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Abstract

We develop a model for determining whether a firm should exercise two real options individually or simultaneously. The simultaneous exercise of both options has positive synergy, such as economies of scale, scope, and networks, while separate exercise of each option benefits from project flexibility. This tradeoff determines the optimal exercise policy. We investigate the static and dynamic management of multiple real options. A firm under static management determines the type of exercise of real options ex ante; on the other hand, a firm under dynamic management makes the decision at the time of exercise. The analysis reveals the gap between the two styles of managing. Most importantly, we highlight the advantage of dynamic management over static management, particularly for weakly correlated markets. We also explain empirical implications regarding a firm's entry into several countries and regions in Asia.

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1 Introduction

The global financial crisis which began in 2007 has increased uncertainty about the future market demand in many industries throughout the world. It has become increasingly important for project managers to take into account both uncertainty and flexibility in the future. The real options approach, in which option pricing theory is applied to capital budgeting decisions, better enables us to find an optimal investment strategy and undertake project valuation in this environment than is possible under more classical methods.

The early literature has investigated a real option that has a rather simple payoff structure, assuming that dynamics of project value follow a one-dimensional stochastic process (e.g., (Dixit and Pindyck 1994)). Naturally, the studies have been developed into a more complicated real options analysis on the basis of a multidimensional process (e.g., (Geltner, et al 1996, Loubergé, et al 2002, Cortazar, et al 2008, Martzoukos 2009, Nishihara 2010)).[1] For example, (Geltner, et al 1996) investigates land development timing with an alternative land use choice, while (Loubergé, et al 2002) investigates timing in switching methods of nuclear waste disposal. These multidimensional models focus primarily on the nature of a single real option that has a complex payoff structure individually.

However, a firm typically possesses a collection of real options at the same time. Because exercising multiple real options, unlike financial options, has the potential to yield synergy, such as economies of scales, scopes, and networks, a firm faces the problem of whether to exercise multiple options individually or simultaneously. To our knowledge, this paper is the first work that attempts to capture the nature of this problem.[2] Several papers (e.g., (Meier, et al 2001, Luehrman 2004, Wang and Hwang 2007)) investigate the management of a portfolio of multiple real options in the context of project portfolio choice. For example, (Meier, et al 2001) proposes both static and dynamic zero-one optimization models for a portfolio of real options, and (Luehrman 2004) presents a conceptual framework for strategic management of real options. However, there is a large gap between these studies and the real options literature on the basis of a multidimensional stochastic process. Indeed, these papers tend to be positioned in the context of portfolio optimization rather than in the context of real options. This paper fills the gap by investigating the problem of how to manage multiple real options in terms of a multidimensional

[1] Another stream of real options development is combined with game theory. Strategic interactions among several firms are investigated in (Grenadier 1996, Grenadier 2002, Lambrecht and Perraudin 2003, Nishihara and Fukushima 2008), while agency problems in a single firm are investigated in (Mauer and Sarkar 2005, Grenadier and Wang 2005, Shibata and Nishihara 2010).

[2] Although (Trigeorgis 1993) investigates the nonadditivity of the value of multiple real options, he does not consider the problem of whether multiple real options are exercised individually or simultaneously. In addition, the analysis is based on a one-dimensional process.

stochastic model.

Our model assumes that a firm has two business opportunities in which it may invest. A firm is able to decide whether two projects are to be carried on individually or simultaneously. Investing in each project individually yields project flexibility, while initiating both projects simultaneously yields positive synergy, including economies of scale, scope, networks, etc. Taking account of this tradeoff, a firm determines the optimal type of investment.

This paper distinguishes two styles of management. One is static management. A firm under static management determines whether it exercises options individually or simultaneously ex ante. This style is likely to apply to a firm which takes a top-down approach to the management decision. The managerial flexibility also depends on the type of project. A project which requires advance preparation contingent on the type of investment forces a firm to make the management decision ex ante. Static management is related to the static optimization approach to a project portfolio choice.

The second style is dynamic management. A firm under dynamic management is capable of deciding whether it invests in projects individually or simultaneously at the time of investment. In comparison to static management, this style is likely to apply to a firm in which the management decision can be made flexibly and with a bottom-up approach. It is presumed for dynamic management that a project does not require advance preparation depending on the type of investment. Dynamic management is closely related to the dynamic optimization approach to the evaluation of an option on multiple assets.

In the model, we reveal the nature of static and dynamic management as well as the gap arising between the two. Our results regarding the exercise region of multiple options under dynamic management can be positioned as an extension of the previous findings by (Geltner, et al 1996, Broadie and Detemple 1997, Detemple 2006, Nishihara 2010). In the comparative statics, we focus on the effects of a correlation among the project values. We demonstrate that a lower correlation among the values gives a firm the incentive to invest individually rather than simultaneously. This finding is contrasted with (Childs, Ott, and Triantis 1998), which shows that a higher correlation increases the value of sequential development rather than parallel development.[3] The difference results from the model assumptions. They focus on the mutually exclusive case in which a firm invests in the development stage of two projects and then may select only a single project to implement. In contrast, we investigate the inclusive case in which a firm can receive profits from both projects. Further, and more importantly, we find that a weaker correlation increases the advantage of dynamic management over static management. This is principally because a weaker correlation increases the possibility that an ex ante choice of the investment type turns out suboptimal ex post.

[3]Their analysis is restricted within static management.

The model applies to the strategic decision concerning market entry into several countries and regions. Below, we focus on a situation in which a firm expands business into several countries and regions in Asia. Recently, an increasing number of corporations are driven by the need to expand business to Asia's markets, primarily because Asia's rapidly growing population will potentially generate the largest markets in the world. For instance, UNIQLO, the Japanese casual wear brand which has already launched operations in China in 2002, Hong Kong and South Korea in 2005, and Singapore in 2009, announced its plans to enter markets in Indonesia, Thailand, and Malaysia within a couple of years.

In expanding business into Asia, a firm must take careful consideration of the diversity which is characteristic of Asia. Even within the same country, the dynamics of the economy vary across regions. In addition to the economies, there are a wide variety of languages, ethnicities, cultural and religious prescriptions, and business practices in Asia. Naturally, a firm entering Asia's markets faces many risks that differ among countries and regions. For example, Indonesia has a risk of political instability, while China's information control greatly affects Internet businesses.

The paper demonstrates that the heterogeneity of market risk in Asia increases the incentive for a firm to enter each market individually, depending on country-specific and region-specific risks, rather than a simultaneous entry into the whole market. This argument supports the overseas expansion strategies of many firms, including UNIQLO. Further, and more importantly, we highlight the advantage of dynamic corporate management over static management for weakly correlated markets. In our view, the dynamic management capability will be a major success determinant for a business in Asia.

The paper is organized as follows. Section 2 presents the properties of the option value and the exercise policy under static management. Then, Section 3 presents those of dynamic management and reveals a gap between the two styles. Section 4 shows further properties of static and dynamic management in numerical examples. Section 5 concludes the paper.

2 Static management

2.1 Model

Consider a firm that plans two projects (denoted by projects $i = 1$ and 2) in which to invest. The risk-adjusted values of the projects, $X(t) = (X_1(t), X_2(t))$, are random and follow a bidimensional time-homogeneous diffusion process

$$\mathrm{d}X_i(t) = \mu_i(X_i(t))\mathrm{d}t + \sigma_i(X_i(t))\mathrm{d}B_i(t), \tag{1}$$

where $(B_1(t), B_2(t))$ is a bidimensional Brownian Motion (BM) with correlation coefficient ρ satisfying $|\rho| < 1$. Coefficients $\mu_i(X_i(t))$ and $\sigma_i(X_i(t))(> 0)$ denote the risk-adjusted growth rate and volatility of the project value, respectively. The firm chooses between individual and simultaneous investment. Investing in project i individually requires an irreversible capital expenditure of $I_i(>0)$, while simultaneous investment in both projects requires an irreversible capital expenditure of $I_{1,2}(>0)$. Assume that $\max(I_1, I_2) < I_{1,2} < I_1 + I_2$. This assumption means that simultaneous investment has positive synergy, including economies of scale, scope, networks, etc. Mathematically, the model is built on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ generated by $(B_1(t), B_2(t))$. The set $\mathcal{F}_t$ represents the set of available information in time t, and the firm finds the optimal policy under this information. The firm's real options are perpetual. The risk-free rate is a constant $r(>0)$.

