 2, \tag{10}$$

where T_i is determined by (9) . Following Guo and Lansing (1998), we assume that the households perceive the rule of progressive taxation on private income, but she takes the transfer payment, T_i, as given. Therefore, taking anticipated sequences of $\{r(t), w(t), Y(t), T_i(t)\}_{t=0}^{\infty}$ and the initial holding of capital, $k_i(0)$, as given, the household of type i maximizes (1) subject to (10).

Using the optimization conditions and (3), we find that the optimal consumption in each moment satisfies the Euler equation such that

$$\dot{c}_i = \frac{c_i}{\sigma_i(c_i)} \left\{ \left[1 - \tau\left(\frac{y_i}{Y}\right) - \frac{y_i}{Y}\tau'\left(\frac{y_i}{Y}\right)\right] f'(K) - \rho \right\}, \quad i = 1, 2, \tag{11}$$

where $\sigma_i(c_i) = -u_i''(c_i) c_i / u_i'(c_i) (> 0)$. The optimal level of consumption should also fulfill the transversality condition

$$\lim_{t \to \infty} q_i(t) k_i(t) e^{-\rho t} = 0, \quad i = 1, 2. \tag{12}$$

Equations (9) and (10) yield

$$\dot{k}_i = \left[1 - \tau \left(\frac{y_i}{Y}\right)\right] y_i - c_i + \theta_1 \tau \left(\frac{y_1}{Y}\right) y_1 + \theta_2 \tau \left(\frac{y_2}{Y}\right) y_2 \tag{13}$$

Summing up the flow budget constraint (10) over all of the households and dividing the both sides by N, we obtain

$$\theta_1 \dot{k}_1 + \theta_2 \dot{k}_2 = \theta_1 y_1 + \theta_2 y_2 - \theta_1 c_1 - \theta_2 c_2.$$

Thus, in view of $y_i = rk_i + w$ and (4) , we obtain the final-good market equilibrium condition for the entire economy:

$$\dot{K} = f(K) - C, \tag{14}$$

where $C = \theta_1 c_1 + \theta_2 c_2$.

3 Macroeconomic Stability

3.1 Dynamic System

From (3) we obtain:

$$y_i = rk_i + w = f(K) + (k_i - K) f'(K).$$

This equation is rewritten as

$$\frac{y_i}{Y} = 1 + \frac{(k_i - K) f'(K)}{f(K)}, \quad i = 1, 2, \tag{15}$$

where $K = \theta_1 k_1 + (1 - \theta_1) k_2$. Substituting (15) into (11) and (13) , we obtain a complete dynamic system with respect to (k_1, k_2, c_1, c_2) . The solution of this dynamic system that fulfills the initial conditions on $k_1(0)$ and $k_2(0)$ as well as the transversality conditions (12) presents the perfect-foresight competitive equilibrium of our model economy.

3.2 Steady-State Equilibrium

In the steady-state equilibrium, k_i and c_i $(i = 1, 2)$ stay constant over time. From (11) and (13) we see that the steady-state conditions are given by

$$c_i^* = y_i^* + \theta_j \left[\tau \left(\frac{y_j^*}{Y^*}\right) y_j^* - \tau \left(\frac{y_i^*}{Y^*}\right) y_i^*\right], \quad i, j = 1, 2, \quad i \neq j, \tag{16}$$

$$\rho = f'(K^*)\left[1 - \tau\left(\frac{y_i^*}{Y^*}\right) - \frac{y_i^*}{Y^*}\tau'\left(\frac{y_i^*}{Y^*}\right)\right], \quad i, j = 1, 2, \tag{17}$$

where c_i^* and k_i^* denote steady-state levels of k_i and c_i.

To simplify analytical argument, we make the following assumption:

Assumption 1. $\tau\left(\frac{y_i}{Y}\right) + \frac{y_i}{Y}\tau'\left(\frac{y_i}{Y}\right)$ $(i = 1, 2)$ *is a monotonic function of the relative income,* y_i/Y.

Since the derivative of the above function with respect to y_i/Y is $2\tau'(y_i/Y) + (y_i/Y)\tau''(y_i/Y)$, from (6) Assumption 1 means that the marginal tax payment, $\partial^2(\tau y_i)/\partial y_i^2$, has the same sign for all feasible levels of y_i/Y. Given Assumption 1, it is easy to confirm the following fact:

Proposition 1. *There is a unique, symmetric steady state in which* $k_1^* = k_2^*$ and $c_1^* = c_2^*$ for $i = 1$ and 2.

Proof. Conditions displayed in (17) yield

$$\tau\left(\frac{y_1^*}{Y^*}\right) + \frac{y_1^*}{Y^*}\tau'\left(\frac{y_1^*}{Y^*}\right) = \tau\left(\frac{y_2^*}{Y^*}\right) + \frac{y_2^*}{Y^*}\tau'\left(\frac{y_2^*}{Y^*}\right).$$

By Assumption 1, the above equation holds if and only if $y_1^* = y_2^*$. Thus from (16) it holds that $c_1^* = c_2^*$. ■

Note that $y_1^* = y_2^* = Y^*$ and $k_1^* = k_2^* = K$ in the symmetric steady state, so that the rate of income tax in the steady-state equilibrium is a given constant, $\tau(1)$. To make the steady state feasible, from (17) we should assume the following:

Assumption 2. *Tax function* $\tau(y_i/Y)$ *satisfies*

$$1 - \tau(1) - \tau'(1) > 0. \tag{18}$$

It is also to be noted that the steady-state wealth distribution is uniquely determined under Assumption 1 even though $\rho_1 \neq \rho_2$. As was discussed in Li and Sarte (2004), if each agent has a different rate of time discount, the steady state conditions are given by

$$\tau\left(\frac{y_1^*}{Y^*}\right) + \frac{y_1^*}{Y^*}\tau'\left(\frac{y_1^*}{Y^*}\right) = \frac{\rho_1}{\rho_2}\left[\tau\left(\frac{y_2^*}{Y^*}\right) + \frac{y_2^*}{Y^*}\tau'\left(\frac{y_2^*}{Y^*}\right)\right],$$

$$1 = \theta_1\frac{y_1^*}{Y^*} + \theta_2\frac{y_2^*}{Y^*}.$$

It is easy to see that. given Assumption 1, the above equations present a unique levels of y_1^*/Y^* and y_2^*/Y^*. Once y_i^*/Y^* is given, (17) determines the steady-state value of aggregate capital, K^*, and thus (15) fixes the steady-state individual income y_i^*.

