



# **Discussion Papers In Economics And Business**

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# Evaluating the occurrence and disappearance of real options\*

Michi NISHIHARA<sup>†</sup>

## Abstract

This paper investigates the decision-making of a firm that has an option to invest in a single project among multiple alternatives. This type of option is called a max-option, and the nature of a max-option has been investigated in several papers. I extend the previous analysis to a model that allows the random occurrence and disappearance of alternative projects in which to invest. The occurrence and disappearance of opportunities in which to invest will be caused by changes in regulation, the exit and entry of rival firms, technological innovation, political risk, catastrophes, etc. For example, the model applies to land development with an alternative land use choice under uncertainty about changes in zoning and development regulations. I show the properties of the exercise region and the value function for the option. Specifically, I demonstrate that the prospective future occurrence of an alternative has the significant effect of increasing the option value and deferring the investment decision.

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**Keywords:** real option, max-option, exercise region, regulatory risk, Poisson arrival.

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# 1 Introduction

The real options approach, in which option pricing theory is applied to capital budgeting decisions, better enables us to find the optimal investment strategy and undertake project valuation under uncertainty than is possible under more classical methods. The early literature has investigated a real option that has a rather simple payoff structure, assuming that the dynamics of project value follow a one-dimensional stochastic process (see (Dixit and Pindyck 1994)). Naturally, the studies have developed into a more complicated analysis on the basis of a multidimensional process (e.g., (Geltner, Riddiough, and Stojanovic 1996, Loubergé, Villeneuve, and Chesney 2002, Martzoukos 2009, Bastian-Pinto, Brandão, and Alves 2010, Nishihara 2010)). For example, (Geltner, Riddiough, and Stojanovic 1996) investigates land development timing with an alternative land use choice. In (Geltner, Riddiough, and Stojanovic 1996), the option's payoff depends on the maximum of several underlying asset prices. This type of option is called a max-option, which has been also investigated in (Stulz 1982, Johnson 1987, Broadie and Detemple 1997, Villeneuve 1999, Detemple 2006).<sup>1</sup>

This paper extends the previous max-option literature to a model that allows the Poisson arrival or death of an alternative project in which to invest. The uncertain occurrence or disappearance of the investment opportunity will be caused by changes in regulation, the exit and entry of rival firms<sup>2</sup>, technological innovation, political risk, catastrophes, etc. The model captures these risks in addition to uncertainty about the future market values of projects. For example, the model applies to land development with an alternative land use choice under uncertainty about changes in zoning and development regulations. Especially in an emerging country, regulatory and political changes frequently happen, and hence, a firm is required to evaluate the option value and optimize the investment strategy taking account of these risks.

Conventionally, numerous studies on financial derivatives have modeled the catastrophic risks as a stock price following a discontinuous stochastic process with jumps. Most of the real options studies have followed this convention. For example, (Martzoukos and Trigeorgis 2002) investigates both a growth option and an extension option involving various types of rare events, assuming the underlying asset value follows a jump diffusion process. (Boyarchenko 2004) presents the option values and the optimal investment strategies for both one-shot investment with fixed costs and incremental capacity expansion, assuming the underlying asset value follows a geometric Lévy process.

A distinction between this paper and the previous works is that this paper directly models the possibility that an opportunity in which to invest occurs or disappears. This

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<sup>1</sup>(Nishihara 2010) reveals the nature of a combination of a max-option and a spread option.

<sup>2</sup>An alternative approach is the game-theoretic approach. Strategic interactions among several firms are investigated in (Grenadier 1996, Grenadier 2002, Weeds 2002, Lambrecht and Perraudin 2003) among others.

direct approach can provide a simpler and more appropriate valuation of the real options than the previous approach. The model directly captures the effects of the uncertain disappearance or occurrence of an alternative. To my knowledge, this is the first paper that reveals the interactions among the random disappearance or occurrence of an alternative project, investment timing, and project choice. Technically, this study links the standard option to the max-option via the Poisson arrival or death. In other words, I investigate an option that may change to the max-option by the Poisson arrival and a max-option that may change to the standard option by the Poisson death.

In the model, I reveal how the possibility of the occurrence or disappearance influences the optimal exercise policy and the option value. Naturally, a higher intensity of the occurrence (disappearance) plays the role in increasing (decreasing) the value of the option to defer the investment timing and discouraging (encouraging) investment. In addition, the properties, such as the monotonicity and convexity shown by the max-option literature (e.g., (Geltner, Riddiough, and Stojanovic 1996, Broadie and Detemple 1997, Detemple 2006)), hold true even if the occurrence or disappearance of the alternative are taken into consideration. The result ensures the robustness of these properties.

Furthermore, the numerical analysis reveals the following characteristics. The possibility of uncertain changes influences the option value and the optimal investment policy for the option that may change to the max-option greater than the max-option that may change to the standard option. Indeed, for the rational parameter values, the prospective future occurrence of an alternative has the potential to enhance the option value by almost 50%. The effect becomes larger, especially for a weaker or negative correlation between the project values, because a weaker or negative correlation increases the value of the max-option that may appear in future.

This paper also entails real-world implications. For example, the results offer rational explanations for the behavior of an owner of farmland which has not been cultivated in many years. Recently, an increase in idled farmland has been a serious issue in Japan. Typically, idled farmland is restricted within the agricultural use because the zoning and development ordinances prohibit nonagricultural uses. However, prospective future regulatory and environmental changes may enable an owner to develop land for residential or commercial uses. That is, an owner of idled farmland has the option that may change to the max-option. In addition to the relatively high value of residential or commercial land, the weak correlation between the alternative use and the agricultural use increases the option value and deters an owner from cultivating farmland.

## 2 Preliminaries

Consider a firm that has an option to invest in a project. There are two exclusive projects  $i = 1$  and  $2$ . The risk-adjusted values of the projects,  $X(t) = (X_1(t), X_2(t))$ , are random and follow GBMs (Geometric Brownian Motion)

$$dX_i(t) = \mu_i X_i(t) dt + \sigma_i X_i(t) dB_i(t), \quad (1)$$

where  $B_1(t), B_2(t)$  are Brownian Motions (BM) with correlation coefficient  $\rho$  satisfying  $|\rho| < 1$ . Constants  $\mu_i$  and  $\sigma_i (> 0)$  denote the risk-adjusted growth rate and volatility of the project value, respectively. Investing in project  $i$  requires an irreversible capital expenditure of  $I_i (> 0)$ . Mathematically, the model is built on the filtered probability space  $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$  generated by  $(B_1(t), B_2(t))$ . The set  $\mathcal{F}_t$  represents the set of available information at time  $t$ , and the firm optimizes the investment policy under this information. The risk-free rate is a constant  $r (> 0)$ . For convergence, I assume that  $r > \mu$ .<sup>3</sup> The maturity of the options is  $T (> 0)$ .

### 2.1 Standard option

As a benchmark, I consider a firm that has an option to invest in a given project  $i$ . A firm cannot invest in project  $j (\neq i)$ . For  $X_i(t) = x_i$ , the option value is equal to the value function of the optimal stopping problem as follows:<sup>4</sup>

$$V_i(x_i, t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{x_i} [1_{\{\tau \leq T\}} e^{-r(\tau-t)} (X_i(\tau) - I_i)], \quad (2)$$

where  $\mathcal{T}_t$  denotes the set of all stopping times  $\tau \geq t$  and  $E_t^{x_i}[\cdot]$  denotes the expectation conditional on  $X_i(t) = x_i$ . The subscript  $i$  denotes the option to invest in project  $i$ . Note that problem (2) is analogous to an American call option written on a stock with a dividend. The optimal stopping time for problem (2) becomes  $\tau_i(t) := \inf\{s \geq t \mid X(s) \in S_i^i(s)\}$ , where the stopping region  $S_i^i(s)$  is defined by

$$S_i^i(s) := \{x \in \mathbb{R}_{++}^2 \mid V_i(x_i, s) = x_i - I_i\}. \quad (3)$$

The superscript  $i$  means the immediate exercise region for project  $i$ . The optimal policy is that a firm makes investment in project  $i$  as soon as  $X(s)$  hits  $S_i^i(s)$ . The following properties are well known (e.g., (Detemple 2006)):

**(Convexity of the value function)**  $V_i(x_i, t)$  is convex with respect to  $x_i$ .

**(Monotonicity of the stopping region)**  $x \in S_i^i(t) \Rightarrow x' \in S_i^i(t) (\forall x'_i \geq x_i, \forall x'_j \in \mathbb{R}_{++})$ , where  $i \neq j$ .

