Discussion Papers In Economics And Business

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Abstract

We develop a dynamic model in which a firm exercises an option to expand production on either a small or large scale with cash reserves and costly external funds. We show that the financing costs greatly distort the firm's financing and investment behavior and result in a policy contingent on the dynamics of the cash flow and reserves. Most notably, we prove that an intermediate level of cash reserves is likely to accelerate investment in the small-scale project by interactions among financing costs, investment timing, and investment sizing. Our results fill the gap between two types of results: (i) empirical findings in a U-shaped relation between the investment volume and internal funds, and (ii) empirical predictions of a U-shaped relation between the investment timing and internal funds.

JEL Classifications Code: G13; G31; G32.

Keywords: Investment timing; Investment size; Costly external financing; Optimal stopping.

*This version: 12 April, 2012.

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1 Introduction

Subsequent to the departures from Modigliani and Miller (1958)'s irrelevance proposition in a frictionless market, there has been a long tradition in corporate finance to investigate the effects of various frictions on financing and investment decisions. Recently, an increasing number of papers have analyzed not only the static but also the dynamic behaviors of corporate financing and investment in the presence of frictions.[1] Among these, a real options approach plays an important role in unveiling investment timing decisions in the presence of such frictions as liquidity constraints (Boyle and Guthrie (2003)), shareholders-debtholders conflicts (Mauer and Sarkar (2005), Sundaresan and Wang (2007)), and asymmetric information (Grenadier and Wang (2005), Shibata and Nishihara (2010) Morellec and Schürhoff (2011), and Grenadier and Malenko (2011)).

We extend this line of research by revealing the interactions of costs of external financing, investment timing, and investment size. The model is as follows: A firm owns an option to expand production on either a small or large scale, where the price of the output follows a geometric Brownian motion. The investment project is financed with cash reserves and costly external funds. The firm's cash reserves gradually increase as the firm's existent production generates cash flows. If the firm waits for a sufficient level of cash reserves for each project, the investment project can be financed entirely with cash reserves. Otherwise, the firm must rely partially on costly external financing. For the small-scale expansion, the sufficient cash level is lower than that of for the large-scale expansion. Considering the trade-off, the firm determines its financing, investment timing, and investment sizing policy.

As in the standard real options literature (e.g., McDonald and Siegel (1986), Dixit and Pindyck (1994)), our model assumes the irreversibility of investing as a friction. A key difference from most of the related papers is that we incorporate the investment sizing decision in addition to the investment timing decision. The assumption of either a small- or large-scale choice builds on Dixit (1993) and Décamps, Mariotti, and Villeneuve (2006). Indeed, our model generalizes their models to a case with costs of external financing. The financing costs are known as one of the most influential frictions in the corporate finance literature (e.g., Altinkilic and Hansen (2000), Hennessy and Whited (2007)). According to the pecking order hypothesis, asymmetric information problems associated with external funding generate higher costs; therefore, managers prefer internal over external financing (Myers (1984), Myers and Majluf (1984)). As a proportional cost accounts for the largest part of external financing costs, we focus primarily on the case with a proportional cost.

Before describing the results, we emphasize a contribution of this paper from the theoretical viewpoint. Most of the related papers demonstrate their results only by numerical examples because the model's complexity precludes analytic results (e.g., Boyle

[1] An incomplete list includes Hennessy and Whited (2005), Hennessy and Whited (2007), Hennessy, Levy, and Whited (2007), Tsyplakov (2008), Tserlukevich (2008), and Morellec and Schürhoff (2011).

and Guthrie (2003), Hirth and Uhrig-Homburg (2010a), Shibata and Nishihara (2012)). Contrasted to the stream of literature, this paper analytically proves the interesting properties of the dynamic corporate financing and investment policy by developing techniques in the mathematical finance literature (e.g., Broadie and Detemple (1997), Detemple (2006), Bobtcheff and Villeneuve (2010)) into a case involving a non-convex payoff and a non-geometric Brownian motion.

The results are summarized as follows. The presence of financing costs, unlike previous results with no financing cost in Décamps, Mariotti, and Villeneuve (2006), leads the firm to take a financing and investment policy contingent not only on the cash flow dynamics but also on the cash reserves dynamics. Specifically, higher financing costs enhance the firm's incentive to wait for a sufficient level of cash reserves and use entirely internal financing, especially for the small-scale project.

The investment threshold for the large-scale project monotonically decreases with cash reserves. This monotonic relation is straightforwardly consistent with conventional views of underinvestment due to financing constraints. On the other hand, the investment region for the small-scale project is not monotonic with cash reserves. The small-scale investment is encouraged with cash reserves until cash reserves reach the investment cost and, after that, the investment is discouraged with cash reserves. The rationale behind the non-monotonic relation is that the firm optimizes not only investment timing but also investment size. Consider the ratio of the total cost associated with the large-scale expansion to that of the small-scale expansion. The ratio, which changes with cash reserves, is maximized when cash reserves are equal to the amount of the small-scale investment cost. Indeed, at that moment the small-scale project requires no external funds while the large-scale project requires a great amount of external funds. The greatest advantage of the small-scale project over the large-scale project plays a role in speeding up the small-scale investment at the intermediate level of cash reserves.

Most notably, our results can link two significant results in corporate finance. The first one is a U-shaped relation between the investment volume and internal funds. Since arguments among Fazzari, Hubbard, and Petersen (1988), Kaplan and Zingales (1997), and Hubbard (1998), investment-cash flow sensitivities have been the center of attention in corporate finance. In particular, recent empirical evidence regarding this issue documented that the investment volume does not necessarily decrease with internal funds but can have a U-shaped relation with internal funds (Cleary, Povel, and Raith (2007), Guariglia (2008)).

The second result is an empirical prediction that the investment threshold has a U-shaped relation with a degree of financial constraints. The prediction has been seen in the recent real option literature. Boyle and Guthrie (2003) examined the effects of a liquidity constraint to the investment timing decision and predicted that the investment threshold has a U-shaped relation with a degree of the liquidity constraint. Shibata and Nishihara (2012), who examined the effects of a debt capacity constraint in a dynamic financing and capital structure model, showed that the investment threshold has a U-shaped relation with a degree of the debt capacity constraint.[2]

If one identifies "earlier" investment as "increased" investment, the two results are inconsistent with each other. However, this argument pays no attention to the point that the investment timing studies consider fixed-scale investment models. Our results can explain both types of results in terms of the interactions of investment timing and sizing decisions with costly external financing. In the presence of financing costs, cash reserves influence the trade-off between the two choices: small- or large-scale expansion. When cash reserves are close to the amount of the small-scale investment cost, the firm has a great incentive to invest in the small-scale project for which the investment threshold is relatively low. When cash reserves are much higher or lower than that level, the firm is likely to undertake the large-scale expansion for which the investment threshold is relatively high. This mechanism can explain U-shaped relations regarding both the investment volume and timing in the previous studies.

In summary, the main contributions of this paper are threefold. First and most importantly, this paper fills the gap between two types of results in the corporate finance literature: (i) empirical evidence regarding a U-shaped relation between the investment volume and internal funds and (ii) predictions of a U-shaped relation between the investment timing and internal funds. Second, this paper complements the investment timing and sizing literature by proving that costs of external financing greatly distort the decision, leading especially to a policy dependent the dynamics of both cash flow and cash reserves. Third, this paper contributes the mathematical finance literature by proving the properties of exercise regions of an optimal stopping problem involving a non-convex payoff function and a non-geometric Brownian motion.

The remainder of this paper is organized as follows. Section 2 presents the setup and the results in the case without financing costs. Section 3 presents the results in the case with a proportional cost and explains empirical implications. Section 4 examines the comparative statics with respect to the price volatility and a case with fixed and proportional costs in numerical examples. Section 5 concludes the paper. All proofs appear in the appendix.