2.2 Valuation of each option

To begin, we evaluate the option to invest in a single project i individually. For $X_i(0) = x_i$, the option value is equal to the value function of the time-homogeneous optimal stopping problem as follows:

$$V_i(x_i) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{x_i}[\mathrm{e}^{-r\tau}(X_i(\tau) - I_i)], \tag{2}$$

where $\mathcal{T}$ denotes the set of all stopping times τ and $E^{x_i}[\cdot]$ is the expectation conditional on $X_i(0) = x_i$. Note that (2) corresponds to a perpetual American call option. Under some plausible assumptions (for details, see (Peskir and Shiryaev 2006)) the optimal stopping time τ_i for problem (2) becomes $\tau_i = \inf\{t \geq 0 \mid X(t) \in S_i\}$, where the stopping region S_i is defined by

$$S_i = \{x \in \mathbb{R}^2 \mid V_i(x_i) = x_i - I_i\}. \tag{3}$$

The optimal policy is that a firm makes investment in project i as soon as $X(t)$ hits S_i.

Next, consider simultaneous investment in both projects. For $X(0) = x = (x_1, x_2)$, the option value is equal to the value function of the time-homogeneous optimal stopping problem as follows:

$$V_{1,2}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{x}[\mathrm{e}^{-r\tau}(X_1(\tau) + X_2(\tau) - I_{1,2})]. \tag{4}$$

Note that (4) corresponds to a perpetual American basket option. Under some plausible assumptions, the optimal stopping time $\tau_{1,2}$ for problem (4) can be expressed as $\tau_{1,2} = \inf\{t \geq 0 \mid X(t) \in S_{1,2}\}$, where the stopping region $S_{1,2}$ is defined by

$$S_{1,2} = \{x \in \mathbb{R}^2 \mid V_{1,2}(x) = x_1 + x_2 - I_{1,2}\}. \tag{5}$$

The optimal policy is that a firm makes simultaneous investment in projects 1 and 2 as soon as $X(t)$ hits $S_{1,2}$.

In general, the value functions $V_i(x_i)$, $V_{1,2}(x)$ and the stopping regions S_i, $S_{1,2}$ cannot be derived in any closed form. It is well known that, for $X(t)$ following either a geometric

Brownian motion (GBM) or a Brownian motion (BM) with a drift, $V_i(x_i)$ and S_i can be derived in closed forms (see (Dixit and Pindyck 1994)). First, consider the case of a GBM. Assume that $\mu_i(X_i(t)) = \mu_i X_i(t), \sigma_i(X_i(t)) = \sigma_i X_i(t), \mu_i < r$ and $X_i(0) = x_i > 0$ for $i = 1, 2$. Then we have

$$V_i(x_i) = \begin{cases} \left(\dfrac{x_i}{x_i^*}\right)^{\beta_i} (x_i^* - I_i) & (0 < x_i < x_i^*) \\ x_i - I_i & (x_i \geq x_i^*) \end{cases} \tag{6}$$

and

$$S_i = \{x \in \mathbb{R}^2_{++} \mid x_i \geq x_i^*\}, \tag{7}$$

where $\beta_i = 1/2 - \mu_i/\sigma_i^2 + \sqrt{(\mu_i/\sigma_i^2 - 1/2)^2 + 2r/\sigma_i^2}(> 1)$, and the investment threshold x_i^* is defined by

$$x_i^* = \frac{\beta_i}{\beta_i - 1} I_i. \tag{8}$$

The option value $V_{1,2}(x)$ and the stopping region $S_{1,2}$ can not be derived in any closed forms, because the sum of GBMs, $X_1(t) + X_2(t)$, does not follow a GBM. Instead, the following properties are well known (e.g., (Broadie and Detemple 1997, Detemple 2006)):
(Convexity of the value function) $V_{1,2}(x)$ is a convex function.
(Convexity of the stopping region) $S_{1,2}$ is a convex set.
(Monotonicity of the stopping region) $x \in S_{1,2} \Rightarrow x' \in S_{1,2}$ $(\forall x_1' \geq x_1, \forall x_2' \geq x_2)$.

Next, suppose that $X(t)$ follows a BM with a drift. Assume that $\mu_i(X_i(t)) = \mu_i, \sigma_i(X_i(t)) = \sigma_i$ for $i = 1, 2$. Then we have the option value

$$V_i(x_i) = \begin{cases} \mathrm{e}^{-\gamma_i(x_i^{**} - x_i)}(x_i^{**} - I_i) & (x_i < x_i^{**}) \\ x_i - I_i & (x_i \geq x_i^{**}) \end{cases} \tag{9}$$

and the stopping region

$$S_i = \{x \in \mathbb{R}^2 \mid x_i \geq x_i^{**}\}, \tag{10}$$

where $\gamma_i = -\mu_i/\sigma_i^2 + \sqrt{(\mu_i/\sigma_i^2)^2 + 2r/\sigma_i^2}(> 0)$, and investment threshold x_i^{**} is defined by

$$x_i^{**} = I_i + \frac{1}{\gamma_i}. \tag{11}$$

The option value $V_{1,2}(x)$ and the stopping region $S_{1,2}$ can also be derived in closed forms, because the sum of BMs, $Y(t) = X_1(t) + X_2(t)$, follows

$$\mathrm{d}Y(t) = (\mu_1 + \mu_2)\mathrm{d}t + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\,\mathrm{d}B_Y(t), \tag{12}$$

where $B_Y(t)$ denotes another BM. Define

$$\gamma_{1,2} = -\frac{\mu_1 + \mu_2}{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2} + \sqrt{\left(\frac{\mu_1 + \mu_2}{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\right)^2 + \frac{2r}{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}(> 0), \tag{13}$$

where $\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2 \neq 0$. We have the option value

$$V_{1,2}(x) = \begin{cases} \mathrm{e}^{-\gamma_{1,2}(x_{1,2}^{**}-(x_1+x_2))}(x_{1,2}^{**} - I_{1,2}) & (x_1 + x_2 < x_{1,2}^{**}) \\ x_1 + x_2 - I_{1,2} & (x_1 + x_2 \geq x_{1,2}^{**}) \end{cases} \tag{14}$$

and the stopping region

$$S_{1,2} = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \geq x_{1,2}^{**}\}, \tag{15}$$

where the investment threshold $x_{1,2}^{**}$ is defined by

$$x_{1,2}^{**} = I_{1,2} + \frac{1}{\gamma_{1,2}}. \tag{16}$$

The following proposition shows the comparative statics with respect to the correlation coefficient ρ.

Proposition 1 Assume that $X(t)$ follows a BM with a drift.
(Monotonicity of the value function) $V_{1,2}(x)$ monotonically increases with ρ.
(Monotonicity of the stopping region) $S_{1,2}$ monotonically decreases with ρ.

Proof By $\partial\gamma_{1,2}/\partial\rho < 0$ and $\partial V_{1,2}(x)/\partial\gamma_{1,2} \leq 0$, we have

$$\frac{\partial V_{1,2}(x)}{\partial\rho} = \frac{\partial V_{1,2}(x)}{\partial\gamma_{1,2}}\frac{\partial\gamma_{1,2}}{\partial\rho} \geq 0,$$

and

$$\frac{\partial x_{1,2}^{**}}{\partial\rho} = -\frac{1}{\gamma_{1,2}^2}\frac{\partial\gamma_{1,2}}{\partial\rho} < 0.$$

□

Proposition 1 can be interpreted as follows. The sum of two project values shows a higher volatility as the correlation between two projects increases. An increase in volatility enhances the option value, as well as the investment threshold. The properties of Proposition 1 tend to hold for a more general diffusion $X(t)$, though it is hard to prove the properties mathematically. We will check the properties numerically for a GBM in Section 4.

2.3 Static management of real options

This section considers static management. A firm under static management decides whether the projects are launched individually or simultaneously ex ante. For $X(0) = x$, firm value under static management is evaluated by

$$V_M(x) = \max\{\underbrace{V_1(x_1) + V_2(x_2)}_{\text{individual investment}}, \underbrace{V_{1,2}(x)}_{\text{simultaneous investment}}\}. \tag{17}$$

The value $V_1(x_1) + V_2(x_2)$ corresponds to the value of individual investment, while the value $V_{1,2}(x)$ corresponds to the value of simultaneous investment. When $X(0) = x$ lies in

$$S_{1,2,M} = \{x \in \mathbb{R}^2 \mid V_{1,2}(x) \geq V_1(x_1) + V_2(x_2)\}, \tag{18}$$

a firm chooses simultaneous investment ex ante and initiates both projects at the time $\tau_{1,2}$ Otherwise, it chooses individual investment ex ante and executes project i individually at the time τ_i. A favorable characteristic of static management is its simplicity, though the value is lower than that of dynamic management. Indeed, we can derive the value and the optimal exercise policy in the manner described in Section 2.2. It should be noted that the static management approach resembles project portfolio selection models. For example, the option value maximization method in (Meier, et al 2001) aims to maximize statically a value of a portfolio of real options.