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<sup>3</sup>Refer to (Dixit and Pindyck 1994) for the economic rationale for this assumption.

<sup>4</sup>When the maturity is infinite, I have only to replace  $1_{\{\tau \leq T\}}$  with  $1_{\{\tau < \infty\}}$ .

The monotonicity implies that  $S_i^i(t)$  can be expressed as  $S_i^i(t) = \{x \in \mathbb{R}_{++}^2 \mid x_i \geq x_i^*(t)\}$ , where  $x_i^*(t)$  denotes the threshold. This type of optimal policy is called the threshold policy.

Next, consider a case in which the investment opportunity for project  $i$  is killed at an instantaneous rate  $\lambda dt$ , where a positive constant  $\lambda$  denotes the intensity of the Poisson death. I assume that the disappearance is independent of  $X(t)$ . The disappearance of an opportunity in which to invest will be caused by the enforcement of new regulations, preemption by rival firms, political changes, natural disasters, etc. For  $X_i(t) = x_i$ , the option value prior to the death is equal to the value function of the optimal stopping problem as follows:

$$V_{i \rightarrow \emptyset}(x_i, t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{x_i} \left[ \int_0^\infty 1_{\{\tau \leq T\}} 1_{\{\tau < t+y\}} e^{-r(\tau-t)} (X_i(\tau) - I_i) \lambda e^{-\lambda y} dy \right], \quad (4)$$

where the subscript  $i \rightarrow \emptyset$  means that project  $i$  may be killed. I have

$$\begin{aligned} & \mathbb{E}_t^{x_i} \left[ \int_0^\infty 1_{\{\tau \leq T\}} 1_{\{\tau < t+y\}} e^{-r(\tau-t)} (X_i(\tau) - I_i) \lambda e^{-\lambda y} dy \right] \\ &= \mathbb{E}_t^{x_i} \left[ 1_{\{\tau \leq T\}} e^{-r(\tau-t)} (X_i(\tau) - I_i) \int_{\tau-t}^\infty \lambda e^{-\lambda y} dy \right] \\ &= \mathbb{E}_t^{x_i} \left[ 1_{\{\tau \leq T\}} e^{-(r+\lambda)(\tau-t)} (X_i(\tau) - I_i) \int_0^\infty \lambda e^{-\lambda y} dy \right] \\ &= \mathbb{E}_t^{x_i} \left[ 1_{\{\tau \leq T\}} e^{-(r+\lambda)(\tau-t)} (X_i(\tau) - I_i) \right]. \end{aligned} \quad (5)$$

Then, problem (4) can be rewritten as follows:

$$V_{i \rightarrow \emptyset}(x_i, t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{x_i} \left[ 1_{\{\tau \leq T\}} e^{-(r+\lambda)(\tau-t)} (X_i(\tau) - I_i) \right]. \quad (6)$$

That is, the problem with the random disappearance is equivalent to the standard problem (2) with the augmented discount rate  $r + \lambda$ . This has also been shown in (Weeds 2002). Then, problem (6) satisfies the same properties, such as the convexity of the value function and the monotonicity of the stopping region, as problem (2). Until the death the option value follows the continuous process  $V_{i \rightarrow \emptyset}(X_i(t), t)$ , and it jumps downward to zero at the random death. In addition, because of  $V_{i \rightarrow \emptyset}(x_i, t) \leq V_i(x_i, t)$ , the stopping region  $S_{i \rightarrow \emptyset}^i(t)$  for problem (6) is larger than  $S_i^i(t)$ . Note that  $V_{i \rightarrow \emptyset}(x_i, t) \downarrow \max\{x_i - I_i, 0\}$ ,  $S_{i \rightarrow \emptyset}^i(t) \uparrow \{x \in \mathbb{R}_{++}^2 \mid x_i > I_i\}$  ( $\lambda \uparrow \infty$ ) and  $V_{i \rightarrow \emptyset}(x_i, t) \uparrow V_i(x_i, t)$ ,  $S_{i \rightarrow \emptyset}^i(t) \downarrow S_i^i(t)$  ( $\lambda \downarrow 0$ ).

Now, consider a case in which the investment opportunity for project  $i$  occurs at an instantaneous rate  $\lambda dt$ , where a positive constant  $\lambda$  denotes the intensity of the Poisson arrival. I assume that the occurrence is independent of  $X(t)$ . The occurrence of an opportunity in which to invest will be caused by deregulation, the exit of rival firms, technological innovation, etc. For  $X_i(t) = x_i$ , the option value prior to its arrival is equal to the following expectation:

$$V_{\emptyset \rightarrow i}(x_i, t) := \mathbb{E}_t^{x_i} \left[ \int_0^\infty e^{-ry} V_i(X_i(t+y), t+y) \lambda e^{-\lambda y} dy \right], \quad (7)$$

where the subscript  $\emptyset \rightarrow i$  means that project  $i$  may be available in future. A firm waits for the arrival of the option, and after the arrival it adopts the optimal policy for problem (2). At the time of arrival, the option value jumps upward from  $V_{\emptyset \rightarrow i}(X_i(t), t)$  to  $V_i(X_i(t), t)$ . Note that  $V_{\emptyset \rightarrow i}(x_i, t) \uparrow V_i(x_i, t)$  ( $\lambda \uparrow \infty$ ) and  $V_{\emptyset \rightarrow i}(x_i, t) \downarrow 0$  ( $\lambda \downarrow 0$ ).

## 2.2 Max-option

As a benchmark, this subsection considers an option to invest in a single project between projects 1 and 2. The model applies not only to a case in which two projects are exclusive (e.g., alternative land use in (Geltner, Riddiough, and Stojanovic 1996)) but also to a case where a firm faces a budget constraint. This type of option is identified as an American max-option. European max-options have been studied in (Stulz 1982, Johnson 1987), while American max-options have been studied in (Geltner, Riddiough, and Stojanovic 1996, Broadie and Detemple 1997, Villeneuve 1999, Detemple 2006). Although a typical max-option has a multidimensional state variable, (Décamps, Mariotti, and Villeneuve 2006) studies a max-option based on a one-dimensional state variable in order to investigate investment timing with an alternative scale choice.

For  $X(t) = x$ , the option value is equal to the value function of the optimal stopping problem as follows:

$$V_{1,2}(x, t) := \sup_{\tau \in \mathcal{I}_t} \mathbb{E}_t^x [1_{\{\tau \leq T\}} e^{-r(\tau-t)} \underbrace{\max_{i=1,2} \{X_i(\tau) - I_i\}}_{\text{project choice}}], \quad (8)$$

where the subscript 1, 2 represents the option to invest between projects 1 and 2.

The optimal stopping time for problem (2) becomes  $\tau_{1,2}(t) := \inf\{s \geq t \mid X(s) \in S_{1,2}(s) = S_{1,2}^1(s) \cup S_{1,2}^2(s)\}$ , where the stopping region  $S_{1,2}^i(s)$  is defined by

$$S_{1,2}^i(s) := \{x \in \mathbb{R}_{++}^2 \mid V_{1,2}(x, s) = x_i - I_i\} \quad (9)$$

for  $i = 1$  and  $2$ . The max-option literature (e.g., (Geltner, Riddiough, and Stojanovic 1996, Broadie and Detemple 1997)) shows the following properties:

**(Convexity of the value function)**  $V_{1,2}(x, t)$  is convex with respect to  $x$ .

**(Convexity of each stopping region)**  $S_{1,2}^i(t)$  is a convex set.

**(Monotonicity of each stopping region)**  $x \in S_{1,2}^i(t) \Rightarrow x' \in S_{1,2}^i(t)$  ( $\forall x'_i \geq x_i, \forall x'_j \leq x_j$ ), where  $i \neq j$ .

**(Behavior on the indifference line)**  $x_1 - I_1 = x_2 - I_2 \Rightarrow x \notin S_{1,2}(t)$ .

The monotonicity of  $S_{1,2}^i(t)$  implies that an increase (decrease) in the value of project  $i$  ( $j \neq i$ ) facilitates investment in project  $i$ . The behavior on the indifference line means that a firm waits and sees which project is better when the values of two projects equal.