[2] Regarding the relation between the investment timing and cash holdings, Hirth and Uhrig-Homburg (2010b), who extended Boyle and Guthrie (2003) to a case with financing costs, pointed out the possibility of various non-monotonic relations, and Nishihara and Shibata (2011) proved that a fixed cost of external financing leads to a non-monotonic relation.

2 Preliminaries

2.1 Setup

Consider a risk-neutral firm that produces a commodity at a constant rate. The output is sold at the market price $X(t)$, which follows a geometric Brownian motion

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}B(t)\ (t > 0), \quad X(0) = x, \tag{1}$$

where $B(t)$ denotes the standard Brownian motion defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mu, \sigma(> 0)$ and $x(> 0)$ are constants. For convergence, we assume that $r > \mu$, where r is a positive and constant interest rate. Assume that the firm owns an option to expand production on either a small or large scale, $A_1(> 1)$ or $A_2(> A_1)$, respectively, at any time. If the small-scale (large-scale) growth option is exercised at time τ, the firm pays a fixed investment cost at time τ and receives an instantaneous cash flow $A_1X(t)$ $(A_2X(t))$ after time τ. Assume that the investment cost is $I_1(> 0)$ $(I_2(> I_1))$ for the small-scale (large-scale) expansion if the whole amount of the cost is internally financed. If part of the investment cost is externally financed, the firm pays a proportional cost $C(\geq 0)$ of external financing. The total cost associated with the investment is expressed as $I_i + C \max(I_i - Y(\tau), 0)$ $(i = 1, 2)$, where $Y(\tau)$ denotes the firm's cash reserves at time τ. Until the investment time τ, cash reserves $Y(t)$ follow

$$\mathrm{d}Y(t) = rY(t)\mathrm{d}t + X(t)\mathrm{d}t,\ (0 < t < \tau) \quad Y(0) = y, \tag{2}$$

where $y(\geq 0)$ is a constant representing the initial cash reserves. Note that $Y(t)$ is an increasing process.

Boyle and Guthrie (2003) assume the dynamics of cash reserves exogenously and consider an option to initiate a new project. In contrast, we relate cash reserves $Y(t)$ to operating cash flows $X(t)$ more directly and consider the option to expand production. In the case of $C = 0$, the setup corresponds to an alternative investment model studied in Dixit (1993) and Décamps, Mariotti, and Villeneuve (2006). For a comprehensive list of typical situations fitting the standard model, refer to Dixit (1993), Dixit and Pindyck (1994), and Décamps, Mariotti, and Villeneuve (2006). We extend the standard model, which presumes that the firm needs no costs of external financing (otherwise, it has sufficient internal funds), to a model involving costs of external financing. Unlike Boyle and Guthrie (2003), who focused on a liquidity constraint in fixed-size investment, we examine the interactions of investment timing, size, and financing costs.[3]

Our assumption of costly external financing is justified as follows. In the pecking order theory, agency and asymmetric information problems cause costs of external financing,

[3] Hirth and Uhrig-Homburg (2010b) extended Boyle and Guthrie (2003) to a case involving both a liquidity constraint and financing costs. Nishihara and Shibata (2011) developed a model involving financing costs, whereas Shibata and Nishihara (2012) focused on a debt issuance constraint. However, these papers assume fixed-scale investment.

which leads to a preference for internal over external finance (Myers (1984), Myers and Majluf (1984)). Practically, financing costs consist of a fixed cost (which is independent of the issue size) and a variable cost (which depends on the issue size). A fixed cost includes taxes, fees, and setup expenses. A variable cost increases with issue size primarily because more underwriting services are required for more funds raised. In the standard view of the literature, a variable cost is convex with respect to the issue size (e.g., Altinkilic and Hansen (2000)). Hennessy and Whited (2007) estimated that proportional costs of equity financing are approximately 5% (10%) for large (small) firms. They showed that a proportional cost can almost completely account for costs of equity financing for large firms, although the fixed cost effect may be not negligible for small firms. In taking account of their results, as well as preserving tractability of the model, we examine the case with only a proportional cost in full detail in Sections 3 and 4.1, and succinctly explain how the results change in the case involving both fixed and proportional costs as a supplement in Section 4.2.

2.2 Case with no financing costs

As a benchmark, this section explains results in the case of $C = 0$. In this case, the firm solves the following problem:

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}^x[\max_{i=1,2} \mathbb{E}^{\tau, X(\tau)}[\int_0^\tau \mathrm{e}^{-rt} X(t)\mathrm{d}t + \int_\tau^\infty \mathrm{e}^{-rt} A_i X(t)\mathrm{d}t - \mathrm{e}^{-r\tau} I_i]], \tag{3}$$

where $\mathcal{T}$ denotes the set of all stopping times and $\mathbb{E}^x[\cdot]$ ($\mathbb{E}^{\tau,X(\tau)}[\cdot]$) denotes the expectation conditional on $t = 0, X(0) = x$ ($t = \tau, X(t) = X(\tau)$). In (3), τ represents the time to expand the scale of production, whereas $\max_{i=1,2} \mathbb{E}^{\tau,X(\tau)}[\cdot]$ represents the sizing choice at time τ. By the strong Markov property of $X(t)$, (3) can be reduced to

$$\frac{x}{r-\mu} + \underbrace{\sup_{\tau \in \mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \max_{i=1,2} \left(\frac{A_i - 1}{r-\mu} X(\tau) - I_i \right)]}_{=:V_0(x) \text{ the growth option value}}.$$

The second term, denoted by $V_0(x)$, represents the growth option value. Décamps, Mariotti, and Villeneuve (2006) derived a closed-form solution for this type of problem.[4] Indeed, we have $V_0(x)$ depending on the relation of A_1, A_2, I_1 and I_2 as follows:

[4]For this type of problem, refer also to Nishihara and Ohyama (2008).

If $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} < I_2/I_1$,

$$V_0(x) = \begin{cases} \left(\frac{(A_1-1)x_1^*}{r-\mu} - I_1\right)\left(\frac{x}{x_1^*}\right)^\beta & (0 < x < x_1^*) \text{ waiting} \\ \frac{(A_1-1)x}{r-\mu} - I_1 & (x_1^* \le x \le x_{21}^*) \text{ small-scale expansion} \\ \frac{(A_1-1)x_2^{*\gamma-1} - (A_2-1)x_1^{*\gamma-1}}{\beta(r-\mu)(x_1^{*\beta-1}x_2^{*\gamma-1} - x_1^{*\gamma-1}x_2^{*\beta-1})}x^\beta & \\ +\frac{(A_1-1)x_2^{*\beta-1} - (A_2-1)x_1^{*\beta-1}}{\gamma(r-\mu)(x_1^{*\gamma-1}x_2^{*\beta-1} - x_1^{*\beta-1}x_2^{*\gamma-1})}x^\gamma & (x_{21}^* \le x \le x_{22}^*) \text{ waiting} \\ \frac{(A_2-1)x}{r-\mu} - I_2 & (x \ge x_{22}^*) \text{ large-scale expansion,} \end{cases} \tag{4}$$

where $\beta := 1/2-\mu/\sigma^2 + \sqrt{(\mu/\sigma^2-1/2)^2+2r/\sigma^2}$ (>1) and $\gamma := 1/2-\mu/\sigma^2 - \sqrt{(\mu/\sigma^2-1/2)^2+2r/\sigma^2}$ (<0) is positive and negative characteristic roots, respectively. Threshold x_1^* is defined by $x_1^* := \beta(r-\mu)I_1/\{(A_1-1)(\beta-1)\}$, which is equal to the threshold for an option to invest only in the small-scale project. On the other hand, thresholds x_{21}^* and x_{22}^* are determined by the value matching (continuous fit) conditions at the boundaries.