3.3 Stability

We are now ready to examine the local stability condition of the steady state equilibrium defined above. Linear approximation of dynamic system, (11) and (13) , around the steady state equilibrium yields the following:

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{k}_1 \\ \dot{k}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \partial\dot{c}_1/\partial k_1 & \partial\dot{c}_1/\partial k_2 \\ 0 & 0 & \partial\dot{c}_2/\partial k_1 & \partial\dot{c}_2/\partial k_2 \\ -1 & 0 & f'(k^*)[1-\theta_2(\tau(1)+\tau'(1))] & \theta_2 f'(k^*)[\tau(1)+\tau'(1)] \\ 0 & -1 & \theta_1 f'(k^*)[\tau(1)+\tau'(1)] & f'(k^*)[1-\theta_1(\tau(1)+\tau'(1))] \end{bmatrix}}_{J} \begin{bmatrix} c_1(t)-c_1^* \\ c_2(t)-c_2^* \\ k_1(t)-k_1^* \\ k_2(t)-k_2^* \end{bmatrix}.$$

Here, $\partial\dot{c}_i/\partial k_j$ $(i,j=1,2)$ evaluated at the steady state are given by

$$\begin{aligned} \frac{\partial\dot{c}_1}{\partial k_1} &= \frac{c^*}{\sigma_1(c^*)}\frac{f'(k^*)^2}{f(k^*)}[\theta_1\Gamma(k^*)(1-\tau(1)-\tau'(1))-\theta_2(\tau''(1)+2\tau'(1))], \\ \frac{\partial\dot{c}_1}{\partial k_2} &= \frac{c^*}{\sigma_1(c^*)}\frac{f'(k^*)^2}{f(k^*)}[\theta_2\Gamma(k^*)(1-\tau(1)-\tau'(1))+\theta_2(\tau''(1)+2\tau'(1))], \\ \frac{\partial\dot{c}_2}{\partial k_1} &= \frac{c^*}{\sigma_2(c^*)}\frac{f'(k^*)^2}{f(k^*)}[\theta_1\Gamma(k^*)(1-\tau(1)-\tau'(1))+\theta_1(\tau''(1)+2\tau'(1))], \\ \frac{\partial\dot{c}_2}{\partial k_2} &= \frac{c^*}{\sigma_2(c^*)}\frac{f'(k^*)^2}{f(k^*)}[\theta_2\Gamma(k^*)(1-\tau(1)-\tau'(1))-\theta_1(\tau''(1)+2\tau'(1))], \end{aligned} \tag{19}$$

where

$$\Gamma(k^*) \equiv \frac{f''(k^*)f(k^*)}{f'(k^*)^2} < 0.$$

Let us write the characteristic equation of J in such a way that

$$\lambda^4 - \mathrm{Tr}J\lambda^3 + \mathrm{W}J\lambda^2 - \mathrm{Z}J\lambda + \mathrm{Det}J = 0, \tag{20}$$

where

$$\mathrm{Tr}J = f'(k^*)[2 - \tau(1) - \tau'(1)], \tag{21a}$$

$$\mathrm{W}J = f'(k^*)\rho + \frac{\partial \dot{c}_1}{\partial k_1} + \frac{\partial \dot{c}_2}{\partial k_2}, \tag{21b}$$

$$\begin{aligned} \mathrm{Z}J = f'(k^*)\Big\{&[1 - \theta_1(\tau(1) + \tau'(1))]\frac{\partial \dot{c}_1}{\partial k_1} + [1 - \theta_2(\tau(1) + \tau'(1))]\frac{\partial \dot{c}_2}{\partial k_2} \\ &- (\tau(1) + \tau'(1))\left[\theta_1 \frac{\partial \dot{c}_1}{\partial k_2} + \theta_2 \frac{\partial \dot{c}_2}{\partial k_1}\right]\Big\}, \end{aligned} \tag{21c}$$

$$\mathrm{Det}J = -\frac{f(k^*)f'(k^*)f''(k^*)\rho}{\sigma(c_1)\sigma(c_2)}[2\tau'(1) + \tau''(1)]. \tag{21d}$$

Since our dynamic system involves two jumpable variables, c_1 and c_2, and two predetermined variables, k_1 and k_2, the presence of stable and determinate equilibrium path requires that the dynamic system exhibits a regular saddlepoint property at least around the steady state equilibrium. Inspecting the characteristic equation given above, we find one of the main results of this paper:

Proposition 2. *Given Assumptions 1 and 2, the steady state satisfies local determinacy if* $2\tau'(1) + \tau''(1) > 0$.

Proof. Let us denote roots of the characteristic equation by $\lambda_s\,(s = 1, 2, 3, 4)$. Assumption 2 means that the trace of J, which equals $\Sigma_{s=1}^4 \lambda_s$, is strictly positive, so that at least one of the characteristic roots has positive real part. In addition, if $2\tau'(1) + \tau''(1) > 0$, the determinant of J $\left(= \Pi_{s=1}^4 \lambda_s\right)$ is positive and, hence, the number of characteristic roots with positive real parts is either two or four. Note that using (19), we may write $\mathrm{Z}J$ in (21c) as

$$\begin{aligned} \mathrm{Z}J = \frac{(f')^3}{\sigma_1(c^*)\sigma_2(c^*)}&\left\{\Gamma(k^*)[1 - \tau(1) - \tau'(1)]^2[\theta_1\sigma_2(c^*) + \theta_2\sigma_1(c^*)]\right. \\ &\left. -[\theta_1\sigma_1(c^*) + \theta_2\sigma_2(c^*)][2\tau'(1) + \tau''(1)]\right\}. \end{aligned}$$

Since $2\tau'(1) + \tau''(1) > 0$ and $\Gamma(k^*) < 0$, ZJ has a negative value. Therefore, remembering that $ZJ = \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2$, we see that at least one root should be negative. Thus there are two stable roots, so that the competitive equilibrium path converging to the steady state is uniquely determined. ■

The above result means that if the marginal tax payment of each agent increases with the individual income, then the economy has a unique converging path towards the symmetric steady state equilibrium where wealth is equally distributed to each agent, regardless of the initial distribution of wealth and form of the utility function of each type of agents. In this sense, the specific form of progressive income taxation assumed in this paper may contribute to establishing income equality in the long run.

If $2\tau'(1) + \tau''(1) < 0$, the dynamic system may not exhibit a regular saddlepoint property. In this case, from $(21d)$ the determinant of J is negative, and, therefore, the number of characteristic root with negative real part is either one or three. If there is only one stable root, the steady state is locally unstable. If matrix J has three stable roots, there is a continuum of converging paths around the steady state equilibrium. In this case the economy may fluctuate around the steady state due to extrinsic uncertainty (sunspots) that affect agents' expectations formation.