### 3 Main Results

This section links the standard option in Section 2.1 and the max-option in Section 2.2 via a random variable distributed exponentially. Section 3.1 investigates a max-option that may change to the standard option, while Section 3.2 investigates an option that may change to the max-option. I show several properties of the option values and exercise regions. The model applies to the decision-making process about land development with an alternative land use choice under regulatory risks.

#### 3.1 Max-option that will change to the standard option

As in Section 2.2, consider an option to invest between projects 1 and 2. Assume that the investment opportunity for project 2 is killed at an instantaneous rate  $\lambda dt$  which is independent of  $X(t)$ .<sup>5</sup> For  $X(t) = x$ , the option value is equal to the value function of the optimal stopping problem as follows:<sup>6</sup>

$$V_{1,2 \rightarrow 1}(x, t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1,2} \{X_i(\tau) - I_i\} + \mathbf{1}_{\{\tau \geq t+y\}} e^{-ry} V_1(X_1(t+y), t+y) \} \lambda e^{-\lambda y} dy \right], \quad (10)$$

where  $y$  denotes the term until the death. The subscript  $1, 2 \rightarrow 1$  represents the max-option that may change to the standard option to invest in project 1.

Using the similar calculation to (5), I can rewrite problem (10) as

$$V_{1,2 \rightarrow 1}(x, t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x \left[ \mathbf{1}_{\{\tau \leq T\}} e^{-(r+\lambda)(\tau-t)} \max_{i=1,2} \{X_i(\tau) - I_i - V_{\emptyset \rightarrow 1}(X_1(\tau), \tau)\} \right] + V_{\emptyset \rightarrow 1}(x_1, t), \quad (11)$$

where  $V_{\emptyset \rightarrow 1}(x_1, t)$  is defined by (7). The optimal stopping time for problem (11) becomes  $\tau_{1,2 \rightarrow 1}(t) := \inf\{s \geq t \mid X(s) \in S_{1,2 \rightarrow 1}(s) = S_{1,2 \rightarrow 1}^1(s) \cup S_{1,2 \rightarrow 1}^2(s)\}$ , where the stopping region  $S_{1,2 \rightarrow 1}^i(s)$  is defined by

$$S_{1,2 \rightarrow 1}^i(s) := \{x \in \mathbb{R}_{++}^2 \mid V_{1,2 \rightarrow 1}(x, s) = x_i - I_i\} \quad (12)$$

for  $i = 1$  and  $2$ .

#### Proposition 1

**(Convexity of the value function)**  $V_{1,2 \rightarrow 1}(x, t)$  is convex with respect to  $x$ .

**(Convexity of each stopping region)**  $S_{1,2 \rightarrow 1}^i(t)$  is a convex set.

**(Monotonicity of each stopping region)**  $x \in S_{1,2 \rightarrow 1}^i(t) \Rightarrow x' \in S_{1,2 \rightarrow 1}^i(t)$  ( $\forall x'_i \geq x_i, \forall x'_j \leq x_j$ ), where  $i \neq j$ .

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<sup>5</sup>(Nishihara 2009) models rival preemption for the max-option endogenously. Indeed, (Nishihara 2009) derives the equilibrium of a preemption game in which two firms compete for two alternatives.

<sup>6</sup>This problem is the same as the max-option problem (8) replaced  $X_2(t)$  with the killed process.

**(Behavior on the indifference line)**  $x_1 - I_1 = x_2 - I_2 \Rightarrow x \notin S_{1,2 \rightarrow 1}(t)$ .

**(Comparison)**  $\max\{V_1(x_1, t), V_{2 \rightarrow \emptyset}(x_2, t)\} \leq V_{1,2 \rightarrow 1}(x, t) \leq V_{1,2}(x, t)$ ,  $S_{1,2}^1(t) \subset S_{1,2 \rightarrow 1}^1(t) \subset S_1^1(t)$ , and  $S_{1,2}^2(t) \subset S_{1,2 \rightarrow 1}^2(t) \subset S_{2 \rightarrow \emptyset}^2(t)$ .

**Proof**

**(Convexity of the value function)** For any  $x, x' \in \mathbb{R}_{++}$  and  $\alpha \in [0, 1]$ , I have

$$\begin{aligned}
& V_{1,2 \rightarrow 1}(\alpha x + (1 - \alpha)x', t) \\
&= \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{(1,1)} \left[ \int_0^\infty 1_{\{\tau \leq T\}} \{1_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1,2} \{(\alpha x_i + (1 - \alpha)x'_i) X_i(\tau) - I_i\} \right. \\
&\quad \left. + 1_{\{\tau \geq t+y\}} e^{-ry} V_1((\alpha x_1 + (1 - \alpha)x'_1) X_1(t+y), t+y)\} \lambda e^{-\lambda y} dy \right] \\
&\leq \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{(1,1)} \left[ \int_0^\infty 1_{\{\tau \leq T\}} \{1_{\{\tau < t+y\}} e^{-r(\tau-t)} (\alpha \max_{i=1,2} \{x_i X_i(\tau) - I_i\} + (1 - \alpha) \max_{i=1,2} \{x'_i X_i(\tau) - I_i\}) \right. \\
&\quad \left. + 1_{\{\tau \geq t+y\}} e^{-ry} (\alpha V_1(x_1 X_1(t+y), t+y) + (1 - \alpha) V_1(x'_1 X_1(t+y), t+y))\} \lambda e^{-\lambda y} dy \right] \quad (13) \\
&\leq \alpha \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{(1,1)} \left[ \int_0^\infty 1_{\{\tau \leq T\}} \{1_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1,2} \{x_i X_i(\tau) - I_i\} \right. \\
&\quad \left. + 1_{\{\tau \geq t+y\}} e^{-ry} V_1(x_1 X_1(t+y), t+y)\} \lambda e^{-\lambda y} dy \right] \\
&\quad + (1 - \alpha) \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{(1,1)} \left[ \int_0^\infty 1_{\{\tau \leq T\}} \{1_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1,2} \{x'_i X_i(\tau) - I_i\} \right. \\
&\quad \left. + 1_{\{\tau \geq t+y\}} e^{-ry} V_1(x'_1 X_1(t+y), t+y)\} \lambda e^{-\lambda y} dy \right] \\
&= \alpha V_{1,2 \rightarrow 1}(x, t) + (1 - \alpha) V_{1,2 \rightarrow 1}(x', t),
\end{aligned}$$

where (13) results from the convexity of  $V_1(x, t)$  and  $\max_{i=1,2} \{x_i - I_i\}$ .

**(Convexity of each stopping region)** For any  $x, x' \in S_{1,2 \rightarrow 1}^i(t)$  and  $\alpha \in [0, 1]$ , I have

$$\begin{aligned}
V_{1,2 \rightarrow 1}(\alpha x + (1 - \alpha)x', t) &\leq \alpha V_{1,2 \rightarrow 1}(x, t) + (1 - \alpha) V_{1,2 \rightarrow 1}(x', t) \quad (14) \\
&= \alpha(x_i - I_i) + (1 - \alpha)(x'_i - I_i) \\
&= \alpha x_i + (1 - \alpha)x'_i - I_i,
\end{aligned}$$

where (14) results from the convexity of  $V_{1,2 \rightarrow 1}(x, t)$ , and the last inequality implies  $\alpha x + (1 - \alpha)x' \in S_{1,2 \rightarrow 1}^i(t)$ .