If $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} \ge I_2/I_1$,

$$V_0(x) = \begin{cases} \left(\frac{(A_2-1)x_2^*}{r-\mu} - I_2\right)\left(\frac{x}{x_2^*}\right)^\beta & (0 < x < x_2^*) \text{ waiting} \\ \frac{(A_2-1)x}{r-\mu} - I_2 & (x \ge x_2^*) \text{ large-scale expansion,} \end{cases} \tag{5}$$

where $x_2^* := \beta(r-\mu)I_2/\{(A_2-1)(\beta-1)\}$. In this case, the problem is reduced to a problem of investing only in the large-scale project because the value (5) dominates the value of the small-scale investment, $(A_1-1)x/(r-\mu) - I_1$, for all $x > 0$.

Note that $\beta/(\beta-1)$ monotonically increases with σ because of $\partial\beta/\partial\sigma < 0$. A higher σ is more likely to lead to the case of $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} \ge I_2/I_1$. The intuition is that a higher σ increases the value of the large-scale growth option so that the value (5) can dominate $(A_1-1)x/(r-\mu) - I_1$ for all $x > 0$. For details, refer to Décamps, Mariotti, and Villeneuve (2006). We also note that

$$V_0(x) \ge \underbrace{\max_{i=1,2}\left\{\sup_{\tau\in\mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau}\left(\frac{A_i-1}{r-\mu}X(\tau) - I_i\right)]\right\}}_{\text{the value by the initial sizing choice}}, \tag{6}$$

where in the left-hand side the sizing decision $i \in \{1,2\}$ is $\mathcal{F}_\tau$-measurable, while in the right-hand side $i \in \{1,2\}$ is $\mathcal{F}_0$-measurable. Dixit (1993) focused only on the right-hand problem, and Décamps, Mariotti, and Villeneuve (2006) complemented his argument by solving the left-hand problem. Because of the difference, in general we have the inequality (6). However, the equality

$$V_0(x) = \max_{i=1,2}\left\{\sup_{\tau\in\mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau}\left(\frac{A_i-1}{r-\mu}X(\tau) - I_i\right)]\right\}, \tag{7}$$

holds for $x \le x_1^*$. This means that, if the initial value $X(0) = x$ is sufficiently low, this problem is unchanged from the case in which the firm chooses the investment size not dynamically but at the initial time.

3 Analytic Results

3.1 Model solutions

This section provides analytic results in the case of $C > 0$. In this case, the growth option value, denoted by $V_C(x, y)$, is expressed as

$$V_C(x,y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{x,y}[\mathrm{e}^{-r\tau} \max_{i=1,2} \left(\frac{A_i - 1}{r - \mu} X(\tau) - I_i - C \max(I_i - Y(\tau), 0) \right)], \tag{8}$$

where where $\mathbb{E}^{x,y}[\cdot]$ denotes the expectation conditional on $t = 0, X(0) = x, Y(0) = y$. The term $C \max(I_i - Y(\tau), 0)$ represents that a proportional cost is required when the firm is short of cash reserves, whereas $\max_{i=1,2}(\cdot)$ means that the firm chooses the optimal size at the investment time τ.

First, we prove several properties of the option value $V_C(x, y)$. Consider the following problems as approximations of $V_C(x, y)$:

$$V_L(x,y) := \sup_{\tau \in \mathcal{T}} \mathbb{E}^{x}[\mathrm{e}^{-r\tau} \max_{i=1,2} \left(\frac{A_i - 1}{r - \mu} X(\tau) - I_i - C \max(I_i - y, 0) \right)],$$
$$V_U(x,y) := \sup_{\tau \in \mathcal{T}} \mathbb{E}^{x,y}[\mathrm{e}^{-r\tau} \max_{i=1,2} \left(\frac{A_i - 1}{r - \mu} X(\tau) - I_i - C(I_i - Y(\tau)) \right)].$$

$V_L(x, y)$ is the same type of problem as $V_0(x)$ so that it allows an explicit solution like (4) or (5). By the strong Markov property of $X(t)$, we can easily show

$$V_U(x,y) = C\left(y + \frac{x}{r-\mu}\right) + \sup_{\tau \in \mathcal{T}} \mathbb{E}^{x,y}[\mathrm{e}^{-r\tau} \max_{i=1,2} \left(\frac{A_i - C - 1}{r - \mu} X(\tau) - (1 + C) I_i \right)].$$

If $A_2 - C - 1 > 0$, we have an explicit solution like (4) or (5). Otherwise, $V_U(x, y) = C(y + x/(r - \mu))$ holds, which means that the growth option will be never exercised. Note that in both $V_L(x, y)$ and $V_U(x, y)$ the exercise policies are independent of $Y(t)$. The following proposition indicates that $V_L(x, y)$ and $V_U(x, y)$ are closed-form bounds of $V_C(x, y)$.

Proposition 1 (Option value)
If $y < I_2$, $V_L(x, y) \le V_C(x, y) \le V_U(x, y)$ is satisfied.
Otherwise, $V_C(x, y) = V_0(x)$ holds.

Proposition 1 states that, once $Y(t)$ reaches the large-scale investment cost I_2, the problem with financing costs is reduced to that of no financing cost. We will also use the following lemma to show the properties of the optimal financing and investment policy.

Lemma 1
If $A_2 - C - 1 > 0$,

$$0 \le V_C(x+\Delta, y) - V_C(x,y) \le \frac{(A_2-1)\Delta}{r-\mu} \tag{9}$$

holds for any positive constant Δ.

$$0 \le V_C(x, y+\Delta) - V_C(x,y) \le C\Delta \tag{10}$$

holds for any positive constant Δ.

Next, we concentrate on the exercise regions for the problem (8). The standard argument proves that the exercise region of the option is expressed as

$$S_C := \{(x,y) \in \mathbb{R}_+^2 \mid V(x,y) = \max_{i=1,2}\{(A_i-1)x/(r-\mu) - I_i - C\max(I_i - y, 0)\}\}. \tag{11}$$

Furthermore, the exercise region S_C can be decomposed into disjoint sets (some may be empty) defined by

$$\begin{aligned}
S_{C,1,E} &:= \{(x,y) \in \mathbb{R}_+^2 \mid V(x,y) = (A_1-1)x/(r-\mu) - I_1 - C(I_1 - y), y < I_1\},\\
S_{C,1,I} &:= \{(x,y) \in \mathbb{R}_+^2 \mid V(x,y) = (A_1-1)x/(r-\mu) - I_1, y \ge I_1\},\\
S_{C,2,E} &:= \{(x,y) \in \mathbb{R}_+^2 \mid V(x,y) = (A_2-1)x/(r-\mu) - I_2 - C(I_2 - y), y < I_2\},\\
S_{C,2,I} &:= \{(x,y) \in \mathbb{R}_+^2 \mid V(x,y) = (A_2-1)x/(r-\mu) - I_2, y \ge I_2\}.
\end{aligned}$$

The regions $S_{C,1,E}$ and $S_{C,1,I}$ represent small-scale investment regions partially with external financing and entirely with internal financing, respectively, whereas $S_{C,2,E}$ and $S_{C,2,I}$ represent large-scale investment regions partially with external financing and entirely with internal financing, respectively. Below, we analytically prove the interesting properties of $S_{C,1,E}$, $S_{C,1,I}$, $S_{C,2,E}$, and $S_{C,2,I}$. It immediately follows from Proposition 1 that the exercise regions coincide with those of $V_0(x)$ for $y \ge I_2$. Note that thresholds $x_1^*, x_{21}^*, x_{22}^*$, and x_2^* were explained in Section 2.2.

Proposition 2 (Case of sufficient cash reserves)
If $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} < I_2/I_1$, $S_{C,1,I} \cap \mathbb{R}_+ \times [I_2, \infty) = [x_1^*, x_{21}^*] \times [I_2, \infty)$ and $S_{C,2,I} = [x_{22}^*, \infty) \times [I_2, \infty)$ hold. Otherwise, $S_{C,1,I} \cap \mathbb{R}_+ \times [I_2, \infty) = \emptyset$ and $S_{C,2,I} = [x_2^*, \infty) \times [I_2, \infty)$ hold.