Since at least one of the characteristic root is positive, the equilibrium path is indeterminate if and only if (20) has three roots with negative real parts. In this case, we may first observe the following fact:

Proposition 3. *Suppose that $2\tau'(1)+\tau''(1) < 0$. Then if agents in both groups have an identical utility function, the steady state equilibrium is asymptotically unstable.*

Proof. See Appendix A. ■

Consequently, if $2\tau'(1) + \tau''(1) < 0$, the existence of multiple equilibrium paths converging to the steady state requires that agents in each group have different forms of utility functions. We find that if each type of agent has different form of utility function, it is hard to obtain the analytical expression of sufficient conditions for the presence of three roots with negative real parts. Hence, in Section 5 we investigate numerical examples to inspect the possibility of equilibrium indeterminacy in the case of $\sigma_1(c_1) \neq \sigma_2(c_2)$.

4 Alternative Fiscal Rules

4.1 Factor-Specific Taxation

So far, we have assumed that the income tax applies to the total revenue of an individual agent. In this section we consider a more general case where the different tax scheme may apply to labor and capital income, respectively. To make the argument parallel to the previous discussion, we assume that the rate of tax levied on each factor income is given by

$$\tau_k^i = \tau_k\left(\frac{rk_i}{rK}\right) = \tau_k\left(\frac{k_i}{K}\right), \qquad \tau_w^i = \tau_w\left(1\right), \quad i = 1, 2,$$

where τ_k^i and τ_w^i respectively denote the rates of tax on capital and labor income applied to the i-th agent. Note that the wage income is the same for all agents, taxation on the wage income is flat.[5] As before, the tax function, $\tau_k\left(.\right)$, is assumed to be monotonically increasing, at least twice differentiable and satisfies, $0 < \tau_k\left(k_i/K\right) < 1$.

Then, the modified capital accumulation constraint in group i is

$$\dot{k}_i = \left[1 - \tau_k\left(\frac{k_i}{K}\right)\right] rk_i + (1 - \tau_w)w - c_i + T_i, \quad i = 1, 2, \tag{22}$$

where T_i represents the government transfer in this model. The government collects the tax revenue by the progressive income tax and returns the lump-sum transfer that amount to the share of each group. Then, the modified flow budget constraint is

$$\theta_1 T_1 + \theta_2 T_2 = \theta_1\left[rk_1\tau_k\left(\frac{k_1}{K}\right) + \tau_w\left(1\right)w\right] + \theta_2\left[rk_2\tau_k\left(\frac{k_2}{K}\right) + \tau_w\left(1\right)w\right].$$

Assuming that the government pay back an identical amount of transfer to each agent, the lump-sum transfers of each group is

$$T_1 = T_2 = \theta_1\left[rk_1\tau_k\left(\frac{k_1}{K}\right) + \tau_w\left(1\right)w\right] + \theta_2\left[rk_2\tau_k\left(\frac{k_2}{K}\right) + \tau_w\left(1\right)w\right]. \tag{23}$$

[5] This conclusion, of course, will not hold if labor-leisure choice is allowed. The distinction between capital and labor income taxation would be more crucial in the model with endogenous labor supply.

It is easy to see that under the factor-specific taxation, the Euler equation for the optimal consumption of the type i agent is given by

$$\dot{c}_i = \frac{c_i}{\sigma(c_i)} \left\{ \left[1 - \tau_k \left(\frac{k_i}{K} \right) - \frac{k_i}{K} \tau_k{}' \left(\frac{k_i}{K} \right) \right] f'(K) - \rho \right\}, \quad i = 1, 2, \tag{24}$$

where $\sigma_i = -u''(c_i)c_i/u'(c_i)(> 0)$. From equations (22) and (23), the dynamic behavior of capital stock held by the type i agents is

$$\dot{k}_i = y_i - c_i + \theta_j \left[r k_j \tau_k \left(\frac{k_j}{K} \right) - r k_i \tau_k \left(\frac{k_i}{K} \right) \right], \quad i, j = 1, 2, \quad i \neq j. \tag{25}$$

Here, K and y_i in (24) and (25) are defined by

$$K = \theta_1 k_1 + \theta_2 k_2, \quad \theta_1 + \theta_2 = 1,$$

$$y_i = f(K) + (k_i - K) f'(K).$$

The steady-state conditions under which $\dot{c}_i = \dot{k}_i = 0$ $(i = 1, 2)$ are the following:

$$c_i^* = f(K^*) + \theta_j f'(K^*) \left[k_j^* \tau_k \left(\frac{k_j^*}{K^*} \right) - k_i^* \tau_k \left(\frac{k_i^*}{K^*} \right) \right], \quad i, j = 1, 2, \quad i \neq j,$$

$$\rho = f'(K^*) \left[1 - \tau_k \left(\frac{k_i^*}{K^*} \right) - \frac{k_i^*}{K^*} \tau_k' \left(\frac{k_i^*}{K^*} \right) \right], \quad i = 1, 2.$$

If $\tau_k(.)$ function satisfies the same property given in Assumption 1, there is a unique, symmetric steady state where the, $k_1^* = k_2^* = K^*$. Consequently, the steady state conditions reduce to

$$c_1^* = c_2^* = f(K^*), \tag{26}$$

$$\rho = f'(K^*) \left[1 - \tau_k(1) - \tau_k'(1) \right]. \tag{27}$$

As before, (27) requires that

$$1 - \tau_k(1) - \tau_k{}'(1) > 0. \tag{28}$$

We can inspect local stability of dynamic system consisting of (24) and (25) in the same way as done in the previous section. The linearized system is given by

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{k}_1 \\ \dot{k}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \partial \dot{c}_1/\partial k_1 & \partial \dot{c}_1/\partial k_2 \\ 0 & 0 & \partial \dot{c}_2/\partial k_1 & \partial \dot{c}_2/\partial k_2 \\ -1 & 0 & f'(k^*)[1 - \theta_2(\tau_k(1) + \tau_k{}'(1))] & f'(k^*)\theta_2[\tau_k(1) + \tau_k{}'(1)] \\ 0 & -1 & f'(k^*)\theta_1[\tau_k(1) + \tau_k'(1)] & f'(k^*)[1 - \theta_1(\tau_k(1) + \tau_k{}'(1))] \end{bmatrix}}_{M} \begin{bmatrix} c_1 - c_1^* \\ c_2 - c_2^* \\ k_1 - k_1^* \\ k_2 - k_2^* \end{bmatrix}.$$