**(Monotonicity of each stopping region)** It follows from the convexity of  $V_1(x_1, t)$  with respect to  $x$  and  $\partial V_1(x_1, t)/\partial x_1 = 1$  for  $x_1 > x_1^*(t)$  that

$$V_1(x'_1, t) \leq V_1(x_1, t) + x'_1 - x_1 \quad (x'_1 \geq x_1) \quad (15)$$

Take any  $x \in S_{1,2 \rightarrow 1}^1(t)$  and  $x'$  satisfying  $x'_1 \geq x_1$  and  $x'_2 \leq x_2$ . I have

$$\begin{aligned}
& V_{1,2 \rightarrow 1}(x', t) \\
&= \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{(1,1)} \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1,2} \{ x'_i X_i(\tau) - I_i \} \right. \\
&\quad \left. + \mathbf{1}_{\{\tau \geq t+y\}} e^{-ry} V_1(x'_1 X_1(t+y), t+y) \} \lambda e^{-\lambda y} dy \right] \\
&\leq \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{(1,1)} \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} (\max_{i=1,2} \{ x_i X_i(\tau) - I_i \} + (x'_1 - x_1) X_1(\tau)) \right. \\
&\quad \left. + \mathbf{1}_{\{\tau \geq t+y\}} e^{-ry} (V_1(x_1 X_1(t+y), t+y) + (x'_1 - x_1) X_1(t+y)) \} \lambda e^{-\lambda y} dy \right] \quad (16) \\
&\leq \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{(1,1)} \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1,2} \{ x_i X_i(\tau) - I_i \} \right. \\
&\quad \left. + \mathbf{1}_{\{\tau \geq t+y\}} e^{-ry} (V_1(x_1 X_1(t+y), t+y)) \} \lambda e^{-\lambda y} dy \right] \\
&\quad + \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^1 \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} (x'_1 - x_1) X_1(\tau) \right. \\
&\quad \left. + \mathbf{1}_{\{\tau \geq t+y\}} e^{-ry} (x'_1 - x_1) X_1(t+y) \} \lambda e^{-\lambda y} dy \right] \\
&\leq V_{1,2 \rightarrow 1}(x, t) + \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^1 \left[ \mathbf{1}_{\{\tau \leq T\}} e^{-r(\tau-t)} (x'_1 - x_1) X_1(\tau) \right] \\
&= x_1 - I_1 + x'_1 - x_1 \\
&= x'_1 - I_1, \quad (17)
\end{aligned}$$

where (16) follows from (15). The last inequality (17) implies  $x' \in S_{1,2 \rightarrow 1}^1(t)$ , and hence, I have the monotonicity of  $S_{1,2 \rightarrow 1}^1(t)$ . Similarly, I can show the monotonicity of  $S_{1,2 \rightarrow 1}^2(t)$ . **(Behavior on the indifference line)** Take any  $x$  satisfying  $x_1 - I_1 = x_2 - I_2$ . First, note that

$$\begin{aligned}
V_{1,2 \rightarrow \emptyset}(x, t) &:= \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1,2} \{ X_i(\tau) - I_i \} \lambda e^{-\lambda y} dy \right] \\
&= \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x \left[ \mathbf{1}_{\{\tau \leq T\}} e^{-(r+\lambda)(\tau-t)} \max_{i=1,2} \{ X_i(\tau) - I_i \} \right] \quad (18) \\
&> x_1 - I_1, \quad (19)
\end{aligned}$$

where I can show (18) by the similar calculation to (5). The last inequality (19) follows from the behavior on the indifference line for the max-option in Section 2.2. Then, I have

$$\begin{aligned}
V_{1,2 \rightarrow 1}(x, t) &\geq V_{1,2 \rightarrow \emptyset}(x, t) \\
&> x_1 - I_1,
\end{aligned}$$

where the last inequality implies that  $x \notin S_{1,2 \rightarrow 1}^1(t)$ . Similarly, I have  $x \notin S_{1,2 \rightarrow 1}^2(t)$ .

**(Comparison)** The inequalities with respect to the value functions are clear. For any  $x \in S_{1,2}^i(t)$ , I have

$$\begin{aligned}
V_{1,2 \rightarrow 1}(x, t) &\leq V_{1,2}(x, t) \\
&= x_i - I_i,
\end{aligned}$$

where the last inequality implies  $x \in S_{1,2 \rightarrow 1}^i(t)$ . Similarly, by the inequalities with respect to the value functions, I can show that  $S_{1,2 \rightarrow 1}^1(t) \subset S_1^1(t)$  and  $S_{1,2 \rightarrow 2}^2(t) \subset S_{2 \rightarrow \emptyset}^2(t)$ .  $\square$

Proposition 1 extends previous findings by (Geltner, Riddiough, and Stojanovic 1996, Broadie and Detemple 1997, Detemple 2006) to a case in which the investment opportunity may be killed. The properties for the max-option in Section 2.2 remain true for the generalized case. The monotonicity of  $S_{1,2 \rightarrow 1}^i(t)$  implies that an increase (decrease) in the value of project  $i$  ( $j \neq i$ ) encourages investment in project  $i$ . The behavior on the indifference line means that a firm will wait and see which project is better when the values of both projects are equal. Clearly,  $V_{1,2 \rightarrow 1}(x, t)$  ( $S_{1,2 \rightarrow 1}^i(t)$ ) monotonically decreases (increases) with the intensity  $\lambda$ . This means that an increased possibility of the disappearance reduces the value of waiting and encourages investment. Note that  $V_{1,2 \rightarrow 1}(x, t) \downarrow \max\{x_2 - I_2, V_1(x_1, t)\}$ ,  $S_{1,2 \rightarrow 1}^1(t) \uparrow S_1^1(t) \setminus \{x \in \mathbb{R}_{++}^2 \mid x_2 - I_2 \geq V_1(x_1, t)\}$ ,  $S_{1,2 \rightarrow 1}^2(t) \uparrow \{x \in \mathbb{R}_{++}^2 \mid x_2 - I_2 > V_1(x_1, t)\}$  ( $\lambda \uparrow \infty$ ) and  $V_{1,2 \rightarrow 1}(x, t) \uparrow V_{1,2}(x, t)$ ,  $S_{1,2 \rightarrow 1}^i(t) \downarrow S_{1,2}^i(t)$  ( $\lambda \downarrow 0$ ).

A higher volatility  $\sigma_i$  increases  $V_{1,2}(x, t)$  and  $V_1(x_1, t)$ , and hence, it increases (decreases)  $V_{1,2 \rightarrow 1}(X(t), t)$  ( $S_{1,2 \rightarrow 1}(t)$ ). On the other hand, an increase in the correlation coefficient  $\rho$  tends to decrease (increase)  $V_{1,2 \rightarrow 1}(X(t), t)$  ( $S_{1,2 \rightarrow 1}(t)$ ). This is because  $V_{1,2}(x, t)$  tends to increase with  $\rho$  (see (Geltner, Riddiough, and Stojanovic 1996, Detemple 2006)). Note that the option value jumps downward from  $V_{1,2 \rightarrow 1}(X(t), t)$  to  $V_1(X_1(t), t)$  at the time of the Poisson death (see Figure 1). The jump size is endogenously determined as  $V_{1,2 \rightarrow 1}(X(t), t) - V_1(X_1(t), t)$ .<sup>7</sup> The jump size decreases with  $\lambda$  and  $\rho$ . Typically,  $V_1(X_1(t), t)$  is more volatile than  $V_{1,2 \rightarrow 1}(X(t), t)$  in which  $X_1(t)$  and  $X_2(t)$  diversify the risk. Accordingly, the volatility of the option value jumps upward on the same timing. This may account for empirical observations (e.g., (Black 1976, Christie 1982)) that the volatility of a stock price increases when the stock price decreases.

### 3.2 Option that will change to the max-option

This subsection considers an option that may change to the max-option. This option contrasts with the option studied in Section 3.1. Assume that the investment opportunity for project 2 is created at an instantaneous rate  $\lambda dt$  which is independent of  $X(t)$ . For  $X(t) = x$ , the option value prior to the Poisson arrival is equal to the value function of the optimal stopping problem as follows:

$$V_{1 \rightarrow 1,2}(x, t) := \sup_{\tau \in \mathcal{I}_t} \mathbb{E}_t^x \left[ \int_0^\infty 1_{\{\tau \leq T\}} \{1_{\{\tau < t+y\}} e^{-r(\tau-t)} (X_1(\tau) - I_1) + 1_{\{\tau \geq t+y\}} e^{-ry} V_{1,2}(X(t+y), t+y)\} \lambda e^{-\lambda y} dy \right] \quad (20)$$

where  $y$  denotes the term until the arrival. The subscript  $1 \rightarrow 1,2$  denotes the option that may change to the max-option. Note that  $V_{1 \rightarrow 1,2}(x, t)$ , unlike  $V_1(x_1, t)$ , depends not

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<sup>7</sup>In contrast, in a model based on the discontinuous stochastic process the jump size must be exogenously presumed.