Now, we examine the properties of the exercise regions in the region $y < I_2$. In this region, the optimal policy is contingent on a combination of $X(t)$ and $Y(t)$, and, hence, it is quite different from the threshold policy depending only on $X(t)$ in the region $y \ge I_2$. The following proposition reveals the properties of the large-scale investment partially with external financing, $S_{C,2,E}$. Note that the region of the large-scale investment entirely with internal financing, $S_{C,2,I}$, is explicitly derived in Proposition 2.

Proposition 3 (Large-scale expansion)
If $A_2 - C - 1 > 0$ is satisfied,

$$S_{C,2,E} = \{(x, y) \in \mathbb{R}_+^2 \mid x \geq x_C^*(y), y < I_2\}, \tag{12}$$

where $x_C^*(\cdot)$ is a continuous and monotonically decreasing function. $x_C^*(y)$ is between the corresponding thresholds in $V_L(x, y)$ and $V_U(x, y)$, and $\lim_{y \uparrow I_2} x_C^*(y)$ is equal to the maximum of x_2^* (or x_{22}^*) and $(C + 1)rI_2/(A_2 - C - 1)$.
Otherwise, $S_{C,2,E} = \emptyset$.

Here we limit our attention to a case in which a proportional cost is relatively low, i.e., $A_2 - C - 1 > 0$. We will explain the case of $A_2 - C - 1 \leq 0$ below Proposition 6. In the case of $A_2 - C - 1 > 0$, the firm invests in the large-scale investment partially with external financing when the output price $X(t)$ exceeds the threshold $x_C^*(Y(t))$. Proposition 3 ensures monotonicity in the lager-scale investment threshold $x_C^*(Y(t))$ with respect to cash reserves $Y(t)$. This monotonicity can be straightforwardly explained as follows. An increase in $Y(t)$ decreases a financing cost $C(I - Y(t))$, which accelerates the large-scale investment.

We now examine the properties of the small-scale investment regions, $S_{C,1,E}$ and $S_{C,1,I}$. Proposition 4 suggests the possibility that $S_{C,1,E}$ and $S_{C,1,I}$ are generated by costs of external financing. Proposition 5 describes the properties of $S_{C,1,E}$ and $S_{C,1,I}$ on the presumption that they exist.

Proposition 4 (Possibility of the small-scale expansion)
If $\{(A_2 - 1)/(A_1 - 1)\}^{\beta/(\beta-1)} < I_2/I_1$, $S_{C,1,E}$ may not be empty and $S_{C,1,I} \neq \emptyset$.
If $I_2/I_1 \leq \{(A_2 - 1)/(A_1 - 1)\}^{\beta/(\beta-1)} < \{I_2 + C(I_2 - I_1)\}/I_1$, $S_{C,1,E}$ and $S_{C,1,I}$ may not be empty.
Otherwise, $S_{C,1,E} = S_{C,1,I} = \emptyset$.

Proposition 5 (Small-scale expansion)
If $(x, y) \in S_{C,1,E}$, $(x, I_1) \in S_{C,1,I}$ and $(x, y') \in S_{C,1,E}$ holds for any $y' \in [y, I_1)$.
If $(x, y) \in S_{C,1,I}$, $(x, y') \in S_{C,1,I}$ holds for any $y' \in [I_1, y]$.
If $\{(A_2 - 1)/(A_1 - 1)\}^{\beta/(\beta-1)} < I_2/I_1$, $\min\{x \geq 0 \mid (x, y) \in S_{C,1,I}\} = x_1^*$ holds for any fixed $y \geq I_1$.

Proposition 5 shows that the small-scale investment region enhances with cash reserves $Y(t)$ until $Y(t) = I_1$ and from the point it decreases with $Y(t)$. This non-monotonic property is in sharp contrast with the monotonicity of the large-scale investment region proved in Proposition 3. Below, we explain the interesting result in terms of the investment sizing choice changing with $Y(t)$.

[Insert Figure 1 about here.]

Figure 1 illustrates how the ratio of the total cost associated with the large-scale expansion to that of the small-scale expansion, $\{I_2+C\max(I_2-Y(t),0)\}/\{I_1+C\max(I_1-Y(t),0)\}$, changes with $Y(t)$. Note that the ratio regarding the profit expansion, $(A_2-1)/(A_1-1)$, is independent of $Y(t)$. The ratio is equal to I_2/I_1 for $Y(t)=0$ because external funds cover entire investment costs in both projects. For $Y(t)\in(0,I_1]$, the internal funds cover $Y(t)$ out of I_1 and I_2 in the small- and large-scale projects, respectively. The coverage ratio in the small-scale project, $Y(t)/I_1$, increases more with $Y(t)$ than that of the large-scale project. Then, the ratio $\{I_2+C\max(I_2-Y(t),0)\}/\{I_1+C\max(I_1-Y(t),0)\}$ monotonically increases up to the maximum level $\{I_2+C(I_2-I_1)\}/I_1$ for $Y(t)=I_1$. This increase leads to the result that a higher $Y(t)$ encourages the firm to invest in the small-scale project until $Y(t)=I_1$. For $Y(t)\in[I_1,I_2)$, the small-scale project requires no external financing, while the large-scale project requires the external funds $I_2-Y(t)$. An increase in $Y(t)$ decreases the total cost associated with the large-scale investment, preserving the small-scale investment cost unchanged. Then, the ratio $\{I_2+C\max(I_2-Y(t),0)\}/\{I_1+C\max(I_1-Y(t),0)\}$ monotonically falls to the minimum level I_2/I_1 for $Y(t)=I_2$. This decrease leads to the novel result that a higher $Y(t)$ discourages the firm to invest in the small-scale project within the region $Y(t)\in[I_1,I_2]$.

As explained by the mechanism above, the possibility of the small-scale expansion is maximized at the point $Y(t)=I_1$. Proposition 4 states that, if the maximum ratio $\{I_2+C(I_2-I_1)\}/I_1$ is smaller than the critical level $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)}$, the firm never undertakes the small-scale investment. Now, compare this result with that of the case with no financing costs. In the absence of financing costs, the firm never undertakes the small-scale investment if $I_2/I_1<\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)}$ is satisfied (see Section 2.2). In the presence of financing costs, on the other hand, the firm has a possibility of investing in the small-scale project in the region $I_2/I_1\leq\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)}<\{I_2+C(I_2-I_1)\}/I_1$. This demonstrates that costs of external financing can trigger the small-scale investment which is never undertaken in the case with no financing costs. This also suggests the counter-intuitive effect that financing costs may speed up investment (but it is small-scale).

We can prove the following proposition regarding the possibility of external financing.

Proposition 6 (Possibility of external financing)
If $C<A_1-1$, $S_{C,1,E}$ may not be empty and $S_{C,2,E}\neq\emptyset$.
If $A_1-1\leq C<A_2-1$, $S_{C,1,E}=\emptyset$ and $S_{C,2,E}\neq\emptyset$.
Otherwise, $S_{C,1,E}=S_{C,2,E}=\emptyset$.

It is clear that high financing costs prevent a firm from financing the project with external funds. Further, and more interestingly, this proposition suggests that the firm is more likely to rely on external funds in the large-scale expansion than in the small-scale expansion. To see this, suppose that the cost of external financing is intermediate, i.e., $A_1-1\leq C<A_2-1$. In this case, the small-scale project is always deferred until the

project can be financed entirely with internal funds. On the other hand, the large-scale project is undertaken partially with external financing when $X(t)$ reaches a sufficiently high level. The reasoning is as follows. In the region $Y(t) < I_1$, the firm receives cost savings of $CX(t)\mathrm{d}t + r(1 + C)I_1\mathrm{d}t$ and loses $(A_1 - 1)X(t)\mathrm{d}t$ by deferring the small-scale investment by an infinitesimally short period $\mathrm{d}t$. Then, the firm never finances the small-scale project with external funds for any $X(t)$ if $C \geq A_1 - 1$ is satisfied. In the case of the large-scale project, the positive effect $CX(t)\mathrm{d}t$ remains unchanged,[5] while the negative effect is enlarged to $(A_2 - 1)X(t)\mathrm{d}t$. This increases the firm's incentive to access external financing in the large-scale expansion.