Let us explore the nature of static management. As will be seen numerically in Section 4, $S_{1,2,M}$ dose not satisfy either monotonicity or convexity. Instead, we can show that $S_1 \cap S_2 \subset S_{1,2,M}$. Indeed, for any $x \in S_1 \cap S_2$, we have $V_1(x_1) + V_2(x_2) = x_1 - I_1 + x_2 - I_2 < x_1 + x_2 - I_{1,2} \leq V_{1,2}(x)$. We can also derive the boundary of $S_{1,2,M}$ for a sufficiently large x_i. For simplicity, assume that $X(t)$ follows either a GBM or a BM with a drift. For a sufficiently large x_i, we have $V_1(x_1) + V_2(x_2) = x_i - I_i + V_j(x_j)$ $(j \neq i)$ and $V_{1,2}(x) = x_1 + x_2 - I_{1,2}$ because x lies in $S_i \cap S_{1,2}$. There exists a unique solution $\hat{x_j} < x_j^*$ (or x_j^{**}) to $V_j(x_j) = x_j - I_{1,2} + I_i$ because of $0 < I_{1,2} - I_i < I_j$. Then, the boundary of $S_{1,2,M}$ coincides with a line $x_j = \hat{x_j}$. In the region $x_j \geq \hat{x_j}$ a firm chooses simultaneous investment, while in the region $x_j < \hat{x_j}$ it chooses individual investment. By proposition 1, we can show the comparative statics with respect to the correlation coefficient ρ.

Proposition 2 Assume that $X(t)$ follows a BM with a drift.
(Monotonicity of the value function) $V_M(x)$ monotonically increases with ρ.
(Monotonicity of the simultaneous investment region) $S_{1,2,M}$ monotonically increases with ρ.

Proof By proposition 1, we have $\partial V_{1,2}(x)/\partial\rho \geq 0$. Then, by (17) we have the monotonicity of $V_M(x)$ with respect to ρ. Because $V_1(x_1) + V_2(x_2)$ is independent of ρ, we also have the monotonicity of $S_{1,2,M}$ with respect to ρ. □

Proposition 2 leads to the straightforward result that a firm is more likely to make simultaneous investment in strongly correlated markets. This result can account for the overseas expansion strategies of many firms entering several countries and regions in Asia. A fine example is UNIQLO, the Japanese casual wear brand. UNIQLO has been operating in China since 2002, but has not yet planned to enter India. On the other hand, it has planned to enter Indonesia and Malaysia almost simultaneously. This is because Indonesia and Malaysia have much in common, while China and India have few similarities.

More generally, there are a wide variety of risks that differ among countries and regions in Asia. Therefore, it is commonly believed that a firm should market different products which meet country-specific and region-specific demands. We complement the conventional argument in terms of the timing of market entry. Diversity, which is a major characteristic in Asia, provides the incentive for a firm to enter each market separately. Note that the properties of Proposition 2, like Proposition 1, tend to hold for a more general diffusion $X(t)$. Relevantly, (Childs, Ott, and Triantis 1998) investigates a model where a firm invests in the development stage of two projects and then may select only a single project to implement. The model compares the values of developing the projects in sequence or in parallel. Because of the assumption of mutual exclusion, their result is in opposition to ours. In their analysis, a firm chooses sequential development rather than parallel development, when projects have highly correlated values.

3 Dynamic management

3.1 Dynamic management of real options

This section considers dynamic management. A firm under dynamic management is capable of determining whether it initiates projects individually or simultaneously at the time of investment. In comparison to static management, a firm requires managerial flexibility. For $X(0) = x$, firm value under dynamic management is evaluated by

$$\begin{aligned} V_D(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \max\{\underbrace{V_1(X_1(\tau)) + V_2(X_2(\tau))}_{\text{option to invest individually}}, \underbrace{V_{1,2}(X(\tau))}_{\text{option to invest simultaneously}}\}] \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \underbrace{V_M(X(\tau))}_{\text{static management}}], \end{aligned} \tag{19}$$

or equivalently,

$$\begin{aligned} V_D(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \max\{&\underbrace{X_1(\tau) + V_2(X_2(\tau)) - I_1}_{\text{individual investment in 1}}, \underbrace{X_2(\tau) + V_1(X_1(\tau)) - I_2}_{\text{individual investment in 2}}, \\ &\underbrace{X_1(\tau) + X_2(\tau) - I_{1,2}}_{\text{simultaneous investment}}\}]. \end{aligned} \tag{20}$$

In (20), $X_i(\tau) + V_j(X_j(\tau)) - I_i$ $(i \neq j)$ is composed of the value of individual investment in project i at the time τ, $X_i(\tau) - I_i$, and the value of the option to invest in project $j (\neq i)$ individually, $V_j(X_j(\tau))$. In (20), $X_1(\tau) + X_2(\tau) - I_{1,2}$ represents the value of simultaneous investment in both projects at the time τ. Under some plausible assumptions, the optimal stopping time τ_D for problem (20) can be expressed as $\tau_D = \inf\{t \geq 0 \mid X(t) \in S_{1,D} \cup S_{2,D} \cup S_{1,2,D}\}$, where the stopping region $S_{i,D}$ are defined by

$$S_{i,D} = \{x \in \mathbb{R}^2 \mid V_D(x) = x_i + V_j(x_j) - I_i\} \quad (j \neq i)$$

for $i = 1, 2$, and the stopping region $S_{1,2,D}$ is defined by

$$S_{1,2,D} = \{x \in \mathbb{R}^2 \mid V_D(x) = x_1 + x_2 - I_{1,2}\}.$$

We first explore the nature of dynamic management for a general diffusion $X(t)$. The following proposition shows the properties of the value function $V_D(x)$ and the stopping regions $S_{i,D}$ and $S_{1,2,D}$.

Proposition 3

$$V_M(x) \leq V_D(x),\ S_{i,D} \subset S_i,\ S_1 \cap S_2 \subset S_{1,2,D} \subset S_{1,2} \cap S_{1,2,M}$$

Proof Clearly, $V_M(x) \leq V_D(x)$ follows from (19). For any $x \in S_{1,D}$, we have

$$V_1(x_1) + V_2(x_2) \leq V_D(x) = x_1 + V_2(x_2) - I_1.$$

Then, we have $V_1(x_1) \leq x_1 - I_1$, which implies $x_1 \in S_1$. Hence, we have $S_{1,D} \subset S_1$. Similarly, we can show $S_{2,D} \subset S_2$, $S_{1,2,D} \subset S_{1,2}$, and $S_{1,2,D} \subset S_{1,2,M}$. For any $x \in S_1 \cap S_2$, we have

$$\begin{aligned}
V_D(x) &\leq \underbrace{\sup_{\tau\in\mathcal{T}} \mathbb{E}^{x_1}[\mathrm{e}^{-r\tau}(X_1(\tau) - I_1)]}_{=V_1(x_1)=x_1-I_1} + \sup_{\tau\in\mathcal{T}} \mathbb{E}^{x_2}[\mathrm{e}^{-r\tau}(X_2(\tau) - I_{1,2} + I_1)] \\
&\leq x_1 - I_1 + \underbrace{\sup_{\tau\in\mathcal{T}} \mathbb{E}^{x_2}[\mathrm{e}^{-r\tau}(X_2(\tau) - I_2)]}_{=V_2(x_2)=x_2-I_2} + \sup_{\tau\in\mathcal{T}} \mathbb{E}^{x_2}[\mathrm{e}^{-r\tau}(I_1 + I_2 - I_{1,2})] \\
&\leq x_1 - I_1 + x_2 - I_2 + I_1 + I_2 - I_{1,2} \\
&= x_1 + x_2 - I_{1,2},
\end{aligned}$$

where the last inequality implies $x \in S_{1,2,D}$, and, hence, we have $S_1 \cap S_2 \subset S_{1,2,D}$. □

For $x \in S_{1,D} \cup S_{2,D} \cup S_{1,2,D}$, $V_M(x)$ agrees with $V_D(x)$, while for $x \notin S_{1,D} \cup S_{2,D} \cup S_{1,2,D}$ $V_D(x)$ is strictly larger than $V_M(x)$. This gap measures the significance of the dynamic management capability. Note that, although dynamic management may require higher costs associated with the difficulty of the optimal exercise policy when compared with static management, the model does not assume any extra costs arising in dynamic management.

We now focus on a case where $X(t)$ follows either a GBM or a BM with a drift to show detailed properties of dynamic management. Before deriving the results, we need the following lemma.