In this case $\partial \dot{c}_i / \partial k_j \ \ (i, j = 1, 2)$ is given by

$$
\begin{aligned}
\frac{\partial \dot{c}_1}{\partial k_1} &= \frac{c^* f'(k^*)}{\sigma_1(c^*) k^*} [\theta_1 \Pi(k^*)(1 - \tau_k(1) - \tau_k'(1)) - \theta_2 (2{\tau_k}'(1) + {\tau_k}''(1))], \\
\frac{\partial \dot{c}_1}{\partial k_2} &= \frac{c^* f'(k^*)}{\sigma_1(c^*) k^*} [\theta_2 \Pi(k^*)(1 - \tau_k(1) - \tau_k'(1)) + \theta_2 (2{\tau_k}'(1) + \tau_k''(1))], \\
\frac{\partial \dot{c}_2}{\partial k_1} &= \frac{c^* f'(k^*)}{\sigma_2(c^*) k^*} [\theta_1 \Pi(k^*)(1 - \tau_k(1) - \tau_k'(1)) + \theta_1 (2\tau_k'(1) + \tau_k''(1))], \\
\frac{\partial \dot{c}_2}{\partial k_2} &= \frac{c^* f'(k^*)}{\sigma_2(c^*) k^*} [\theta_2 \Pi(k^*)(1 - \tau_k(1) - \tau_k'(1)) - \theta_1 (2{\tau_k}'(1) + {\tau_k}''(1))],
\end{aligned}
\tag{29}
$$

where

$$
\Pi(k^*) \equiv \frac{f''(k^*) k^*}{f'(k^*)} < 0.
$$

The characteristic equation of M is given by

$$
\lambda^4 - \mathrm{Tr}M \lambda^3 + \mathrm{W}M \lambda^2 - \mathrm{Z}M \lambda + \mathrm{Det}M = 0,
$$

where

$$
\begin{aligned}
\mathrm{Tr}M &= f'(k^*)[2 - \tau_k(1) - {\tau_k}'(1)], \\
\mathrm{W}M &= f'(k^*)^2 [1 - \tau_k(1) - {\tau_k}'(1)] + \frac{\partial \dot{c}_1}{\partial k_1} + \frac{\partial \dot{c}_2}{\partial k_2}, \\
\mathrm{Z}M &= f'(k^*) \Big\{ [1 - \theta_1(\tau_k(1) + \tau'(1))] \frac{\partial \dot{c}_1}{\partial k_1} + [1 - \theta_2(\tau_k(1) + \tau_k'(1))] \frac{\partial \dot{c}_2}{\partial k_2} \\
&\quad - [\tau_k(1) + {\tau_k}'(1)] \left(\theta_2 \frac{\partial \dot{c}_2}{\partial k_1} + \theta_1 \frac{\partial \dot{c}_1}{\partial k_2} \right) \Big\}, \\
\mathrm{Det}M &= -\frac{f''(k^*) f'(k^*) f(k^*)^2 \sigma_1(c^*) \sigma(c^*) [1 - \tau_k(1) - \tau_k'(1)]}{k^*} [2{\tau_k}'(1) + {\tau_k}''(1)].
\end{aligned}
$$

We find that $\mathrm{Z}M$ given above is written as

$$
\begin{aligned}
\mathrm{Z}M = \frac{(f')^2 f}{k^* \sigma_1(c^*) \sigma_2(c^*)} \Big\{ &\Pi(k^*)[1 - \tau_k(1) - \tau_k'(1)]^2 [\sigma_2(c^*)\theta_1 + \sigma_1(c^*)\theta_2] \\
&- [\sigma_1(c^*)\theta_1 + \sigma_2(c^*)\theta_2][2{\tau_k}'(1) + {\tau_k}''(1)] \Big\},
\end{aligned}
$$

It is easy to show that if we replace $\tau(y_i/Y)$ function with $\tau_k(k_i/K)$, then Proposition 2 also holds for the case of factor-specific taxation. First, if $2{\tau_k}'(1) + {\tau_k}''(1) > 0$, there is a unique, symmetric steady state. In addition, (28) and our assumption, $2{\tau_k}'(1) + {\tau_k}''(1) > 0$, means that Tr $M > 0$, Det $M > 0$ and $\mathrm{Z}M < 0$. Therefore, as shown by the proof for Proposition 3, we may claim the following results:

Proposition 4. *Under the factor-specific income taxation, the steady-state equilibrium is uniquely given and satisfies local determinacy, if the marginal tax payment from capital income monotonically increases with relative capital holding, k_i/K.*

4.2 Government Consumption

One of the key assumptions of our discussion is that the tax revenue of the government is equally distributed back to the households as lump-sum transfers. Our main results may depend on this assumption. To check this, suppose that all the tax revenue is spent for consumption by the government. If this is the case, the flow budget constraint for the government is given by

$$\theta_1 \tau \left(\frac{y_1}{Y}\right) y_1 + \theta_2 \tau \left(\frac{y_2}{Y}\right) y_2 = G, \tag{31}$$

where G denotes the government consumption of the final goods. Since there is no transfer from the government, the budget constraint for type i agent is

$$\dot{k}_i = \left[1 - \tau\left(\frac{rk_i + w}{Y}\right)\right](rk_i + w) - c_i, \quad i = 1, 2,$$

and the aggregate dynamics of capital is

$$\dot{K} = f(K) - C - G.$$

Here, we again assume that the income tax is levied on capital and labor income uniformly.

In this case it is easy to see that the linearized dynamic system can be written as follows:

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{k}_1 \\ \dot{k}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \partial \dot{c}_1/\partial k_1 & \partial \dot{c}_1/\partial k_2 \\ 0 & 0 & \partial \dot{c}_2/\partial k_1 & \partial \dot{c}_2/\partial k_2 \\ -1 & 0 & f'(k^*)[1-(\tau(1)+\tau'(1))] & 0 \\ 0 & -1 & 0 & f'(k^*)[1-(\tau(1)+\tau'(1))] \end{bmatrix}}_{N} \begin{bmatrix} c_1(t) - c_1^* \\ c_2(t) - c_2^* \\ k_1(t) - k_1^* \\ k_2(t) - k_2^* \end{bmatrix}.$$

Inspecting this system, we first find that the stability results shown in Proposition 2 and 3 still hold:

Proposition 5. *(i) If the government consumes its tax revenue and if the tax function satisfies* $2\tau'(1) + \tau''(1) > 0$, *then the steady state equilibrium satisfies regular saddle-point stability.*

(ii) *If the government consumes its tax revenue and if* $\sigma_1(c_1^*) = \sigma_2(c_2^*)$, *then the steady-state equilibrium satisfies saddle-point stability.*

Proof. See Appendix B. ■

When $2\tau'(1) + \tau''(1) < 0$ and $\sigma_1 \neq \sigma_2$, the steady-state equilibrium is again either totally unstable or locally indeterminate. Comparing J with N, we see that each matrix involves different elements for $\partial \dot{k}_i / \partial k_j \quad (i, j = 1, 2)$. This means that sufficient conditions for equilibrium indeterminacy for J and N would be different from each other. Therefore, the introduction of government consumption affects equilibrium dynamics quantitatively rather than qualitatively.