3.2 Empirical implications

In this subsection, we provide empirical implications obtained from the propositions in Section 3.1 and clarify our contributions to the literature.

[Insert Figure 2 about here.]

Figure 2 summarizes the properties of the financing and investment policy in Propositions 3–6.[6] The first, second, and third rows correspond to the cases with high, intermediate, and low cost of external financing, respectively. The first, second, and third columns correspond to the cases with high, intermediate, and low values of I_2/I_1 (or $(A_1 - 1)/(A_2 - 1)$). In the hatched regions the firm invests entirely with internal financing, while in the other regions it relies partially on external financing. The figure also indicates the investment size the firm chooses in the exercise region.

When I_2/I_1 (or $(A_1-1)/(A_2-1)$) is sufficiently low, i.e., in the first column, the financing and investment policy is the same as that of the case with only the large-scale project (fixed-scale investment model). This case is studied in details by Nishihara and Shibata (2011), and, hence, we omit the explanation. The second and third columns present more interesting panels, in which the firm optimizes the investment size as well as investment timing. The panels show that costs of external financing bring many differences from the case with no financing constraint in Section 2.2 (refer also to Décamps, Mariotti, and Villeneuve (2006)). Among all, a key difference is regarding the necessity of dynamically deciding the investment size. Except for Panel (ix) the firm cannot determine the investment size at the initial time even if the initial value $X(0) = x$ is low. The firm invests in either the small- or large-scale project depending on the dynamics of $(X(t), Y(t))$. Recall

[5]The term $r(1 + C)I_2\mathrm{d}t$ is negligible when $X(t)$ is high.

[6]In all figures in this paper, we set the axes in the same way as Boyle and Guthrie (2003), Hirth and Uhrig-Homburg (2010b), and Nishihara and Shibata (2011) for comparison. This figure illustrates the cases in which x_2^* (or x_{22}^*) in problem $V_0(x)$ is larger than $(C+1)rI_2/(A_2 - C - 1)$ in Proposition 3. Otherwise, there is a gap between $\lim_{y\uparrow I_2} x_C^*(y)$ and x_2^* (or x_{22}^*) for $Y(t) = I_2$. The characteristics remain unchanged except for the gap at the point $Y(t) = I_2$.

that in Section 2.2 the dynamic choice is not necessary for a sufficiently low $X(0) = x$. Our results complement Dixit (1993) and Décamps, Mariotti, and Villeneuve (2006) by revealing the significance of the firm's dynamic sizing choice in the presence of costs of external financing.

Next, we clarify a significant contribution to the corporate finance literature. In corporate finance, there have been long-term arguments about sensitivities of investment to cash flow since seminal works by Fazzari, Hubbard, and Petersen (1988), Kaplan and Zingales (1997), and Hubbard (1998). Recently, several empirical studies regarding this issue have demonstrated a counter-intuitive result that the investment volume does not necessarily decrease with a degree of financing constraints. Specifically, Cleary, Povel, and Raith (2007) and Guariglia (2008) documented a U-shaped relation between the investment volume and internal funds. This suggests that an intermediate level of financing constraint can lead to underinvestment.

On the other hand, a recent stream of real options literature has provided another prediction: an intermediate level of financing constraint can lead a firm to hasten investment. Boyle and Guthrie (2003) showed that the investment threshold has a U-shaped relation with a degree of a liquidity constraint because of a firm's incentive to avoid the risk of a cash shortfall. Shibata and Nishihara (2012) showed that a firm's consideration of the optimal capital structure leads to a U-shaped relation between the investment threshold and a debt capacity constraint. If "hastened" investment is equal to "increased" investment, the prediction from the real options literature is contrary to the empirical evidence. However, this argument is not precise because the investment timing studies consider fixed-scale investment models. Indeed, by integrating both investment timing and sizing problems, we can demonstrate that the two results do not conflict but complement each other.

[Insert Figure 3 about here.]

To see this clearly, we concentrate on the most interesting case, Panel (viii) (or (v)) in Figure 2.[7] Figure 3 illustrates the minimum threshold price in which the firm invests and the project that is chosen at the threshold in this case. This figure indicates that the firm with low or high cash reserves undertakes the large-scale project on the later timing, while, with intermediate cash reserves ($Y(t) \approx I_1$), it invests in the small-scale project on the earlier timing. An intermediate level of cash holdings can play both roles in accelerating the investment time and decreasing the investment size. In the presence of the cost of external financing, as explained with Figure 1, an intermediate level of cash reserves provides the greatest incentive for the firm to undertake the small-scale expansion rather than the large-scale expansion. Then, from the viewpoint of investment

[7]In other cases, our model can also explain the conventional results, i.e., the monotonic relations regarding the investment volume and timing. In the real options literature, Milne and Robertson (1996), Nishihara and Shibata (2010), and Hirth and Uhrig-Homburg (2010a) are in line with the monotonicity.

size, the investment is scaled down from the first-best level of a case with no financing cost. At the same time, from the viewpoint of investment timing, the investment takes place earlier than the first-best timing because the small-scale investment has a lower value of the deferring option than that of the large-scale investment. Unlike Boyle and Guthrie (2003) and Shibata and Nishihara (2012), by the interaction of financing costs, investment timing, and investment size, we can explain both non-monotonic relations regarding the investment volume and timing. This mechanism fills the gap between the investment volume and timing literature.

Lastly, we explain a contribution to the literature from a technical viewpoint. This paper, unlike most of the related papers relying on numerical simulations, attains analytic results. Although some of the techniques used in the proof are inspired by the mathematical finance literature (e.g., Broadie and Detemple (1997), Detemple (2006), Bobtcheff and Villeneuve (2010)), several developments are attained by this paper.[8] While the mathematical finance studies investigated the exercise regions of American options that involve a multi-dimensional geometric Brownian motion, the stochastic process $Y(t)$ in our model is not a geometric Brownian motion; instead, it is defined by (2). Furthermore, the payoff function of problem (8) is not convex, which makes the proofs more difficult. These technical developments can be potentially applied to a variety of investment timing and sizing problems.

4 Numerical examples

As mentioned in Section 1, the main contribution of this paper is to attain analytic results regarding the firm's optimal financing and investment policy in Section 3. We supplement the results from two aspects in numerical examples. Section 4.1 presents the comparative statics results with respect to the output price volatility σ. Section 4.2 we present the results in a case with a fixed cost in addition to a proportional cost of external financing.

4.1 Base case

We explore the effects of output price uncertainty to the financing and investment policy. The base parameter values are set as follows:

$$r = 0.07, \mu = 0.03, \sigma = 0.2, I_1 = 50, I_2 = 100, A_1 = 1.385, A_2 = 1.5, C = 0.1. \tag{13}$$

For comparison, we set the parameter values for the large-scale expansion at the same values as Nishihara and Shibata (2011). These parameter values are also similar to those of the standard real options literature.

[8] Nishihara and Shibata (2011) developed similar techniques to prove the properties of the optimal financing and investment policy in the presence of a proportional cost of external financing, but they assumed a fixed investment size.

For the base parameter values (13), we have $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} = 1.9569 < I_2/I_1 = 2$ regarding the possibility of the small-scale expansion (Proposition 4). This means that the firm may undertake the small-scale investment. We also have $C = 0.1 < A_1 - 1 = 0.285$ regarding the possibility of external financing (Proposition 6). This means that the projects may be financed partially with external financing. In the computation, we make a tri-nomial lattice model that approximates to a geometric Brownian motion (1), and we use a value function iteration algorithm.

[Insert Figure 4 about here.]