Lemma 1 Assume that $X(t)$ follows either a GBM or a BM with a drift.

$$0 \leq V_i(x_i') - V_i(x_i) \leq x_i' - x_i \quad (x_i' \geq x_i).$$

Proof We can easily check that $0 < \mathrm{d}V_i(x_i)/\mathrm{d}x_i \leq 1$ holds for all x_i. Then, the statement follows from the mean value theorem. □

Using Lemma 1, we can show the following properties of the value function $V_D(x)$ and the stopping regions $S_{i,D}$ and $S_{1,2,D}$.

Proposition 4 Assume that $X(t)$ follows either a GBM or a BM with a drift.
(Convexity of the value function) $V_D(x)$ is a convex function.
(Convexity of the simultaneous exercise region) $S_{1,2,D}$ is a convex set.
(Monotonicity of the simultaneous exercise region) $x \in S_{1,2,D} \Rightarrow x' \in S_{1,2,D}$ $(\forall x'_1 \geq x_1, \forall x'_2 \geq x_2)$.
(Semi-monotonicity of the individual exercise regions) $x \in S_{i,D} \Rightarrow x' \in S_{i,D}$ $(\forall x'_i \geq x_i, x'_j = x_j\ (j \neq i))$.
(Behavior on the indifference lines) $x_1+V_2(x_2)-I_1 = x_2+V_1(x_1)-I_2 \geq x_1+x_2-I_{1,2}$ $\Rightarrow x \notin S_{1,D} \cup S_{2,D} \cup S_{1,2,D}$. $x_i + V_j(x_j) - I_i = x_1 + x_2 - I_{1,2} \geq x_j + V_i(x_i) - I_j$ $(j \neq i)$ $\Rightarrow x \notin S_{1,D} \cup S_{2,D} \cup S_{1,2,D}$.

Proof For simplicity, we denote the payoff function of problem (20) by[4]

$$f(x) = \max\{x_1 + V_2(x_2) - I_1, x_2 + V_1(x_1) - I_2, x_1 + x_2 - I_{1,2}, 0\}.$$

(Convexity of the value function) By the convexity of $V_i(x_i)$, the payoff function $f(x)$ is also convex. Because of the convexity of the payoff function the value function $V_D(x)$ is convex (by Proposition A.6 in (Broadie and Detemple 1997), or equivalently, Proposition 88 in (Detemple 2006)), when $X(t)$ follows a GBM. Consider $X(t)$ following a BM with a drift. Similar to the case of a GBM, we can show the convexity of the value function as follows. For any $\lambda \in (0,1), x \in \mathbb{R}$, and $y \in \mathbb{R}$, we have

$$\begin{aligned}
V_D(\lambda x + (1-\lambda)y) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau} f(\lambda(x + X(\tau)) + (1-\lambda)(y + X(\tau)))] \\
&\leq \sup_{\tau \in \mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau} \lambda f(x + X(\tau)) + \mathrm{e}^{-r\tau}(1-\lambda) f(y + X(\tau))] \qquad (21) \\
&\leq \lambda \sup_{\tau \in \mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau} f(x + X(\tau))] + (1-\lambda) \sup_{\tau \in \mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau} f(y + X(\tau))] \\
&= \lambda V_D(x) + (1-\lambda) V_D(y),
\end{aligned}$$

where we use the convexity of $f(x)$ in (21).
(Convexity of the simultaneous exercise region) Take any $\lambda \in (0,1), x \in S_{1,2,D}$, and $y \in S_{1,2,D}$. By the convexity of $V_D(x)$, we have

$$\begin{aligned}
V_D(\lambda x + (1-\lambda)y) &\leq \lambda V_D(x) + (1-\lambda) V_D(y) \\
&= \lambda(x_1 + x_2 - I_{1,2}) + (1-\lambda)(y_1 + y_2 - I_{1,2}) \\
&= \lambda x_1 + (1-\lambda) y_1 + \lambda x_2 + (1-\lambda) y_2 - I_{1,2},
\end{aligned}$$

[4]For technical reasons we define $f(x)$ as the nonnegative function. This does not matter because a firm never exercises the option which yields a negative payoff.

where the last inequality implies $\lambda x+(1-\lambda)y \in S_{1,2,D}$, and, hence, we have the convexity of the stopping region $S_{1,2,D}$.

(Monotonicity of the simultaneous exercise region) First, assume that $X(t)$ follows a GBM. Take any $x \in S_{1,2,D}$, $x_1' \geq x_1$, and $x_2' \geq x_2$.

$$
\begin{aligned}
V_D(x') &= \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(1,1)}[\mathrm{e}^{-r\tau}\max\{x_1'X_1(\tau)+V_2(x_2'X_2(\tau))-I_1, x_2'X_2(\tau)+V_1(x_1'X_1(\tau))-I_2,\\
&\quad x_1'X_1(\tau)+x_2'X_2(\tau)-I_{1,2}\}]\\
&\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(1,1)}[\mathrm{e}^{-r\tau}\max\{(x_1'-x_1)X_1(\tau)+(x_2'-x_2)X_2(\tau)+x_1X_1(\tau)+V_2(x_2X_2(\tau))-I_1,\\
&\quad (x_1'-x_1)X_1(\tau)+(x_2'-x_2)X_2(\tau)+x_2X_2(\tau)+V_1(x_1X_1(\tau))-I_2,\\
&\quad (x_1'-x_1)X_1(\tau)+(x_2'-x_2)X_2(\tau)+x_1X_1(\tau)+x_2X_2(\tau)-I_{1,2}\}] \qquad (22)\\
&\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(1,1)}[\mathrm{e}^{-r\tau}(x_1'-x_1)X_1(\tau)]+\sup_{\tau\in\mathcal{T}} \mathbb{E}^{(1,1)}[\mathrm{e}^{-r\tau}(x_2'-x_2)X_2(\tau)]\\
&\quad +\sup_{\tau\in\mathcal{T}} \mathbb{E}^{(1,1)}[\mathrm{e}^{-r\tau}\max\{x_1X_1(\tau)+V_2(x_2X_2(\tau))-I_1, x_2X_2(\tau)+V_1(x_1X_1(\tau))-I_2,\\
&\quad x_1X_1(\tau)+x_2X_2(\tau)-I_{1,2}\}] \qquad (23)\\
&= x_1'-x_1+x_2'-x_2+\underbrace{V_D(x)}_{=x_1+x_2-I_{1,2}}\\
&= x_1'+x_2'-I_{1,2},
\end{aligned}
$$

where we use Lemma 1 in (22), and the last inequality implies $x' \in S_{1,2,D}$. Hence, we have $x \in S_{1,2,D} \Rightarrow x' \in S_{1,2,D}$ $(\forall x_1' \geq x_1, \forall x_2' \geq x_2)$ in the case of a GBM.

Similarly, we can show the monotonicity in the case of a BM with a drift as follows. For any $x \in S_{1,2,D}$, $x_1' \geq x_1$, and $x_2' \geq x_2$, we have

$$
\begin{aligned}
V_D(x') &= \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau}\max\{x_1'+X_1(\tau)+V_2(x_2'+X_2(\tau))-I_1,\\
&\quad x_2'+X_2(\tau)+V_1(x_1'+X_1(\tau))-I_2, x_1'+X_1(\tau)+x_2'+X_2(\tau)-I_{1,2}\}]\\
&\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau}\max\{x_1'-x_1+x_2'-x_2+x_1+X_1(\tau)+V_2(x_2+X_2(\tau))-I_1,\\
&\quad x_1'-x_1+x_2'-x_2+x_2+X_2(\tau)+V_1(x_1+X_1(\tau))-I_2,\\
&\quad x_1'-x_1+x_2'-x_2+x_1+X_1(\tau)+x_2+X_2(\tau)-I_{1,2}\}] \qquad (24)\\
&\leq \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau}(x_1'-x_1)]+\sup_{\tau\in\mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau}(x_2'-x_2)]\\
&\quad +\sup_{\tau\in\mathcal{T}} \mathbb{E}^{(0,0)}[\mathrm{e}^{-r\tau}\max\{x_1+X_1(\tau)+V_2(x_2+X_2(\tau))-I_1,\\
&\quad x_2+X_2(\tau)+V_1(x_1+X_1(\tau))-I_2, x_1+X_1(\tau)+x_2+X_2(\tau)-I_{1,2}\}] \qquad (25)\\
&= x_1'-x_1+x_2'-x_2+\underbrace{V_D(x)}_{=x_1+x_2-I_{1,2}}\\
&= x_1'+x_2'-I_{1,2},
\end{aligned}
$$

where we use Lemma 1 in (24), and the last inequality implies $x' \in S_{1,2,D}$.