5 Numerical Examples

In Sections 3.3 and 4.1 we have confirmed that if the marginal tax payments decreases with the relative income, the steady-state equilibrium is either locally indeterminate or totally unstable. Unless the two groups of agents have an identical utility function, it is hard to obtain analytical conditions that determine whether or not the steady state equilibrium is indeterminate. We thus present clearer conditions for equilibrium indeterminacy by examining numerical examples. In our examples the instantaneous utility function of each agent and the aggregate production function are respectively specified as

$$u_i(c_i) = \frac{c_i^{1-\sigma_i} - 1}{1 - \sigma_i}, \quad \sigma_i > 0, \quad i = 1, 2 \tag{32}$$

$$\bar{Y} = F(\bar{K}, N) = \bar{K}^{\alpha} N^{1-\alpha}, \quad 0 < \alpha < 1. \tag{33}$$

We first consider the case of uniform taxation by use of the following tax function:

$$\tau\left(\frac{y_i}{Y}\right) = \frac{(y_i/Y)^{\xi}}{b + m(y_i/Y)^{\xi}}, \tag{34}$$

where

$$b + m > 0, \quad b\xi > 0, \quad \text{and} \quad (b + \xi)^2 > b(1 + \xi) + m.$$

It is to be noted that this functional form satisfies all of our assumptions on the tax function including Assumption 1.[6] Given this tax function, the key values evaluated

[6] Guo and Lansing (1998) and Li and Sarte (2004) specify the tax function in such a way that

$$\tau\left(\frac{y_i}{Y}\right) = \tau_0 \left(\frac{y_i}{Y}\right)^{\phi}, \quad 0 < \tau_0 < 1, \quad \phi < 1.$$

at the steady state equilibrium are given by the following:

$$\tau(1) = \frac{1}{b+m} > 0,$$

$$\tau'(1) = \frac{b\xi}{(b+m)^2} > 0,$$

$$\tau''(1) = \frac{b\xi\{b(\xi-1) - m(1+\xi)\}}{(b+m)^3},$$

$$1 - \tau(1) - \tau'(1) = \frac{(b+m)^2 - b(1+\xi) - m}{(b+m)^2} > 0.$$

As for the bench mark case, we set

$$\alpha = 0.3, \quad b = 0.58, \quad m = 2.2, \quad \xi = 5.8, \quad \rho = 0.02.$$

Then the before-tax rate of return to capital, r, is 0.9756 and the rate of the income tax is 0.3579 so that $1 - \tau(1) - \tau'(1)$ has a positive value.[7] In what follows, we focus on the elasticity of intertemporal substitution in consumption, $1/\sigma_i$, as well as on the population share of each group, θ_i, in order to explore the possibility of equilibrium indeterminacy around the steady state. In so doing, we depict the region of (σ_1, σ_2) space under alternative values of θ_1 in which the characteristic equation of J has three roots with negative real parts. Figure 1 shows the boundary between instability and indeterminacy regions for the case that $\theta_1 = 0.5$.[8] As this figure demonstrates, indeterminacy emerges when σ_1 is sufficiently smaller than σ_2.[9] Notice that the figure focuses on the case that $\sigma_1 < \sigma_2$. Conversely, when σ_1 is

---

Using this functional form, we obtain

$$\frac{\partial\left(\tau\left(y_i/Y\right)y_i\right)/\partial y_i}{\tau\left(y_i/Y\right)} = 1 + \phi,$$

$$2\tau'\left(1\right) + \tau''\left(1\right) = \phi\left(\phi + 1\right).$$

Therefore, in this case $2\tau'\left(1\right) + \tau''\left(1\right)$ cannot have a negative sign, unless income taxation is regressive, i.e. $\phi < 0$. In addition, the above function monotonically increases with y_i/Y, it may violates $\tau\left(y_i/Y\right) < 1$. Function (34) is free from those problems.

[7] Since we have ignore capital depreciation, the before tax rate of return to capital in the steady state has a rather high value.

[8] To depict the graphs in Figure 1, we change σ_i from 0.1 to 5.0 with an intervals of 0.05.

[9] Additionally, if we raise α from 0.3 to a higher value such as 0.8, then indeterminacy tends to disappear.

sufficiently higher than σ_2, we also find the combinations of σ_1 and σ_2 under which local indeterminacy holds around the steady state.

Panel (a) in Figure 2 depicts the same graph as that of Figure 1 for alternative levels of θ_1. Inspecting these graphs, we may infer that the indeterminacy region first expands as θ_1 rises, and then it shrinks as θ_1 increases further. Panel (b) in Figure 1 confirms this intuition. Fixing σ_2 at 4.0, this figure depicts the relation between θ_1 and the upper bounds of σ_1 under which indeterminacy emerges. The graph indicates that the mass of one type of agent should not dominate the other to yield equilibrium indeterminacy. In fact, when θ_1 is close to either 0 or 1.0, then the economy resembles to the one with representative agent. The representative-agent economy with our taxation scheme will not exhibit multiple converging paths. In fact, if two groups are identical, the tax rate is always fixed at $\tau(1)$ even out of the steady state and the government budget satisfies $T = \tau Y$. Thus the aggregate dynamic system under our fiscal rule may be summarized as

$$\dot{C} = \frac{C}{\sigma(C)} \left[\left(1 - \tau(1) - \tau'(1) \right) f'(K) - \rho \right],$$
$$\dot{K} = f(K) - C,$$

so that the regular saddlepoint stability is guaranteed for all $\tau(1) \in [0, 1)$. Our numerical examples, therefore, mean that sufficient degree of heterogeneity is needed to hold indeterminacy. Since in our setting each agent holds an identical level of wealth at the steady state, the presence of equilibrium indeterminacy requires that there exists a large degree of heterogeneity of preferences. This fact suggests that if each type of agent has different time discount rate so that inequality of wealth distribution remains in the steady state (so that we have additional heterogeneity), then the difference in preference structure between two groups necessary for indeterminacy would be mitigated.