Figure 4 plots the exercise regions with varying levels of σ. We can see from Figure 4 that a higher σ decreases the exercise regions, which implies that the investment threshold and the option value enhance with σ. This volatility effect is straightforwardly consistent with the standard results with neither financing costs nor sizing choice.

Further and more notably, Figure 4 demonstrates that the volatility effect is stronger for the small-scale investment than for the large-scale investment. As σ increases up to 0.2, the region of the small-scale investment partially with external financing, $S_{C,1,E}$, disappears first. As σ increases up to 0.25, the region of the small-scale investment entirely with internal financing, $S_{C,1,I}$, disappears either. Then, for $\sigma = 0.25$ and 0.3, the financing and investment policy is the same as that of a case in which the investment size is fixed at the large scale. The rationale is based on Proposition 4. Indeed, we have $\{(A_2 - 1)/(A_1-1)\}^{\beta/(\beta-1)} < I_2/I_1$ for $\sigma = 0.1, 0.15, 0.2$, $I_2/I_1 \le \{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} < \{I_2 + C(I_2 - I_1)\}/I_1$ for $\sigma = 0.213$, and $\{I_2 + C(I_2-I_1)\}/I_1 \le \{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)}$ for $\sigma = 0.25, 0.3$. Note that $\beta/(\beta-1)$ monotonically increases with σ. As a result, the financing and investment policy becomes like panel (ix) $\to$ panel (vi) $\to$ panel (v) $\to$ panel (iv) in Figure 2 with an increase in σ. A higher σ, like a higher I_2/I_1 or $(A_1-1)/(A_2-1)$, increases the advantage of the large-scale expansion over the small-scale expansion. It should be also noted that the panel of $\sigma = 0.213$ corresponds to the most interesting case, panel (v) in Figure 2 (or Figure 3).

4.2 Extension to a case with fixed and proportional costs

So far we have concentrated on a proportional cost of external financing because a proportional cost accounts for the greatest part of financing (e.g., Hennessy and Whited (2007)). However, according to Hennessy and Whited (2007), the effects of a fixed cost may be observable for small firms. To check the robustness, we present a case with both fixed and proportional costs of external financing.

[Insert Figure 5 about here.]

Figure 5 illustrates the exercise regions in the extended case with varying levels of σ. In the numerical examples, the fixed cost is set at 1. This means that the total financing cost is equal to $0.1 \max(I_i - Y(t), 0) + 1_{\{I_i - Y(t) > 0\}}$, where $1_{\{\cdot\}}$ denotes the indicator function. For $Y(t) \uparrow I_i$, a fixed cost provides the greatest incentive for the firm to wait and invest entirely with internal funds. Then, the firm never invests when $Y(t)$ reach nearly $I_1 = 50$ or $I_2 = 100$, which leads to disconnected exercise regions. This fixed cost effect is documented in Nishihara and Shibata (2011) who examined the fixed cost effects in a fixed-scale investment model. Furthermore, we can see from Figure 5 that the fixed cost effect is strong particularly in the small-scale project. Indeed, compared to Figure 4, Figure 5 has no small-scale investment region partially with external financing, $S_{C,1,E}$, at all. The presence of a fixed cost removes the possibility of investing in the small-scale project partially with external financing. Figure 5 inherits the other characteristics from Figure 4. Most notably, the panel of $\sigma = 0.213$ still holds an island-like small-scale investment region even if a fixed cost is considered.

5 Conclusion

This paper examined the interactions of financing costs, investment timing, and investment sizing in a dynamic model. We assumed that a firm can invest in either a small- or large-scale project with cash reserves, which are increasing with time, and external funds that require a proportional cost. We, unlike most of the related papers, analytically proved the properties of the corporate financing and investment policy. The results are summarized as follows.

Financing costs lead to the financing and investment policy strongly contingent on the dynamics of both the cash flow and reserves. In particular, in the presence of financing costs, the firm is more likely to invest in the small-scale project entirely using internal funds. The investment threshold for the large-scale project monotonically decreases with cash reserves, while the investment region for the small-scale project has a non-monotonic relation with cash reserves. Most notably, an intermediate level of cash reserves can play both roles in speeding up the investment and in decreasing the investment size. Our results can explain two types of results in corporate finance: (i) empirical findings that the investment volume has a U-shaped relation with internal funds, and (ii) empirical predictions that the investment threshold has a U-shaped relation with internal funds.

Acknowledgment

This work was supported by KAKENHI (22710146, 22710142, 23310103, 21241040).

A Proof of Proposition 1

Since $Y(t)$ monotonically increases from the initial value y, we have

$$I_i - Y(t) \leq \max(I_i - Y(t), 0) \leq \max(I_i - y, 0) \quad (i = 1, 2)$$

at any time t. This implies that $V_L(x, y) \leq V_C(x, y) \leq V_U(x, y)$ holds. The problem (8) can be reduced to the problem with no financing cost when $y \geq I_2$.

B Proof of Lemma 1

First, we prove (9). Note that $\mathbb{E}^1[\cdot]$ represents the expectation with $X(t)$ starting from $X(0) = 1$. For any positive constant Δ, we have

$$\begin{aligned}
&V_C(x+\Delta, y) \\
=&\sup_{\tau\in\mathcal{T}} \mathbb{E}^1[\mathrm{e}^{-r\tau} \max_{i=1,2} \left(\frac{(A_i - 1)(x+\Delta)}{r-\mu} X(\tau) - I_i - C \max(I_i - \mathrm{e}^{r\tau} y - \int_0^\tau \mathrm{e}^{r(\tau - s)} (x+\Delta) X(s) \mathrm{d}s, 0) \right)] \\
\leq&\sup_{\tau\in\mathcal{T}} \mathbb{E}^1[\mathrm{e}^{-r\tau} \left(\max_{i=1,2} \left(\frac{(A_i - 1)x}{r-\mu} X(\tau) - I_i - C \max(I_i - \mathrm{e}^{r\tau} y - \int_0^\tau \mathrm{e}^{r(\tau - s)} x X(s) \mathrm{d}s, 0) \right) \right. \\
&\left. + \frac{(A_2 - 1)\Delta}{r-\mu} X(\tau) + C\Delta \int_0^\tau \mathrm{e}^{r(\tau - s)} X(s) \mathrm{d}s \right)] \qquad (14) \\
\leq&\underbrace{\sup_{\tau\in\mathcal{T}} \mathbb{E}^1[\mathrm{e}^{-r\tau} \max_{i=1,2} \left(\frac{(A_i - 1)x}{r-\mu} X(\tau) - I_i - C \max(I - \mathrm{e}^{r\tau} y - \int_0^\tau \mathrm{e}^{r(\tau - s)} x X(s) \mathrm{d}s, 0) \right)]}_{=V_C(x,y)} \\
&+ \sup_{\tau\in\mathcal{T}} \mathbb{E}^1[\mathrm{e}^{-r\tau} \left(\frac{(A_2 - 1)\Delta}{r-\mu} X(\tau) + C\Delta \int_0^\tau \mathrm{e}^{r(\tau - s)} X(s) \mathrm{d}s \right)] \\
=&V_C(x, y) + \sup_{\tau\in\mathcal{T}} \mathbb{E}^1[\mathrm{e}^{-r\tau} \left(\frac{(A_2 - 1)\Delta}{r-\mu} X(\tau) \right) + C\Delta \left(\int_0^\infty \mathrm{e}^{-rs} X(s) \mathrm{d}s - \int_\tau^\infty \mathrm{e}^{-rs} X(s) \mathrm{d}s \right)] \\
=&V_C(x, y) + \frac{C\Delta}{r-\mu} + \frac{(A_2 - 1 - C)\Delta}{r-\mu} \underbrace{\sup_{\tau\in\mathcal{T}} \mathbb{E}^1[\mathrm{e}^{-r\tau} X(\tau)]}_{=1} \qquad (15) \\
=&V_C(x, y) + \frac{(A_2 - 1)\Delta}{r-\mu},
\end{aligned}$$

where (14) follows from $A_1 < A_2$, and in (15) $\sup_{\tau\in\mathcal{T}} \mathbb{E}^1[\mathrm{e}^{-r\tau} X(\tau)] = 1$ follows from $\mu < r$.