(Semi-monotonicity of the individual exercise regions) First, consider the case of a GBM. Take any $x \in S_{1,D}$, $x_1' \geq x_1$, and $x_2' = x_2$. In the same manner as the proof of the monotonicity of the simultaneous exercise region, we have

$$\begin{aligned} V_D(x') &\le (23) \\ &= x_1' - x_1 + \underbrace{V_D(x)}_{=x_1+V_2(x_2)-I_1} \\ &= x_1' + V(x_2) - I_1, \end{aligned} \tag{26}$$

where the last inequality implies $x' \in S_{1,D}$. Hence, we have $x \in S_{1,D} \Rightarrow x' \in S_{1,D}$ $(\forall x_1' \geq x_1, x_2' = x_2)$ in the case of a GBM. By the symmetry, we have the semi-monotonicity of $S_{2,D}$.

Next, assume that $X(t)$ follows a BM with a drift. Take any $x \in S_{1,D}$, $x_1' \geq x_1$, and $x_2' = x_2$. In the same manner as the proof of the monotonicity of the simultaneous exercise region, we have

$$\begin{aligned} V_D(x') &\le (25) \\ &= x_1' - x_1 + \underbrace{V_D(x)}_{=x_1+V_2(x_2)-I_1} \\ &= x_1' + V(x_2) - I_1, \end{aligned}$$

where the last inequality implies $x' \in S_{1,D}$. By the symmetry, we have the semi-monotonicity of $S_{2,D}$.

(Behavior on the indifference lines) Assume that $x_1+V_2(x_2)-I_1 = x_2+V_1(x_1)-I_2 \geq x_1 + x_2 - I_{1,2}$. Note that $x \notin S_1 \cup S_2$ because of $I_{1,2} < I_1 + I_2$. By Proposition 3, we have $x \notin S_{1,D} \cup S_{2,D}$, which implies

$$V_D(x) > x_1 + V_2(x_2) - I_1 = x_2 + V_1(x_1) - I_2 \geq x_1 + x_2 - I_{1,2}.$$

Thus, we have $x \notin S_{1,D} \cup S_{2,D} \cup S_{1,2,D}$.

Assume that $x_1 + V_2(x_2) - I_1 = x_1 + x_2 - I_{1,2} \geq x_2 + V_1(x_1) - I_2$. Note that $x \notin S_2$ because of $I_{1,2} < I_1 + I_2$. First, consider the case of a GBM. By the convexity of $V_2(x_2')$, we have

$$V_2(x_2') \geq c_1 x_2' + c_2 \quad (x_2' \in \mathbb{R}_{++}) \tag{27}$$

where

$$c_1 = \frac{\mathrm{d}V(x_2)}{\mathrm{d}x_2} = \frac{\beta_2}{x_2}\left(\frac{x_2}{x_2^*}\right)^{\beta_2}(x_2^* - I_2) \in (0,1)$$

and

$$c_2 = -x_2\frac{\mathrm{d}V(x_2)}{\mathrm{d}x_2} + V(x_2) = -(\beta_2 - 1)\left(\frac{x_2}{x_2^*}\right)^{\beta_2}(x_2^* - I_2) \in (-I_2, 0).$$

Note that the right-hand side of (27) is the first order Taylor approximation to $V_2(x_2')$ around the point x_2. In (27), the equality holds if and only if x_2' is equal to x_2. For any

$t \geq 0$, by (27) we have

$$\begin{aligned} V_D(x) &\geq \mathbb{E}^x[\mathrm{e}^{-rt} \max\{X_1(\tau) + c_1X_2(t) + c_2 - I_1, X_1(t) + X_2(t) - I_{1,2}, 0\}] \\ &\geq \underbrace{\mathrm{e}^{-rt}\mathbb{E}^x[X_1(\tau) + c_1X_2(t) + c_2 - I_1]}_{\to x_1+V_2(x_2)-I_1 \ (t\downarrow 0)} \\ &\quad + \underbrace{\mathrm{e}^{-rt}\mathbb{E}^x[\max\{(1-c_1)X_2(t) - I_{1,2} - c_2 + I_1, 0\}]}_{\downarrow 0 \ (t\downarrow 0)}. \end{aligned} \tag{28}$$

In (28), the first term $\to x_1+c_1x_2+x_2-I_1 = x_1+V_2(x_2)-I_1$ $(t \downarrow 0)$ at a finite rate while the second term $\downarrow 0$ $(t \downarrow 0)$ at a rate that increases to infinity in the limit. Note that the second term corresponds to the value of an at-the-money European call option with maturity t. Therefore, there exists some $t > 0$ such that (28) is strictly larger than $x_1 + V_2(x_2) - I_1$. This implies that $V_D(x) > x_1 + V_2(x_2) - I_1 = x_1 + x_2 - I_{1,2}$, i.e., $x \notin S_{1,D} \cup S_{2,D} \cup S_{1,2,D}$. By the symmetry, we have $x_2 + V_1(x_1) - I_2 = x_1 + x_2 - I_{1,2} \geq x_1 + V_2(x_2) - I_1 \Rightarrow x \notin S_{1,D} \cup S_{2,D} \cup S_{1,2,D}$. Similarly, using the first order Taylor approximation, we can show that the statement holds for a BM with a drift. □

Proposition 4 extends previous findings by (Geltner, et al 1996, Broadie and Detemple 1997, Bobtcheff and Villeneuve 2005) to a case allowing a convex function $V_i(x)$. Technically, the proof of the behavior on the indifference lines has been accomplished in the same manner as (Nishihara 2010). The monotonicity of $S_{1,2,D}$ leads to the straightforward prediction that simultaneously increased values of two projects encourage simultaneous investment. Although we prove neither the convexity of $S_{i,D}$ nor a property that $x \in S_{i,D} \Rightarrow x' \in S_{i,D}$ $(\forall x'_i = x_i, x'_j \leq x_j \ (j \neq i))$, these properties will be verified numerically in Section 4. Then, the increased value of either project tends to provide the incentive for a firm to undertake the better project individually. The behavior on the indifference lines means that a firm must delay the decision on the investment type when the two types of investment yield the same value. This is in line with the previous findings in the max-options analysis (e.g., (Geltner, et al 1996, Detemple 2006)). By Proposition 2, we can also show the following comparative statics with respect to the correlation coefficient ρ.

Proposition 5 Assume that $X(t)$ follows a BM with a drift.
(Monotonicity of the value function) $V_D(x)$ monotonically increases with ρ.
(Monotonicity of the exercise regions) $S_{i,D}$ and $S_{1,2,D}$ monotonically decrease with ρ.

Proof The monotonicity of $V_D(x)$ immediately follows from Proposition 2. We attach superscript ρ to $V_D(x)$ and $S_{1,D}$ to avoid confusing. For any $x \in S^{\rho}_{1,D}$ and $\rho' < \rho$, we have

$$V^{\rho'}_D(x) \leq V^{\rho}_D(x) = x_1 + V_2(x_2) - I_1,$$

which implies $x \in S^{\rho'}_{1,D}$. Similarly, we can show the monotonicity of $S_{2,D}$ and $S_{1,2,D}$. □

Note that a decrease in $S_{1,2,D}$ with ρ does not mean a decrease in the possibility that a firm chooses simultaneous investment. An increase in $V_D(x)$ with ρ results from an increase in $V_{1,2}(x)$. Then, as in static management (cf. Proposition 2), a firm tends to make simultaneous investment in strongly correlated markets. We also make a brief comment regarding the one-dimensional model by (Décamps, et al 2006). Problem (20) is similar to that of (Décamps, et al 2006), when $X_1(t)$ and $X_2(t)$ follow the same dynamics (which means that $\rho = 1$) with different initial values. In this case, as in (Décamps, et al 2006), we can derive $V_M(x)$ and $V_D(x)$ in closed forms, and it can readily demonstrate that there is no gap between $V_M(x)$ and $V_D(x)$ for a sufficiently small x. Accordingly, we recognize that an imperfect correlation is a source of the gap between static and dynamic management. This effect can be intuitively explained as follows. As the correlation becomes weaker, a static choice of the investment type is more likely to result in an incorrect choice ex post. Then, the gap between static and dynamic management increases with a weaker correlation. This result implies that managerial flexibility can be a key to success in market entry into several countries and regions in Asia with wide diversity. This view will be examined numerically in Section 4.