In the model with factor specific taxation, (34) is replaced with

$$\tau_k \left(\frac{rk_i}{rK} \right) = \frac{(k_i/K)^{\xi}}{b + m(k_i/K)^{\xi}},$$

$$\tau_w \left(\frac{wN_i}{wN} \right) = \frac{(\theta_i)^{\varepsilon}}{b' + m'(\theta_i)^{\varepsilon}}.$$

Using those tax functions, we conduct numerical experiments to obtain the graphs displayed in Figures 3 (a) and (b). As these figures show, the results are similar to the case of uniform taxation: indeterminacy tends to emerge when σ_2 is sufficiently larger than σ_1 (or σ_1 is sufficiently larger than σ_2). Panel (b), however, shows that the region of (σ_1, σ_2) in which indeterminacy holds is smaller than that in the case of uniform tax. Therefore, in our model economy, the factor-specific taxation may reduce the possibility of expectations-driven economic fluctuations caused by multiplicity of perfect-foresight competitive equilibrium.

6 Conclusion

This paper has studied equilibrium dynamics of a Ramsey economy with heterogenous agents in which income taxation is progressive. We have assumed that the rate of income tax depends on an individual taxable income relative to the average income of the economy at large and that the tax payments are equally distributed back to each agent. In this setting, it is shown that under weak restrictions on the tax function, the steady-state equilibrium is uniquely given and there exists a unique converging path at least around the steady state unless the marginal tax payment of each household diaereses with its relative income. Otherwise, the steady state is either unstable or locally indeterminate. If the latter holds, there is a continuum of converging path around the steady state, so that expectations-driven fluctuations may be present. Using numerical examples, we have confirmed that the presence of equilibrium indeterminacy requires that the elasticity of intertemporal substitution in consumption of each type of agent is sufficiently different from each other. The central message of our study is that the stabilizing power of progressive income taxation demonstrated in representative-agent models may not be always effective if there are heterogenous agents with different preferences.

The analytical framework of this paper is one of the simplest settings. We have assumed that there are only two types of agents and each agent supplies a fixed level of labor. In addition, we have focused on the symmetric steady state equilibrium in which all the agents hold the identical levels of wealth and income. Among the possible extensions of our discussion, an argent task is to introduce endogenous labor-leisure choice of the households. Such a generalization would be particularly interesting for comparing uniform taxation with factor-specific taxation discussed in Sections 3 and 4, because the factor-specific taxation may play a more prominent role when labor supply is flexible.

Appendices

Appendix A

Letting I be 4×4 unit matrix, the characteristic equitation matrix J is expressed in the following manner:

$$\det\left[I\lambda-J\right]=\det\begin{bmatrix}\lambda & 0 & -\frac{\omega}{\sigma_1}\left[\theta\Gamma\Delta-(1-\theta)\,T\right] & -\frac{\omega}{\sigma_1}(1-\theta)\left[\Gamma\Delta+T\right]\\ 0 & \lambda & -\frac{\omega}{\sigma_2}\theta\left[\Gamma\Delta+T\right] & -\frac{\omega}{\sigma_2}\left[(1-\theta)\,\Gamma\Delta-\theta T\right]\\ 1 & 0 & \lambda-f'[1-(1-\theta)\,(1-\Delta)] & -(1-\theta)\,f'[1-\Delta]\\ 0 & 1 & -\theta f'[1-\Delta] & \lambda-f'[1-\theta(1-\Delta)]\end{bmatrix}$$

$$=\det\begin{bmatrix}\begin{array}{c}-\frac{\omega}{\sigma_1}\left[\theta\Gamma\Delta-(1-\theta)\,T\right]-\lambda^2\\ +f'[1-(1-\theta)\,(1-\Delta)]\lambda\end{array} & \begin{array}{c}-\frac{\omega}{\sigma_1}(1-\theta)\left[\Gamma\Delta+T\right]\\ +(1-\theta)\,f'[1-\Delta]\lambda\end{array}\\ -\frac{\omega}{\sigma_2}\theta\left[\Gamma\Delta+T\right]+\theta f'[1-\Delta]\lambda & \begin{array}{c}-\frac{\omega}{\sigma_2}\left[(1-\theta)\,\Gamma\Delta-\theta T\right]\\ -\lambda^2+f'[1-\theta(1-\Delta)]\lambda\end{array}\end{bmatrix}$$

In the above, we define:

$$\begin{aligned}&\theta=\theta_1=1-\theta_2\quad\text{so }\theta_2=1-\theta\\ &\Gamma=\frac{f''(k^*)\,f(k^*)}{f(k^*)^2}<0,\qquad\Delta=1-\tau(1)-\tau'(1)>0,\\ &T=2\tau'(1)+\tau''(1),\quad\omega=\frac{c^*f'(k^*)^2}{f(k^*)}>0.\end{aligned}$$

It is now easy to confirm that, if $\sigma_1=\sigma_2=\sigma$, then the characteristic equation can be expressed as

$$\begin{aligned}\det\left[I\lambda-J\right]&=-\left(\lambda^2-f'\lambda+\frac{\omega}{\sigma}\Gamma\Delta\right)\Big\{-\frac{\omega}{\sigma}\left[(1-\theta)\,\Gamma\Delta-\theta T\right]-\lambda^2\\ &\quad+f'[1-\theta(1-\Delta)]\lambda+\frac{\omega}{\sigma}(1-\theta)\left[\Gamma\Delta+T\right]-(1-\theta)\,f'[1-\Delta]\lambda\Big\}\\ &=\left[\lambda^2-f'\lambda+\frac{\omega}{\sigma}\Gamma\Delta\right]\left[\lambda^2-\Delta f'\lambda-\frac{\omega}{\sigma}T\right].\end{aligned}$$

Thus the characteristic equation, $\det\left[I\lambda-J\right]=0$, is given by the following:

$$\left[\lambda^2-f'\lambda+\frac{c^*f'f''}{\sigma}\left(1-\tau-\tau'\right)\right]\left[\lambda^2-\left(1-\tau-\tau'\right)f'\lambda-\frac{c^*f'^2}{\sigma f}\left(2\tau'+\tau''\right)\right]=0$$

Notice that equation

$$\lambda^2-f'(k^*)\,\lambda+\frac{c^*f'(k^*)\,f''(k^*)}{\sigma(c^*)}\left(1-\tau(1)-\tau'(1)\right)=0$$

has one positive and one negative roots, while both roots of

$$\lambda^2 + \bigl(1 - \tau(1) - \tau'(1)\bigr) f'\lambda + \frac{c^* f'^2(k^*)}{\sigma(c^*) f(k^*)} \bigl(2\tau'(1) + \tau''(1)\bigr) = 0$$

have positive real parts under the assumption of $2\tau'(1) + \tau''(1) < 0$. Therefore, the characteristic equation of J has one negative and three roots with positive real parts, which means that there is no converging path around the steady state when the initial values of k_1 and k_2 diverge from their steady state values of k_1^* and k_2^*.