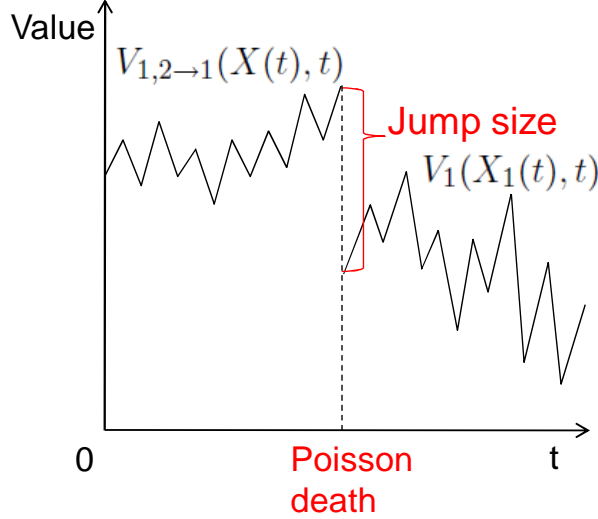


Figure 1: The downward jump caused by the disappearance of the alternative.

only  $x_1$  but also  $x_2$  because of the potential arrival of project 2.

Using the similar calculation to (5), I can rewrite problem (20) as

$$V_{1 \rightarrow 1,2}(x, t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x [1_{\{\tau \leq T\}} e^{-(r+\lambda)(\tau-t)} (X_1(\tau) - I_1 - V_{\emptyset \rightarrow 1,2}(X(\tau), \tau))] + V_{\emptyset \rightarrow 1,2}(x, t), \quad (21)$$

where  $V_{\emptyset \rightarrow 1,2}(x, t)$  is defined by

$$V_{\emptyset \rightarrow 1,2}(x, t) = \mathbb{E}_t^x \left[ \int_0^\infty e^{-ry} V_{1,2}(X(t+y), t+y) \lambda e^{-\lambda y} dy \right]. \quad (22)$$

The optimal stopping time for problem (21) becomes  $\tau_{1 \rightarrow 1,2}(t) := \inf\{s \geq t \mid X(s) \in S_{1 \rightarrow 1,2}^1(s)\}$ , where the stopping region  $S_{1 \rightarrow 1,2}^1(s)$  is defined by

$$S_{1 \rightarrow 1,2}^1(s) := \{x \in \mathbb{R}_{++}^2 \mid V_{1 \rightarrow 1,2}(x, s) = x_1 - I_1\}. \quad (23)$$

Note that, the exercise region  $S_{1 \rightarrow 1,2}^1(s)$  exists only for project 1 although it also depends on the value of project 2.

### Proposition 2

**(Convexity of the value function)**  $V_{1 \rightarrow 1,2}(x, t)$  is convex with respect to  $x$ .

**(Convexity of the stopping region)**  $S_{1 \rightarrow 1,2}^1(t)$  is a convex set.

**(Monotonicity of the stopping region)**  $x \in S_{1 \rightarrow 1,2}^1(t) \Rightarrow x' \in S_{1 \rightarrow 1,2}^1(t)$  ( $\forall x'_1 \geq x_1, \forall x'_2 \leq x_2$ ).

**(Comparison)**  $V_1(x_1, t) \leq V_{1 \rightarrow 1,2}(x, t) \leq V_{1,2}(x, t)$  and  $S_{1,2}^1(t) \subset S_{1 \rightarrow 1,2}^1(t) \subset S_1^1(t)$ .

### Proof

**(Convexity of the value function)** Using the convexity of  $V_{1,2}(x, t)$ , I can construct the same proof as the convexity of the value function in Proposition 1.

**(Convexity of the stopping region)** Using the convexity of  $V_{1 \rightarrow 1,2}(x, t)$ , I can construct the same proof as the convexity of the stopping region in Proposition 1.

**(Monotonicity of the stopping region)** Using

$$V_{1,2}(x', t) \leq V_{1,2}(x, t) + x'_1 - x_1 \quad (\forall x'_1 \geq x_1, \forall x'_2 \leq x_2),$$

I can construct the same proof as the monotonicity of the stopping region in Proposition 1.

**(Comparison)** The relationships among the value functions are clear. Then, I can easily show the relationships among the stopping regions.  $\square$

To my knowledge, there has been no studies that investigates the option that will change to the max-option. Proposition 2 presents the first result for this type of option. Propositions 1 and 2 bridge the gap between the standard option and the max-option from both sides. The exercise region  $S_{1 \rightarrow 1,2}^1(t)$  has the same convexity and monotonicity as  $S_{1,2}^1(t)$  for the max-option. The monotonicity of  $S_{1 \rightarrow 1,2}^1(t)$  implies that an increase (decrease) in the value of project 1 (2) accelerates investment in project 1. In the presence of the potentially available project 2, a high value of project 2 increases the option value and delay investment in project 1. It follows from expression (21) that  $S_{1 \rightarrow 1,2}^1(t)$  is contained in  $\{x \in \mathbb{R}_{++} \mid x_1 \geq I_1 + V_{\emptyset \rightarrow 1,2}(x, t)\}$ . In particular, by

$$\begin{aligned} V_{\emptyset \rightarrow 1,2}(x, t) &\geq \mathbb{E}_t^x \left[ \int_0^\infty e^{-ry} (X_2(t+y) - I_2) \lambda e^{-\lambda y} dy \right] \\ &= \frac{\lambda x_2}{r + \lambda - \mu} - \frac{\lambda I_2}{r + \lambda}, \end{aligned}$$

I can show that  $x_1 \geq I_1 + \lambda x_2 / (r + \lambda - \mu) - \lambda I_2 / (r + \lambda)$  for  $x \in S_{1 \rightarrow 1,2}^1(t)$ . This indicates that for a high  $x_2$ ,  $S_{1 \rightarrow 1,2}^1(t)$  is much smaller than  $S_1^1(t) = \{x \in \mathbb{R}_{++}^2 \mid x_1 \geq x_1^*(t)\}$ . Clearly,  $V_{1 \rightarrow 1,2}(x, t)$  ( $S_{1 \rightarrow 1,2}(t)$ ) monotonically increases (decreases) with the intensity  $\lambda$ . This means that an increased possibility of the arrival enhances the value of waiting and discourages investment. I have  $V_{1 \rightarrow 1,2}(x, t) \uparrow V_{1,2}(x, t)$ ,  $S_{1 \rightarrow 1,2}^1(t) \downarrow S_{1,2}^1(t)$  ( $\lambda \uparrow \infty$ ) and  $V_{1 \rightarrow 1,2}(x, t) \downarrow V_1(x_1, t)$ ,  $S_{1 \rightarrow 1,2}^1(t) \uparrow S_1^1(t)$  ( $\lambda \downarrow 0$ ).

As in Section 3.1,  $V_{1 \rightarrow 1,2}(X(t), t)$  ( $S_{1 \rightarrow 1,2}(t)$ ) increases (decreases) with the volatility  $\sigma_i$  and decreases (increases) with the correlation coefficient  $\rho$ . Note that the option value jumps upward from  $V_{1 \rightarrow 1,2}(X(t), t)$  to  $V_{1,2}(X(t), t)$  at the time of the Poisson arrival (see Figure 2). The jump size, which is endogenously determined as  $V_{1,2}(X(t), t) - V_{1 \rightarrow 1,2}(X(t), t)$ , decreases with  $\lambda$ .