3.2 Extensions and limitations

This section investigates the robustness of the results in Section 3.1 with respect to changes in the model assumptions. First, we consider the effects of strategic interactions. Separate investment may entail a higher risk of rival preemption than simultaneous investment. This is because the first investment induces potential rivals to invest in the remaining business opportunity. We can incorporate this into our model as follows.[5] Assume that the second investment opportunity for project i is killed at an instantaneous rate $\lambda_i \mathrm{d}t$, where a positive constant λ_i denotes the intensity. Let $\tilde{V}_i(x)$ denotes the value function (2) for the killed process of $X_i(t)$. For $X(0) = x$, firm value under dynamic management is evaluated by

$$\begin{aligned} V_D(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \max\{X_1(\tau) + \tilde{V}_2(X_2(\tau)) - I_1, X_2(\tau) + \tilde{V}_1(X_1(\tau)) - I_2, \\ X_1(\tau) + X_2(\tau) - I_{1,2}\}]. \end{aligned} \tag{29}$$

Clearly, Proposition 3 holds for the killed problem (29). It is also clear that $\tilde{V}_i(x)$ monotonically decreases with λ_i. Then, similar to Proposition 5, we can show a monotonic decrease in $V_D(x)$ and monotonic increases in $S_{i,D}$ and $S_{1,2,D}$ with respect to λ_i. We can easily derive $\tilde{V}_i(x)$ in a closed form when $X(t)$ follows either a GBM or a BM with a drift. In this case, Propositions 4 and 5 hold for the killed problem (29). Accordingly, the results in Section 3.2 are relatively robust with respect to consideration of strategic

[5] Another approach to strategic interactions is the game-theoretic approach (e.g., (Grenadier 1996, Huisman 2001)). For example, (Nishihara 2009) investigates a duopoly real options game concerning two projects.

interactions. In addition, for $X(t)$ following a nonnegative process, including a GBM, the value function (29) approaches to $V_{1,2}(x)$ as $\lambda_i \to \infty$ $(i = 1, 2)$. This means that the threat of rival preemption provides the incentive for a firm to undertake both projects simultaneously.

So far, we assume that $I_{1,2} < I_1 + I_2$ to capture the positive synergy of simultaneous investment. However, synergy may change not only the costs but also the profits. When the value of simultaneous investment can be expressed as a linear combination of $X_1(t)$ and $X_2(t)$ with positive coefficients, few difficulties arise from the technical viewpoint. Propositions 4 and 5 hold for the case. When there is nonlinear synergy of simultaneous investment, it is mathematically difficult to show the properties of the value function and the stopping regions. In such cases, the results depend on parameter values, and we must calibrate the model carefully.

The effect of learning is another important issue that should be addressed. When a firm undertakes projects sequentially, it may benefit in learning from the first investment. From the first investment, a firm may acquire skill, know-how, reputation, etc. If this is the case, a firm will make the second investment more efficiently. We can capture the effect by assuming that the second investment requires the sunk costs $\tilde{I}_j$, which is lower than I_j.[6] As $\tilde{I}_j$ decreases, the possibility of individual investment increases. In particular, when $I_i + \tilde{I}_j$ $(i \neq j)$ decreases below $I_{1,2}$, the positive synergy of simultaneous investment is offset by the positive effect of learning in separate investment. In this case, a firm always chooses individual investment with the benefit from project flexibility.

In this paper, we consider two projects. One of its natural extensions is to take into consideration more than two projects. Because the number of combinations of projects which are undertaken simultaneously increases exponentially with the number of projects, the formulation and computation of the optimal exercise policy become much more difficult. However, the theoretical results in Propositions 3–5 remain essentially unchanged.

4 Numerical examples

This section reveals further properties of static and dynamic management in numerical examples. Assume that $X(t)$ follows a GBM. We use base parameter values as follows[7]:

$$r = 8\%, \ \mu_1 = \mu_2 = 0\%, \ \sigma_1 = \sigma_2 = 20\%. \tag{30}$$

[6] An alternative modeling for learning is the filtering approach (e.g., (Bernardo and Chowdhry 2002, Décamps, et al 2005, Shibata 2008)). Extending our model to a filtering model will be a difficult but important challenge in future work.

[7] These parameter values are similar to (Geltner, et al 1996, Detemple 2006). We carried out a lot of computations with varying parameter values and distilled robust results into this section.

For expositional purposes, we set $I_1 = I_2 = 10$, $I_{1,2} = 15$. The positive synergy of simultaneous investment is $(20 - 15)/20 = 25\%$. We can derive $V_i(x)$ in a closed form, but we must rely on numerical methods to compute $V_{1,2}(x), V_M(x)$, and $V_D(x)$. We make a bivariate lattice model[8] that approximates to a GBM, and we execute a value function iteration algorithm to compute $V_{1,2}(x), V_M(x)$, and $V_D(x)$.

First, we explore the nature of static management. Table 1 shows the option values $V_i(x), V_{1,2}(x), V_M(x) = \max\{V_1(x) + V_2(x), V_M(x)\}, V_D(x)$ and the investment threshold x_i^* with varying levels of the correlation coefficient ρ. We set $x = (10, 10)$, which is the same as the sunk cost $I_1 = I_2 = 10$ for individual investment. Note that $V_i = 16.404$ and $x_i^* = 1.802$ do not depend on ρ. For $\rho = -0.5$, $x = (10, 10)$ lies in the stopping region $S_{1,2,D}$, and, hence, $V_{1,2}(x) = V_M(x) = V_D(x) = x_1 + x_2 - I_{1,2} = 5$ holds. For other levels of ρ, $x = (10, 10)$ lies in $S_{1,2,M} \setminus (S_{1,D} \cap S_{2,D} \cap S_{1,2,D})$, which means that $V_{1,2}(x) = V_M(x) < V_D(x)$. In Table 1, the value of the option to invest simultaneously, $V_{1,2}(x)$, monotonically increases with ρ. This is because a higher ρ makes $X_1(t) + X_2(t)$ more volatile and increases the option value. Figure 1 illustrates the stopping region $S_{1,2}$ for the basket option. As mentioned in Section 2.2, $S_{1,2}$ satisfies the convexity and the monotonicity. We see a monotonic decrease in $S_{1,2}$ with respect to ρ. This is analogous to the monotonic increase in $V_{1,2}(x)$ with respect to ρ. This monotonicity of $V_{1,2}(x)$ and $S_{1,2}$ is the same as the case of a BM with a drift (cf. Proposition 1).

Figure 4 illustrates the simultaneous investment region under static management, $S_{1,2,M}$, with varying levels of ρ. The region $S_{1,2,M}$ monotonically decreases with ρ because of the monotonicity of $V_{1,2}(x)$. Similar to the case of a BM with a drift (cf. Proposition 2), a firm is more likely to make simultaneous investment in strongly correlated markets. This result is consistent with empirical observations. For a large x_i, as mentioned in Section 2.3, the boundary of $S_{1,2,M}$ coincides with a line $x_j = \hat{x_j} = 5.2729$ $(j \neq i)$, where $\hat{x_j}$ is a unique solution to $V_j(x_j) = x_j - I_{1,2} + I_i$. For a large ρ, $S_{1,2,M}$ shows neither monotonicity nor convexity. As ρ approaches 100%, $S_{1,2,M}$ approaches the region $a_1 \leq x_1/x_2 \leq a_2$ for a small x, where a_i is a constant. This is because the proportion of x_1 to x_2 matters to a GBM. In contrast, the difference between x_1 and x_2 matters to a BM with a drift. Then, when ρ is very large, $S_{1,2,M}$ is like the region $a_1 \leq x_1 - x_2 \leq a_2$ for a small x. The patterns of static management are revealed first by our analysis.

Next, we explore the nature of dynamic management. Figure 3 illustrates the stopping regions $S_{i,D}$ and $S_{1,2,D}$ for $\rho = 0\%$. For comparison, we plot $S_{1,2,M}$ under static management. We can check all of the properties in Propositions 3 and 4. The convexity and monotonicity of $S_{i,D}$ can be also verified numerically. The continuation region becomes much smaller as x_i increases. Indeed, we see that the continuation region approaches to a

[8] We make a discretization with 100 time steps per 1 year following a bivariate version of the lattice binomial method (see (Boyle 1988)).

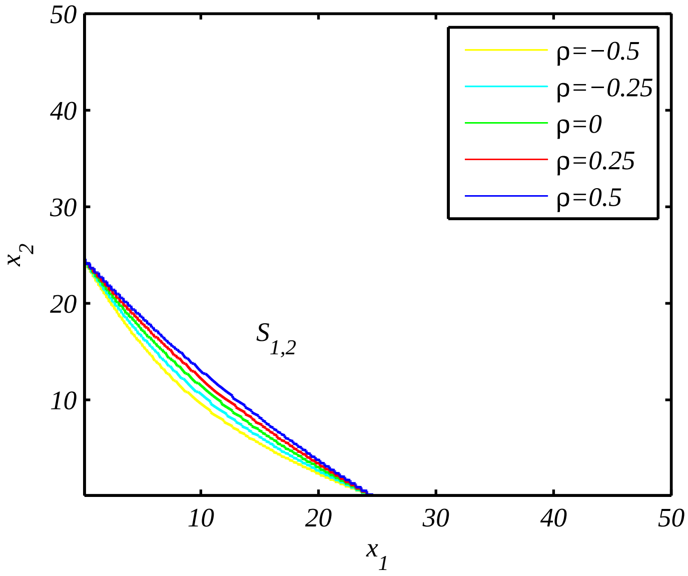

Figure 1: The exercise region for the option to invest simultaneously. This figure plots $S_{1,2}$ with varying levels of ρ. The parameter values are set at the base case (30).