Appendix B

The characteristic equation of matrix N is

$$\det[\lambda I - N] = \begin{bmatrix} \lambda & 0 & -\frac{\omega}{\sigma_1}[\theta\Gamma\Delta - (1-\theta)T] & -\frac{\omega}{\sigma_1}(1-\theta)[\Gamma\Delta + T] \\ 0 & \lambda & -\frac{\omega}{\sigma_2}\theta[\Gamma\Delta + T] & -\frac{\omega}{\sigma_2}[(1-\theta)\Gamma\Delta - \theta T] \\ 1 & 0 & \lambda - f'(k^*)[1 - (\tau(1) + \tau'(1))] & 0 \\ 0 & 1 & 0 & \lambda - f'(k^*)[1 - (\tau(1) + \tau'(1))] \end{bmatrix}$$

$$= \det \begin{bmatrix} \begin{array}{c} -\frac{\omega}{\sigma_1}[\theta\Gamma\Delta - (1-\theta)T] - \lambda^2 \\ +f'[1 - (1-\theta)(1-\Delta)]\lambda \end{array} & -\frac{\omega}{\sigma_1}(1-\theta)[\Gamma\Delta + T] \\ -\frac{\omega}{\sigma_2}\theta[\Gamma\Delta + T] & \begin{array}{c} -\frac{\omega}{\sigma_2}[(1-\theta)\Gamma\Delta - \theta T] \\ -\lambda^2 + f'[1 - \theta(1-\Delta)]\lambda \end{array} \end{bmatrix},$$

where T, Γ, Δ, ω and θ are the same defined in Appendix A. Thus the characteristic equitation, $\det[\lambda I - N] = 0$, is given by

$$\begin{aligned} &\left[\lambda^2 - f'(1 - (1-\theta)(1-\Delta))\lambda + \frac{\omega}{\sigma_1}[\theta\Gamma\Delta - (1-\theta)T]\right] \\ &\times \left[\lambda^2 - f'(1 - \theta(1-\Delta))\lambda + \frac{\omega}{\sigma_2}(1-\theta)(\Gamma\Delta + T)\right] - \frac{\omega^2}{\sigma_1\sigma_2}(\Gamma\Delta + T)^2 \\ &= 0 \end{aligned}$$

Applying the same logic used in the proof of Proposition 1, we can confirm that this equation has two roots with negative real parts if $T = 2\tau'(1) + \tau''(1) > 0$..