### 3.3 Extensions and limitations

This section explains several extensions of the results. First, consider the max option that has  $n$  investment opportunities. Assume that projects  $n, n-1, \dots, m+1$  will be killed

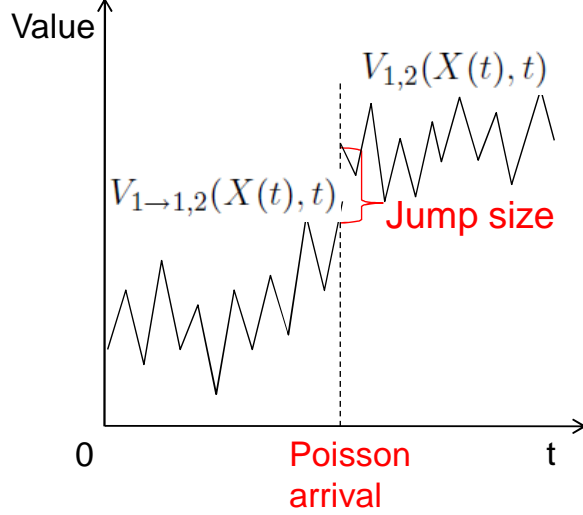


Figure 2: The upward jump caused by the occurrence of the alternative.

sequentially with the intensity  $\lambda$ . For  $X(t) = x$ , the option value prior to the Poisson death is equal to the value function of the optimal stopping problem as follows:

$$\begin{aligned}
& V_{1, \dots, n \rightarrow 1, \dots, n-1 \rightarrow \dots \rightarrow 1, \dots, m}(x, t) \\
:= & \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1, \dots, n} \{X_i(\tau) - I_i\} \right. \\
& \left. + \mathbf{1}_{\{\tau \geq t+y\}} e^{-ry} V_{1, \dots, n-1 \rightarrow 1, \dots, n-2 \rightarrow \dots \rightarrow 1, \dots, m}(X(t+y), t+y) \} \lambda e^{-\lambda y} dy \right] \quad (n > m),
\end{aligned}$$

which is defined backward from

$$\begin{aligned}
& V_{1, \dots, m \rightarrow 1, \dots, m}(x, t) \\
:= & V_{1, \dots, m}(x, t) \\
:= & \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x \left[ \mathbf{1}_{\{\tau \leq T\}} e^{-r(\tau-t)} \max_{i=1, \dots, m} \{X_i(\tau) - I_i\} \right].
\end{aligned}$$

This is a generalized version of problem (10). Note that  $V_{1, \dots, m}(x, t)$  is convex with respect to  $x$ . Then, using backward induction, I can show the same properties as Proposition 1.

Similarly, a generalized version of problem (20) is expressed as

$$\begin{aligned}
& V_{1, \dots, m \rightarrow 1, \dots, m+1 \rightarrow \dots \rightarrow 1, \dots, n}(x, t) \\
:= & \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x \left[ \int_0^\infty \mathbf{1}_{\{\tau \leq T\}} \{ \mathbf{1}_{\{\tau < t+y\}} e^{-r(\tau-t)} \max_{i=1, \dots, m} \{X_i(\tau) - I_i\} \right. \\
& \left. + \mathbf{1}_{\{\tau \geq t+y\}} e^{-ry} V_{1, \dots, m+1 \rightarrow \dots \rightarrow 1, \dots, n}(X(t+y), t+y) \} \lambda e^{-\lambda y} dy \right].
\end{aligned}$$

I can show the same properties as Proposition 2 by the backward induction. Of course, it does not matter if the intensities vary over projects. Furthermore, I can show the same properties even if the order of the disappearances or occurrences is not presumable.

For the problem in Section 3.2, a new value  $X_2(t)$  may be unobservable until the arrival. For example, value of an alternative that proceeds from some technical innovation may not be evaluated correctly prior to the innovation. This partial information problem is essentially different from problem (20) in that the option value and the investment strategy depend only on  $X_1(t)$ .

## 4 Numerical Examples

This section provides numerical examples of the options studied in the previous section. I use base parameter values as follows<sup>8</sup>:

$$r = 8\%, \mu_1 = \mu_2 = 0\%, \sigma_1 = \sigma_2 = 20\%, \rho = 0\%. \quad (24)$$

For expositional purposes, I set the investment cost  $I_1 = I_2 = 100$  and the option values are computed at the money, i.e.,  $x = (100, 100)$ . The maturity of the option is set at  $T - t = 3$  years. For the intensity  $\lambda$ , the probability that an opportunity in which to invest disappears or occurs within 3 years is expressed as

$$\int_0^3 \lambda e^{-\lambda t} dt = 1 - e^{-3\lambda}.$$

I set this probability at 25%, 50%, and 75%. These correspond to  $\lambda = 0.096, 0.231$ , and  $0.462$ . In the numerical procedure, I make a discretization with 200 time steps per 1 year, and use a bivariate version of the lattice binomial method (e.g., (Boyle 1988)). Technically, I compute the lattice model for maturity  $T = 4$  years, and draw the investment regions for  $t = 1$  year.

First, I set the intensity  $\lambda = 0.231$ . Figure 3 illustrates the exercise regions  $S_{1,2 \rightarrow 1}(t) = S_{1,2 \rightarrow 1}^1(t) \cup S_{1,2 \rightarrow 1}^2(t)$  and  $S_{1 \rightarrow 1,2}^1(t)$  for the base parameter values (24). For comparison, the figure also presents the exercise regions  $S_{1,2}(t) = S_{1,2}^1(t) \cup S_{1,2}^2(t)$  and  $S_1^1(t)$ . I can check the convexity and monotonicity of  $S_{1,2 \rightarrow 1}^i(t)$  and  $S_{1 \rightarrow 1,2}^1(t)$ , as well as the relationship that  $S_{1,2}^1(t) \subset S_{1,2 \rightarrow 1}^1(t) \subset S_1^1(t)$ ,  $S_{1,2}^2(t) \subset S_{1,2 \rightarrow 1}^2(t)$ , and  $S_{1,2}^1(t) \subset S_{1 \rightarrow 1,2}^1(t) \subset S_1^1(t)$ .

In Figure 3, let us take a look at the max-option that will change to the standard option. When  $X(t)$  hits the boundary of  $S_{1,2 \rightarrow 1}^1(t)$  (the lower-right curve), a firm invests in project 1. On the contrary, when  $X(t)$  hits the boundary of  $S_{1,2 \rightarrow 1}^2(t)$  (the upper-left curve), the firm invests in project 2. The firm delays the decision on project choice for  $X(t)$  between the two curves, although project 2 may be killed in the waiting time. The figure indicates that the difference between  $S_{1,2 \rightarrow 1}^2(t)$  and  $S_{1,2}^2(t)$  is greater than the difference between  $S_{1,2 \rightarrow 1}^1(t)$  and  $S_{1,2}^1(t)$ . The prospective disappearance of project 2 encourages investment in project 2 prior to the disappearance, while it does not have a significant

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<sup>8</sup>These parameter values are similar to (Geltner, Riddiough, and Stojanovic 1996, Detemple 2006). I carried out a lot of computations with varying parameter values and distilled robust results into this section.



influence on the investment timing in project 1. The boundary of  $S_{1,2 \rightarrow 1}^2(t)$  approaches the threshold  $x_2^*(t)$  replaced  $r = 0.08$  with  $r + \lambda = 0.08 + 0.231 = 0.311$  when  $x_1 \rightarrow 0$ . Then, for  $x_1 \approx 0$ ,  $S_{1,2 \rightarrow 1}^2(t)$  is much larger than  $S_{1,2}^2(t)$ . On the other hand, both boundaries of  $S_{1,2}^1(t)$  and  $S_{1,2 \rightarrow 1}^1(t)$  converges to the threshold  $x_1^*(t)$  when  $x_2 \rightarrow 0$ . Then, for  $x_2 \approx 0$ ,  $S_{1,2 \rightarrow 1}^1(t)$  is almost the same as  $S_{1,2}^1(t)$ .

In Figure 3, let us now turn to the option that may change to the max-option. When  $X(t)$  hits the boundary of  $S_{1 \rightarrow 1,2}^1(t)$  (the lower-right curve), a firm invests in project 1. Otherwise, the firm delays investment, and in the waiting time project 2 may become available. I see that the gap between  $S_{1 \rightarrow 1,2}^1(t)$  and  $S_1^1(t)$  increases with  $x_2$ . This is because a higher  $x_2$  increases an incentive for the firm to wait for the occurrence of project 2. On the other hand, for  $x_2 \approx 0$  the boundary of  $S_{1 \rightarrow 1,2}^1(t)$  converges to the threshold  $x_1^*(t)$ , and hence there is no gap between them.

Now, consider the comparative statics. First, I explore the effects of the intensity  $\lambda$ . Figures 4 and 5 draw the exercise regions  $S_{1,2 \rightarrow 1}(t) = S_{1,2 \rightarrow 1}^1(t) \cup S_{1,2 \rightarrow 1}^2(t)$  and  $S_{1 \rightarrow 1,2}^1(t)$  with varying levels of  $\lambda$ . As mentioned in Section 3.2,  $S_{1,2 \rightarrow 1}^i(t)$  increases with  $\lambda$ , while  $S_{1 \rightarrow 1,2}^1(t)$  decreases with  $\lambda$ . For the max-option that may change to the standard option for project 1, a higher  $\lambda$  increases an incentive for a firm to invest in project 2 prior to the disappearance. On the other hand, for the option that may change to the max-option, a higher  $\lambda$  increases the value of waiting for project 2. Comparing Figures 4 and 5, I find that the impact of  $\lambda$  on the exercise policy is stronger for the option that will change to the max-option. This suggests that a firm should take into careful consideration an alternative which may be available in future rather than an alternative which may disappear in future.