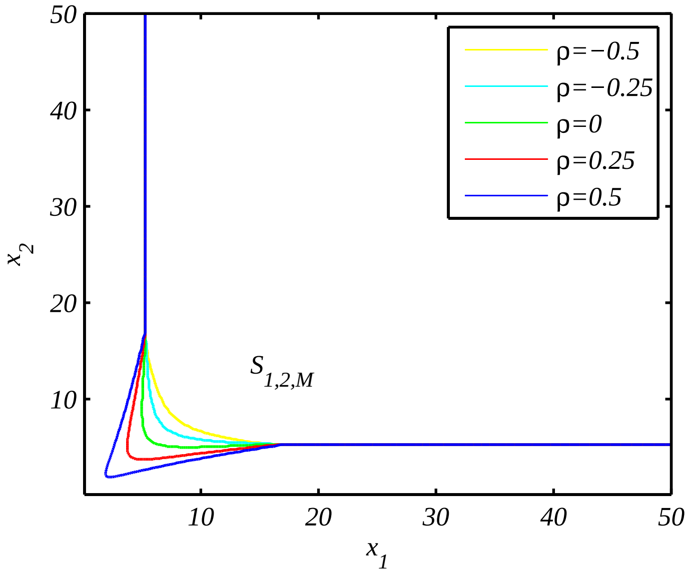

Figure 2: The simultaneous investment region under static management. This figure plots $S_{1,2,M}$ with varying levels of ρ. The parameter values are set at the base case (30).

Table 1: The option values.

$\rho\backslash$ Value	x_i^*	$V_i(x_i)$	$V_{1,2}(x)$	$V_M(x)$	$V_D(x)$
-50%	16.404	1.802	5	5	5
-25%	16.404	1.802	5.023	5.023	5.054
0%	16.404	1.802	5.118	5.118	5.15
25%	16.404	1.802	5.239	5.239	5.259
50%	16.404	1.802	5.371	5.371	5.38

line $x_j = \hat{x_j} = 5.2729$ $(j \neq i)$ when $x_i \to \infty$. This means that for a large x_i, a firm undertakes investment in a short time, whether individually or simultaneously, even if the two types of investment have the same value. This finding contrasts with that of max-option analysis (Geltner, et al 1996, Detemple 2006, Nishihara 2010), in which the waiting time is rather long when two values are equivalent. Figure 4 shows the comparative statics of $S_{i,D}$ and $S_{1,2,D}$ with respect to ρ. A monotonic decrease in $S_{1,2,D}$ can be seen clearly, while $S_{i,D}$ is robust with respect to changes in ρ (cf. Proposition 5). This is because changes in ρ influence $V_{1,2}(x)$ rather than $V_i(x)$. Table 1 shows that $V_D(x)$, like $V_M(x)$, monotonically increases with ρ. This follows from a monotonic increase in $V_{1,2}(x)$ with respect to ρ.

We proceed with an analysis of a gap between dynamic and static management. Unless x lies in the stopping regions $S_{i,D}$ and $S_{1,2,D}$, $V_D(x)$ is strictly higher than $V_M(x)$. This gap measures the impact of the managerial flexibility on firm value. Figures 5 and 6 plot contour lines of $V_D(x)/V_M(x)$. Figure 5 shows the comparative statics of $V_D(x)/V_M(x)$ with respect to ρ. We see from each panel that $V_D(x)/V_M(x)$ becomes large on the boundary of $S_{1,2,M}$ for a small x. This finding can be interpreted as follows. When the option values of individual and simultaneous investment are similar, a firm must wait and see which type is more efficient. In this case, there is a remarkable advantage to dynamic corporate management over static management. Otherwise, it does not matter whether a firm chooses the investment type statically or dynamically.

From Figure 5, we recognize that $V_D(x)/V_M(x)$ is rather robust with respect to changes in ρ. To examine it more accurately, $V_D(x)/V_M(x)$ tends to be higher when ρ approaches 0%. This is mainly because a weaker correlation increases the possibility that an initial decision of whether to exercise options individually or simultaneously leads to inefficiency ex post. This result predicts that the dynamic management capability will be a major success determinant for a firm's expansion into Asia's emerging markets which involve a variety of risks depending on countries and regions. The gap $V_D(x)/V_M(x)$ also depends on r, μ_i, and σ_i. For example, Figure 5 shows the comparative statics with respect to σ_i. We see that $V_D(x)/V_M(x)$ decreases with σ_i. This is because the positive synergy of simultaneous investment becomes smaller relative to the option value, which greatly increases with σ_i. Similarly, an increase in μ and a decrease in r enhance the option value and then reduce $V_D(x)/V_M(x)$.

5 Conclusion

The paper proposed a model for management of multiple real options to fill a great gap between the project portfolio and real options literature. In particular, we focused on the problem of whether a firm should exercise two real options individually or simultaneously. The model assumes that simultaneous exercise of both options has positive synergy, such as economies of scale, scope, and networks. The analysis revealed the characteristics of two styles of management, namely static and dynamic management. A firm under static management determines whether it exercises options individually or simultaneously ex ante, while a firm under dynamic management makes the choice at the time of exercise. We verified the natural intuition that a lower correlation among project values gives a firm the incentive to invest individually rather than simultaneously. Further, and more importantly, we emphasized the significance of dynamic corporate management to a firm entering weakly correlated markets. The model applies to a firm's strategic decision on business expansion into several countries and regions in Asia. Our results imply that the heterogeneous dynamics of Asia's markets across countries and regions increase the incentive for a firm to enter each market individually rather than the whole market simultaneously. Further, our results imply that the dynamic management capability will be a major success determinant for a business in Asia.

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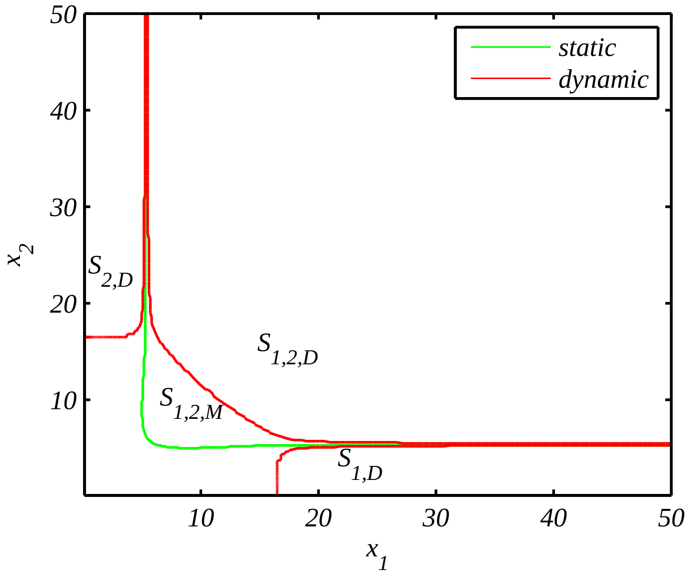

Figure 3: Static and dynamic management. This figure plots the stopping regions $S_{i,D}$ and $S_{1,2,D}$ under dynamic management along with $S_{1,2,M}$ under static management. The parameter values are set at the base case (30) with $\rho = 0\%$.

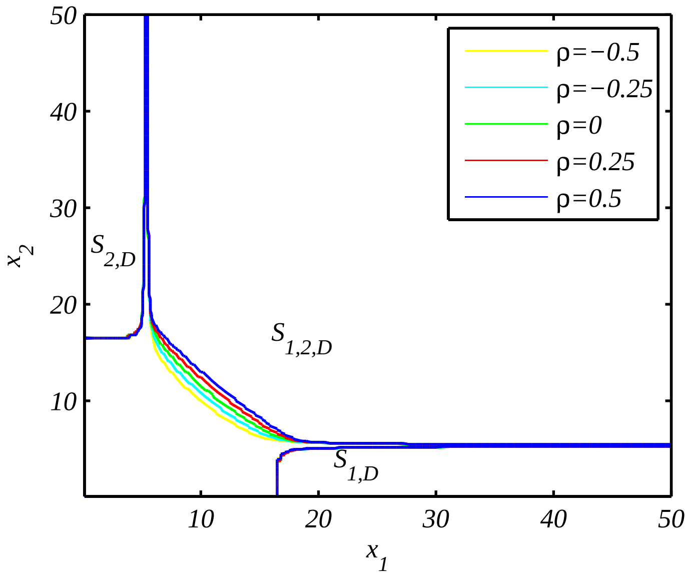

Figure 4: The exercise regions under dynamic management. This figure plots $S_{i,D}$ and $S_{1,2,D}$ with varying levels of ρ. The parameter values are set at the base case (30).