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Figure 1: uniform taxation

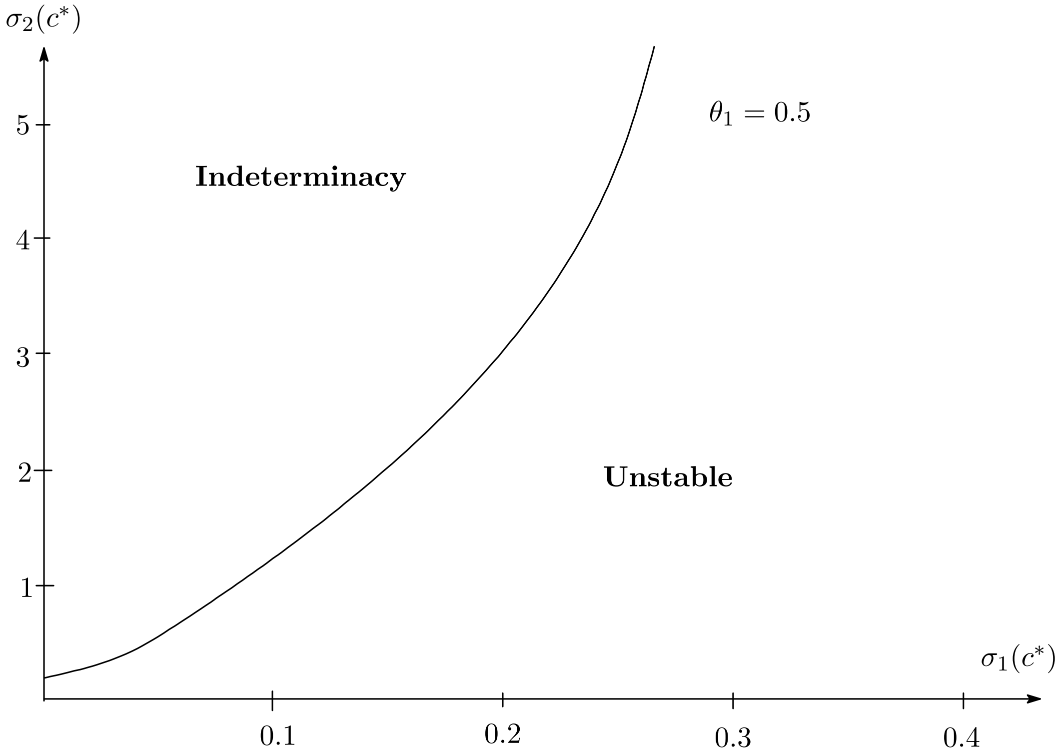

Figure 2(a): uniform taxation

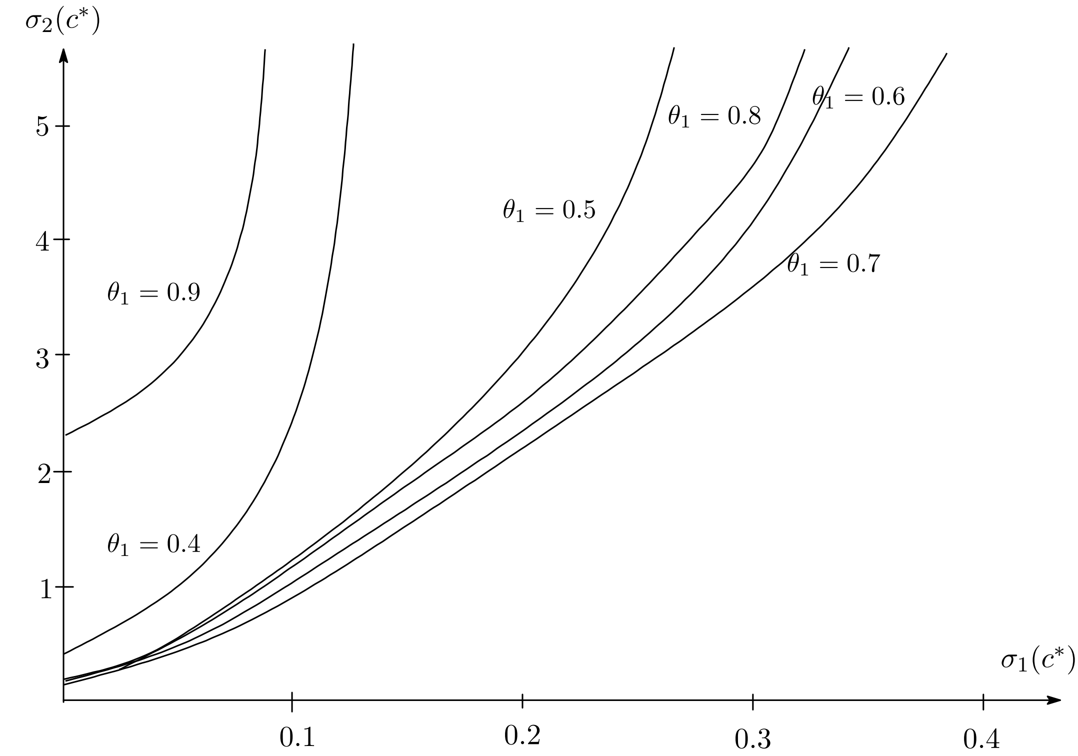

Figure 2(b): uniform taxation ($\sigma_2 = 4.0$)

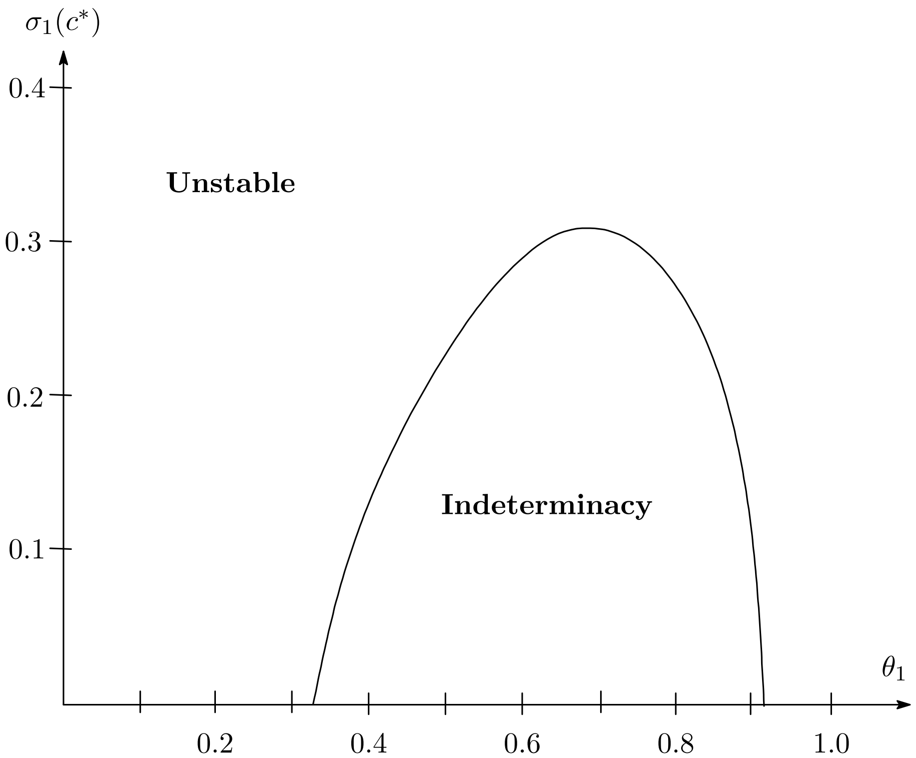

Figure 3(a): factor–specific taxation

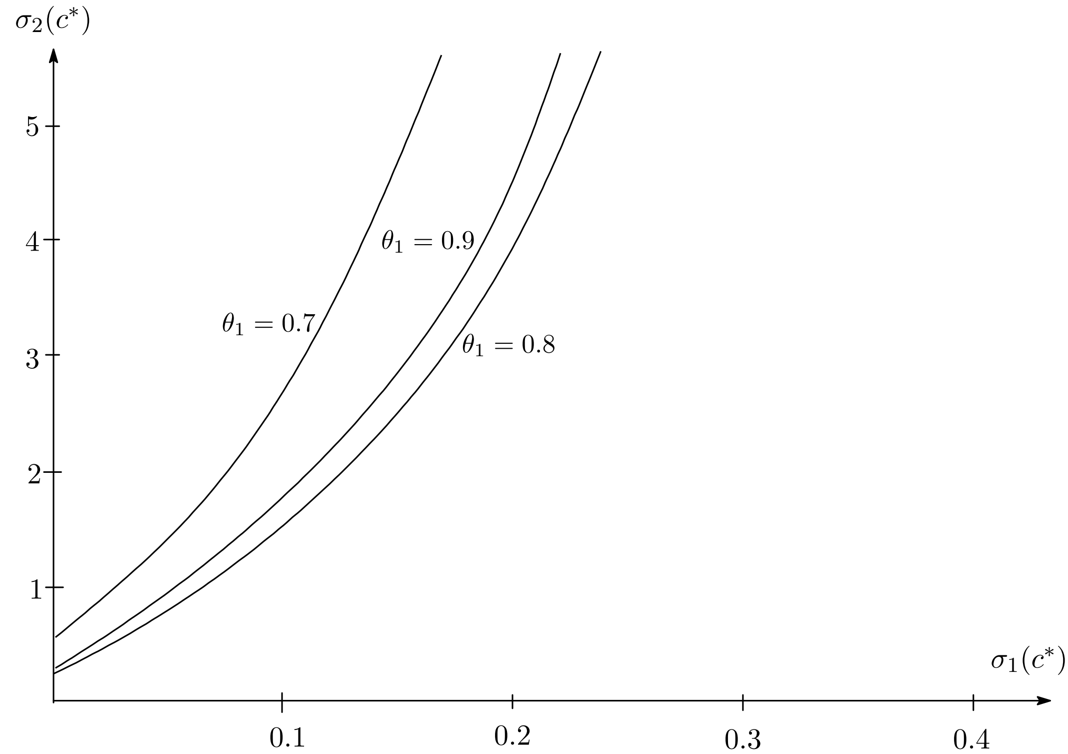

Figure 3(b): factor–specific taxation ($\sigma_2 = 4.0$)