Next, I explore the effects of the correlation coefficient  $\rho$ . Figures 6 and 7 illustrate the exercise regions  $S_{1,2 \rightarrow 1}(t) = S_{1,2 \rightarrow 1}^1(t) \cup S_{1,2 \rightarrow 1}^2(t)$  and  $S_{1 \rightarrow 1,2}^1(t)$  with varying levels of  $\rho$ . In both figures, the intensity  $\lambda$  is fixed at  $\lambda = 0.231$ . I see that  $S_{1,2 \rightarrow 1}^i(t)$  and  $S_{1 \rightarrow 1,2}^1(t)$  monotonically enlarge with  $\rho$ . It is known (e.g., (Geltner, Riddiough, and Stojanovic 1996, Detemple 2006)) that  $S_{1,2}^i(t)$  tends to increase with  $\rho$  because a higher  $\rho$  decreases the value of the option to delay the decision concerning project choice. Figures 6 and 7 demonstrate that the same result holds even if the disappearance or occurrence of an alternative is taken into consideration. This means the robustness of the previous findings.

The effects of  $\rho$  also appear in Tables 1 and 2. The tables present the option values  $V_{1,2 \rightarrow 1}(x, t)$  and  $V_{1 \rightarrow 1,2}(x, t)$  for varying levels of  $\lambda$  and  $\rho$ . Note that  $V_{1,2 \rightarrow 1}(x, t)$  and  $V_{1 \rightarrow 1,2}(x, t)$  are computed at the money, i.e.,  $x = (100, 100)$ . The option values monotonically decrease with  $\rho$ . The impact of  $\rho$  on  $V_{1,2 \rightarrow 1}(x, t)$  weakens with  $\lambda$ , while the impact of  $\rho$  on  $V_{1 \rightarrow 1,2}(x, t)$  strengthens with  $\lambda$ . This can be explained as follows. When  $\lambda \rightarrow \infty$ ,  $V_{1,2 \rightarrow 1}(x, t)$  approaches  $V_1(x, t)$  which is independent of  $\rho$ . On the other hand,  $V_{1 \rightarrow 1,2}(x, t)$  approaches  $V_1(x, t)$  when  $\lambda \rightarrow 0$ . Then, a higher (lower)  $\lambda$  weakens the impact of  $\rho$  on  $V_{1,2 \rightarrow 1}(x, t)(V_{1 \rightarrow 1,2}(x, t))$ . The impact of  $\lambda$  is significant to  $V_{1 \rightarrow 1,2}(x)$  especially for a low

Table 1: The option values  $V_{1,2 \rightarrow 1}(x, t)$ .

$\lambda \setminus \rho$	-75%	-50%	-25%	0%	25%	50%	75%
0	22.8	22.1	21.28	20.32	19.2	17.82	15.98
0.096	21.07	20.46	19.75	18.93	17.97	16.8	15.26
0.231	19.36	18.83	18.23	17.55	16.75	15.79	14.53
0.462	17.5	17.07	16.59	16.04	15.42	14.68	13.73

Table 2: The option values  $V_{1 \rightarrow 1,2}(x, t)$ .

$\lambda \setminus \rho$	-75%	-50%	-25%	0%	25%	50%	75%
0	11.7	11.7	11.7	11.7	11.7	11.7	11.7
0.096	14.42	14.24	14.02	13.77	13.47	13.12	12.65
0.231	17.16	16.81	16.39	15.9	15.33	14.62	13.7
0.462	19.93	19.41	18.79	18.07	17.22	16.18	14.8

$\rho$ . Indeed, for  $\rho = -50\%$ , the Poisson arrival with the intensity  $\lambda = 0.096, 0.231$ , and  $0.462$  enhances the option value by 21%, 44%, and 66%, respectively. For  $\rho = 0\%$  the Poisson arrival with the intensity  $\lambda = 0.096, 0.231$ , and  $0.462$  increases the option value by 18%, 36%, and 54%, respectively.

Let us examine Tables 1 and 2 from a different aspect. As mentioned in Section 3.1 and 3.2 (see Figures 1 and 2), the gap between  $V_{1,2 \rightarrow 1}(x, t)$  and  $V_1(x, t) = 11.7$  is equal to the jump size at the time of the Poisson death, while the gap between  $V_{1 \rightarrow 1,2}(x, t)$  and  $V_{1,2}(x, t)$  is equal to the jump size at the time of the Poisson arrival. Comparing Tables 1 and 2, I find that the downward jump size tends to be larger than the upward jump size. This finding may be related to empirical findings (e.g., (Skinner 1994)) that the stock price response to bad news is larger than the stock price response to good news.

## 5 Conclusion

The paper has investigated the nature of two types of option; the max-option that will change to the standard option by the Poisson death, and the option that will change to the max-option by the Poisson arrival. These uncertain changes in real options can be caused by changes in regulatory, political, competitive, and technological environment. The model, unlike the previous studies, directly captures these catastrophic risks in capital budgeting. In the model, I have revealed how the possibility of the disappearance or occurrence influences the optimal exercise policy and the option value. The results are summarized as follows.

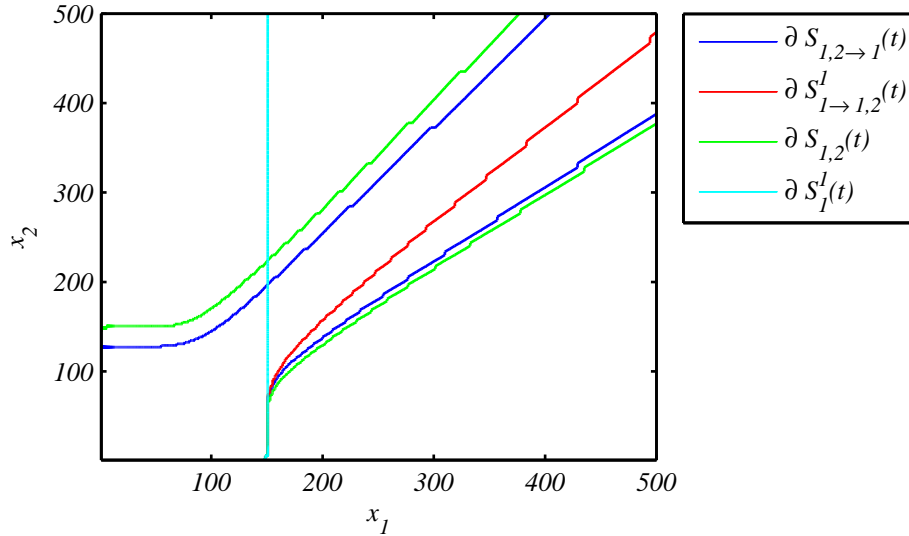


Figure 3: The exercise regions for the four options. This figure plots the boundaries of the exercise regions  $S_{1,2\rightarrow 1}(t) = S_{1,2\rightarrow 1}^1(t) \cup S_{1,2\rightarrow 1}^2(t)$ ,  $S_{1\rightarrow 1,2}(t)$ ,  $S_{1,2}(t) = S_{1,2}^1(t) \cup S_{1,2}^2(t)$  and  $S_1^1(t)$ . The parameter values are set at the base case (24) with the intensity  $\lambda = 0.231$ .

A higher intensity of the disappearance (occurrence) decreases (increases) the option value and encourages (discourages) investment. The properties, such as the monotonicity and convexity shown by the max-option literature, remain true when the random change is taken into consideration. The impact of the uncertain change is relatively greater for the option that may change to the max-option. Notably, the impact is significant for a weaker or negative correlation between the project values. The results offer rational explanations for the behavior of an owner of farmland which has not been cultivated in many years. The results have the potential to account for the asymmetric market reaction to good and bad news, although the model is not intended to investigate the stock price reactions.