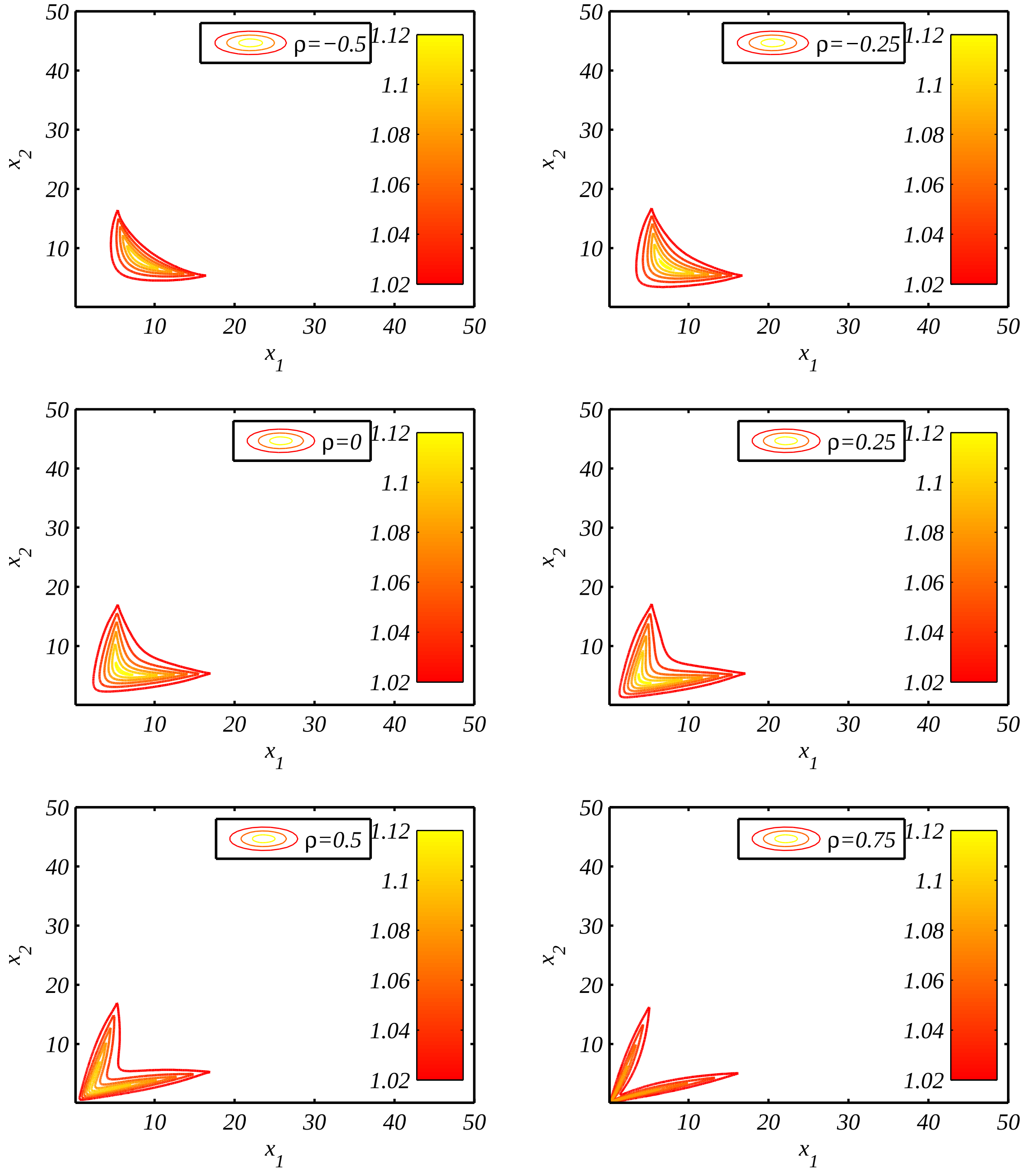

Figure 5: The gap between static and dynamic management. These panels plot contour lines of $V_D(x)/V_M(x)$ for $\rho = -50\%, -25\%, 0\%, 25\%, 50\%$, and 75%. The contour levels are set at $1.02, 1.04, 1.06, 1.08, 1.1$, and 1.12. The parameter values are set at the base case (30).

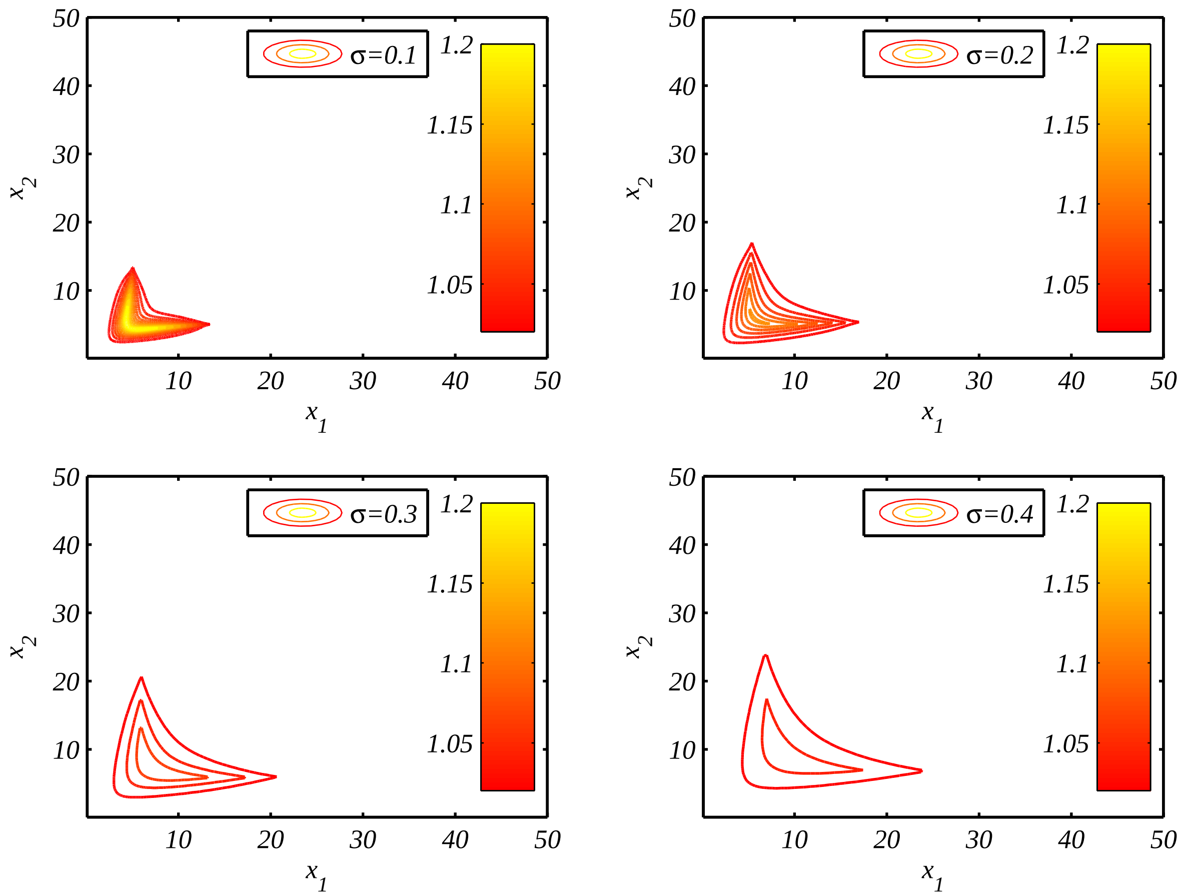

Figure 6: The gap between static and dynamic management. These panels plot contour lines of $V_D(x)/V_M(x)$ for $\sigma_1 = \sigma_2 = 10\%, 20\%, 30\%,$ and 40%. The contour levels are set at $1.02, 1.04, 1.06, \ldots, 1.2$. The other parameter values are set at the base case (30) with $\rho = 0\%$.

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