Next, we prove (10). For any positive constant Δ, we have

$$
\begin{aligned}
&V_C(x, y+\Delta) \\
=&\sup_{\tau\in\mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \max_{i=1,2}\left(\frac{(A_i-1)}{r-\mu}X(\tau) - I_i - C\max(I_i - \mathrm{e}^{r\tau}(y+\Delta - \int_0^\tau \mathrm{e}^{r(\tau-s)}X(s)\mathrm{d}s, 0)\right)] \\
\le&\sup_{\tau\in\mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \left(\max_{i=1,2}\left(\frac{(A_i-1)}{r-\mu}X(\tau) - I_i - C\max(I_i - \mathrm{e}^{r\tau}y - \int_0^\tau \mathrm{e}^{r(\tau-s)}X(s)\mathrm{d}s, 0)\right) + \mathrm{e}^{r\tau}C\Delta\right)] \\
=&\underbrace{\sup_{\tau\in\mathcal{T}} \mathbb{E}^x[\mathrm{e}^{-r\tau} \max_{i=1,2}\left(\frac{(A_i-1)}{r-\mu}X(\tau) - I_i - C\max(I_i - \mathrm{e}^{r\tau}y - \int_0^\tau \mathrm{e}^{r(\tau-s)}X(s)\mathrm{d}s, 0)\right)]}_{=V_C(x,y)} + C\Delta \\
=&V_C(x,y) + C\Delta.
\end{aligned}
$$

C Proof of Proposition 2

Note that $Y(t)$ monotonically increases from the initial value y. Once $Y(t)$ reached I_2, $V_C(x,y)$ can be reduced to $V_0(x,y)$. Then, the exercise region agrees with the exercise region for the benchmark problem $V_0(x,y)$ which is explicitly obtained in Section 2.2.

D Proof of Proposition 3

First, consider the case of $A_2 - C - 1 > 0$. Fix $(x,y) \in S_{C,2,E}$ and (x',y') satisfying $x \le x'$ and $y \le y' < I_2$. Using Lemma 1, we have

$$
\begin{aligned}
V_C(x',y') &= V_C(x',y') - V_C(x,y') + V_C(x,y') - V_C(x,y) + V_C(x,y) \\
&\le \frac{(A_2-1)(x'-x)}{r-\mu} + C(y'-y) + V_C(x,y) \\
&= \frac{(A_2-1)(x'-x)}{r-\mu} + C(y'-y) + \frac{(A_2-1)x}{r-\mu} - I_2 - C(I_2-y) \\
&= \frac{(A_2-1)x'}{r-\mu} - I_2 - C(I_2-y'),
\end{aligned}
$$

where the last inequality implies $(x',y') \in S_{C,2,E}$. This proves that $S_{C,2,E}$ is expressed as (12) with the decreasing function $x_C^*(\cdot)$. By Proposition 1, we can immediately show that $x_C^*(y)$ is between the corresponding thresholds in $V_L(x,y)$ and $V_U(x,y)$.

Next, we derive $\lim_{y\uparrow I_2} x_C^*(y)$. Clearly we have $\lim_{y\uparrow I_2} x_C^*(y) \ge x_2^*$. Here we explain the case of $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} \ge I_2/I_1$. Replace x_2^* with x_{22}^* in the case of $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} < I_2/I_1$. Denote

$$
f_2(x,y) := \frac{(A_2-1)x}{r-\mu} - I_2 - C(I_2-y). \tag{16}
$$

We have

$$\begin{aligned}
&\mathcal{L}f_2(x,y) - rf_2(x,y) \le 0 \\
\Leftrightarrow &\frac{(A_2-1)\mu x}{r-\mu} + C(x+ry) - r\left(\frac{(A_2-1)x}{r-\mu} - I_2 - C(I_2-y)\right) \le 0 \\
\Leftrightarrow &x \ge \frac{(C+1)rI_2}{A_2-C-1},
\end{aligned} \tag{17}$$

where $\mathcal{L}$ denotes the generating operator of $(X(t), Y(t))$, i.e.,

$$\mathcal{L} := \mu x \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (x+ry)\frac{\partial}{\partial y}. \tag{18}$$

Since the general theory of optimal stopping ensures $\mathcal{L}V_C(x,y) - rV_C(x,y) \le 0$ (refer to Peskir and Shiryaev (2006)), (17) implies that $f_2(x,y)$ is not equal to $V_C(x,y)$ for $x < (C+1)rI_2/(A_2-C-1)$ and $y < I_2$. In other words, the option is not exercised in the region $\{(x,y) \in \mathbb{R}_+^2 \mid x < (C+1)rI_2/(A_2-C-1), y < I_2\}$. This proves that $\lim_{y\uparrow I_2} x_C^*(y) \ge (C+1)rI_2/(A_2-C-1)$.

[Insert Figure 6 about here.]

Now, suppose that $(C+1)rI_2/(A_2-C-1) \le x_2^* < \lim_{y\uparrow I_2} x_C^*(y)$. See Figure 6. We can lead to contradiction as follows. Consider problem

$$\sup_{\tau\in\mathcal{T},\tau\le T} \mathbb{E}^{x,y}[\mathrm{e}^{-r\tau} f_2(X(\tau), Y(\tau))] \tag{19}$$

with a finite maturity T. Generally, the exercise region of an American option converges to the region $\mathcal{L}f_2 - rf_2 \le 0$, when the remaining life of the option goes to zero (refer to Detemple (2006)). Then, because of (17), the exercise region of problem (19) with a finite maturity T converges to $\{(x,y) \in \mathbb{R}_+^2 \mid x \ge (C+1)rI_2/(A_2-C-1)\}$ when $T \downarrow 0$. Consider the exercise region of problem (8) for a fixed x satisfying $x_2^* < x < \lim_{y\uparrow I_2} x_C^*(y)$ and $y \uparrow I_2$. Note that $\inf\{t \ge 0 \mid X(t) \ge x_2^*, Y(t) \ge I_2\}$ converges to 0 as $y \uparrow I_2$. Accordingly, the exercise region of the problem (8) for the fixed x and $y \uparrow I_2$ converges to that of problem (19) with $T \downarrow 0$. This implies that $\lim_{y\uparrow I_2} x_C^*(y) = (C+1)rI_2/(A_2-C-1)$, which contradicts the assumption of $(C+1)rI_2/(A_2-C-1) < \lim_{y\uparrow I_2} x_C^*(y)$. Similarly we can lead to contradiction if $x_2^* < (C+1)rI_2/(A_2-C-1) < \lim_{y\uparrow I_2} x_C^*(y)$ is supposed. Thus, we have $\lim_{y\uparrow I_2} x_C^*(y) = \max(x_2^*, (C+1)rI_2/(A_2-C-1))$.