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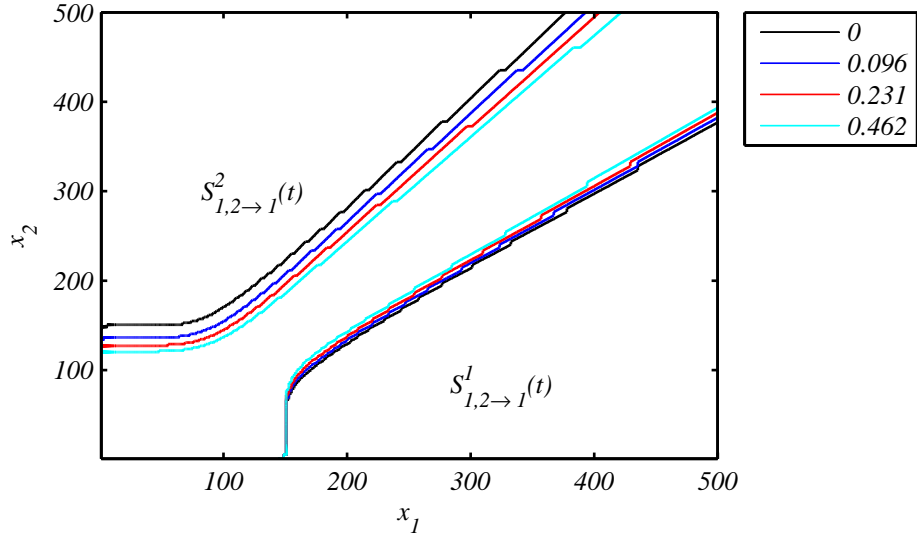


Figure 4: The exercise regions for the max-option that will change to the standard option. This figure plots the boundaries of the exercise regions  $S_{1,2 \rightarrow 1}(t) = S_{1,2 \rightarrow 1}^1(t) \cup S_{1,2 \rightarrow 1}^2(t)$  for the intensity  $\lambda = 0, 0.096, 0.231$ , and  $0.462$ . For  $\lambda = 0$ , the region is equal to the exercise region for the max-option,  $S_{1,2}(t)$ . The parameter values are set at the base case (24).

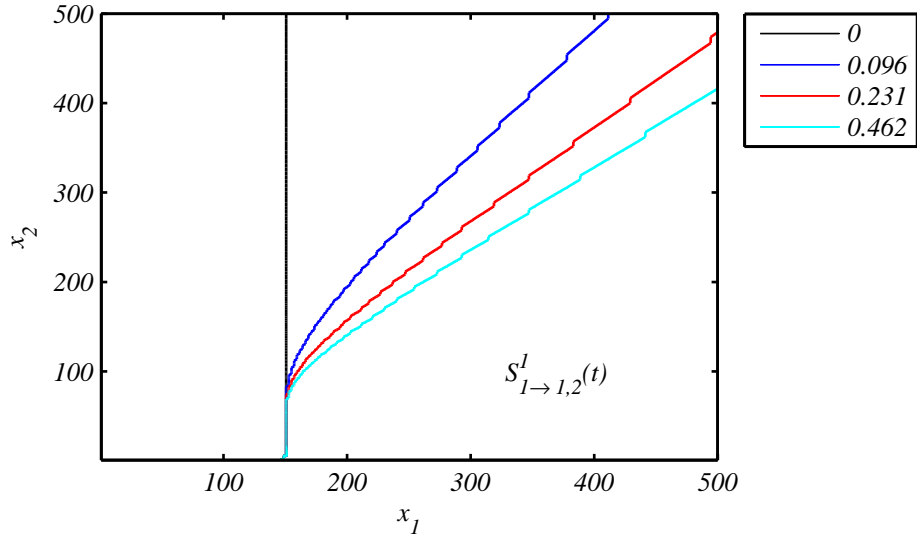


Figure 5: The exercise regions for the option that will change to the max-option. This figure plots the boundaries of the exercise regions  $S_{1 \rightarrow 1,2}^1(t)$  for the intensity  $\lambda = 0, 0.096, 0.231$ , and  $0.462$ . For  $\lambda = 0$ , the region is equal to the exercise region for the standard option,  $S_1^1(t)$ . The parameter values are set at the base case (24).

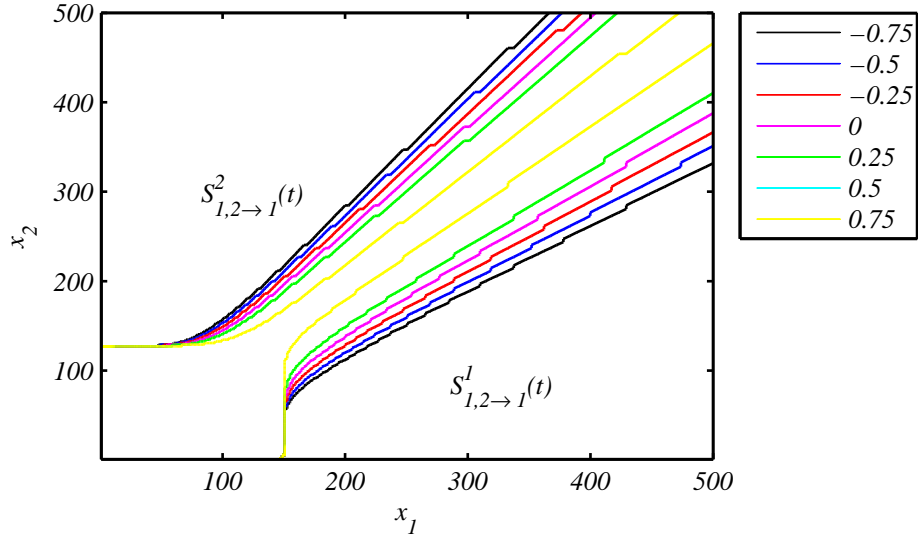


Figure 6: The exercise regions for the max-option that will change to the standard option. This figure plots the boundaries of the exercise regions  $S_{1,2 \rightarrow 1}(t) = S_{1,2 \rightarrow 1}^1(t) \cup S_{1,2 \rightarrow 1}^2(t)$  for the correlation coefficient  $\rho = -75\%, -50\%, -25\%, 0\%, 25\%, 50\%$ , and  $75\%$ . The intensity is set at  $\lambda = 0.231$ . The other parameter values are set at the base case (24).

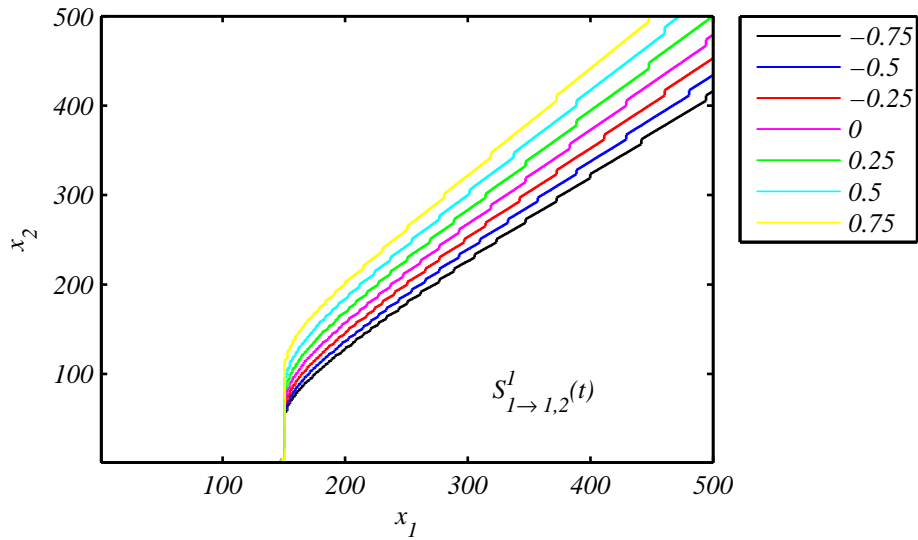


Figure 7: The exercise regions for the option that will change to the max-option. This figure plots the boundaries of the exercise regions  $S_{1 \rightarrow 1,2}^1(t)$  for the correlation coefficient  $\rho = -75\%, -50\%, -25\%, 0\%, 25\%, 50\%$ , and  $75\%$ . The intensity is set at  $\lambda = 0.231$ . The other parameter values are set at the base case (24).