We can show the continuity of $x_C^*(\cdot)$ as follows. By Lemma 1 we have the continuity of $V_C(x,y)$. Since $V_C(x,y)$ and $(A_2-1)x/r-\mu-I_2-C\max(I_2-y,0)$ are both continuous, $S_{C,2,E} \cup S_{C,2,I}$ is a closed set. Consider any fixed $y(< I_2)$. Then, we have $\lim_{\epsilon\downarrow 0}(x_C^*(y+\epsilon), y+\epsilon) \in S_{C,2,E}$, which leads to $\lim_{\epsilon\downarrow 0} x_C^*(y+\epsilon) \ge x_C^*(y)$. We have $\lim_{\epsilon\downarrow 0} x_C^*(y+\epsilon) \le x_C^*(y)$ because $x_C^*(\cdot)$ is a decreasing function. Thus, we obtain the right-continuity of $x_C^*(\cdot)$. Now, suppose that there exists $y(< I_2)$ satisfying $x_C^*(y) < \lim_{\epsilon\downarrow 0} x_C^*(y-\epsilon)$. We can lead to contradiction as the same method as the proof of $\lim_{y\uparrow I_2} x_C^*(y) = \max(x_2^*, (C+1)rI_2/(A_2-C-1))$. Consider the exercise region of problem (19) for a fixed x satisfying

$x_C^*(y) < x < \lim_{\epsilon\downarrow 0} x_C^*(y-\epsilon)$ and $y-\epsilon$. Note that $\inf\{t \geq 0 \mid X(t) \geq x_C^*(Y(t))\}$ converges to 0 as $\epsilon \downarrow 0$. Then, the exercise region converges to that of problem (19) with $T \downarrow 0$. This implies $\lim_{\epsilon\downarrow 0} x_C^*(y-\epsilon) = (C+1)rI_2/(A_2-C-1)$, which contradicts $(C+1)rI_2/(A_2-C-1) \leq x_C^*(y) < \lim_{\epsilon\downarrow 0} x_C^*(y-\epsilon)$. Thus, we obtain the left-continuity of $x_C^*(\cdot)$.

Lastly, consider the case of $A_2 - C - 1 \leq 0$. In this case, we have for any $(x,y) \in \mathbb{R}_+^2$

$$\mathcal{L}f_2(x,y) - rf_2(x,y) = -(A_2 - C - 1)x + (C+1)rI_2 > 0,$$

where $\mathcal{L}$ is the generating operator defined by (18) and $f_2(x,y)$ is defined by (16). Since $\mathcal{L}V_C(x,y) - rV_C(x,y) \leq 0$ follows from the general theory of optimal stopping, $f_2(x,y)$ does not agree with $V_C(x,y)$. This implies $S_{C,2,E} = \emptyset$. The proof is completed.

E Proof of Proposition 4

Note that $V_C(x,y)$ can be reduced to $V_0(x,y)$ for $y \geq I_2$. In the case of $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} < I_2/I_1$, the problem $V_0(x,y)$ has the small-scale expansion region $[x_1^*, x_{21}^*]$ (recall Section 2.2). Then, we have $[x_1^*, x_{21}^*] \times [I_2, \infty) \subset S_{C,1,I}$, which implies that $S_{C,1,I} \neq \emptyset$.

Next, consider the case of $\{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} \geq (I_2 + C(I_2 - I_1))/I_1$. By Proposition 1, we have $V_C(x,y) \geq V_L(x,y)$. Since we have

$$\begin{aligned}\left(\frac{A_2-1}{A_1-1}\right)^{\frac{\beta}{\beta-1}} &\geq \frac{I_2 + C(I_2 - I_1)}{I_1} \\ &\geq \frac{I_2 + C\max(I_2 - y, 0)}{I_1 + C\max(I_1 - y, 0)}\end{aligned}$$

for any y, the problem $V_L(x,y)$ has no small-scale expansion region (cf. Section 2.2). This implies that

$$V_L(x,y) > \frac{(A_1-1)x}{r-\mu} - I_1 - C\max(I_1 - y, 0). \tag{20}$$

By (20) and $V_C(x,y) \geq V_L(x,y)$, we have $S_{C,1,E} = S_{C,1,I} = \emptyset$.

Note that, in the case of $I_2/I_1 \leq \{(A_2-1)/(A_1-1)\}^{\beta/(\beta-1)} < (I_2 + C(I_2 - I_1))/I_1$, We have no clear results, but, as seen in the argument above, $S_{C,1,E}$ and $S_{C,1,I}$ may exist within the region $0 \leq y < I_2$.

F Proof of Proposition 5

Take any $(x,y) \in S_{C,1,E}$ and $y' \in [y, I_1)$. By (10) in Lemma 1, we have

$$\begin{aligned}V_C(x,y') &\leq V_C(x,y) + C(y'-y) \\ &= \frac{(A_1-1)x}{r-\mu} - I_1 - C(I_1 - y) + C(y'-y) \\ &= \frac{(A_1-1)x}{r-\mu} - I_1 - C(I_1 - y'),\end{aligned}$$

where the last inequality implies $(x, y) \in S_{C,1,E}$. Similarly, we have

$$V_C(x, I_1) \leq \frac{(A_1 - 1)x}{r - \mu} - I_1,$$

which means $(x, I_1) \in S_{C,1,I}$.

Next, consider any $(x, y) \in S_{C,1,I}$ and $y' \in [I_1, y)$. We have

$$\begin{aligned} V_C(x, y') &\leq V_C(x, y) \\ &= \frac{(A_1 - 1)x}{r - \mu} - I_1, \end{aligned}$$

where the last inequality implies $(x, y) \in S_{C,1,I}$.

Lastly, we focus on the case of $\{(A_2 - 1)/(A_1 - 1)\}^{\beta/(\beta-1)} < I_2/I_1$. In this case, it follows from (4) that

$$\begin{aligned} V_C(x, y) &\leq V_0(x) \\ &= \left(\frac{(A_1 - 1)x_1^*}{r - \mu} - I_1 \right) \left(\frac{x}{x_1^*} \right)^{\beta}, \end{aligned} \tag{21}$$

for $x < x_1^*$. Consider the problem $V_C(x, y)$ with a initial point $(X(0), Y(0)) = (x, y)$ satisfying $x < x_1^*$ and $y \geq I_1$. For the problem, we can realize the right-hand side of (21) by the stopping time $\inf\{t \geq 0 \mid X(t) \geq x_1^*\}$. Then, this threshold policy $\inf\{t \geq 0 \mid X(t) \geq x_1^*\}$ is optimal. In other words, $\min\{x \geq 0 \mid (x, y) \in S_{C,1,I}\} = x_1^*$ holds for any fixed $y \geq I_1$.

G Proof of Proposition 6

We have already proved the properties of $S_{C,2,E}$ in Proposition 3, and, hence, we focus on the properties of $S_{C,1,E}$ below. In the same manner as the proof of Proposition 3, we can show the properties. Define

$$f_1(x, y) := \frac{(A_1 - 1)x}{r - \mu} - I_1 - C(I_1 - y),$$

and consider the case of $C \geq A_1 - 1$. We have for any $(x, y) \in \mathbb{R}_+^2$

$$\mathcal{L} f_1(x, y) - r f_1(x, y) = -(A_1 - C - 1)x + (C + 1) r I_1 > 0,$$

where $\mathcal{L}$ is the generating operator defined by (18). Since $\mathcal{L} V_C(x, y) - r V_C(x, y) \leq 0$ follows from the general theory of optimal stopping, $f_1(x, y)$ does not agree with $V_C(x, y)$ for any $(x, y) \in \mathbb{R}_+^2$. This implies that $S_{C,1,E} = \emptyset$. The proof is completed.

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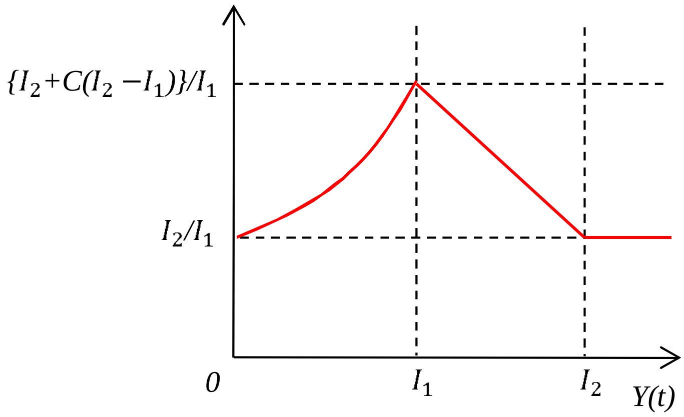

Figure 1: The ratio $\{I_2+C\max(I_2-Y(t),0)\}/\{I_1+C\max(I_1-Y(t),0)\}$. The ratio is unimodal and has the maximum value $\{I_2+C(I_2-I_1)\}/I_1$ for $Y(t)=I_1$.

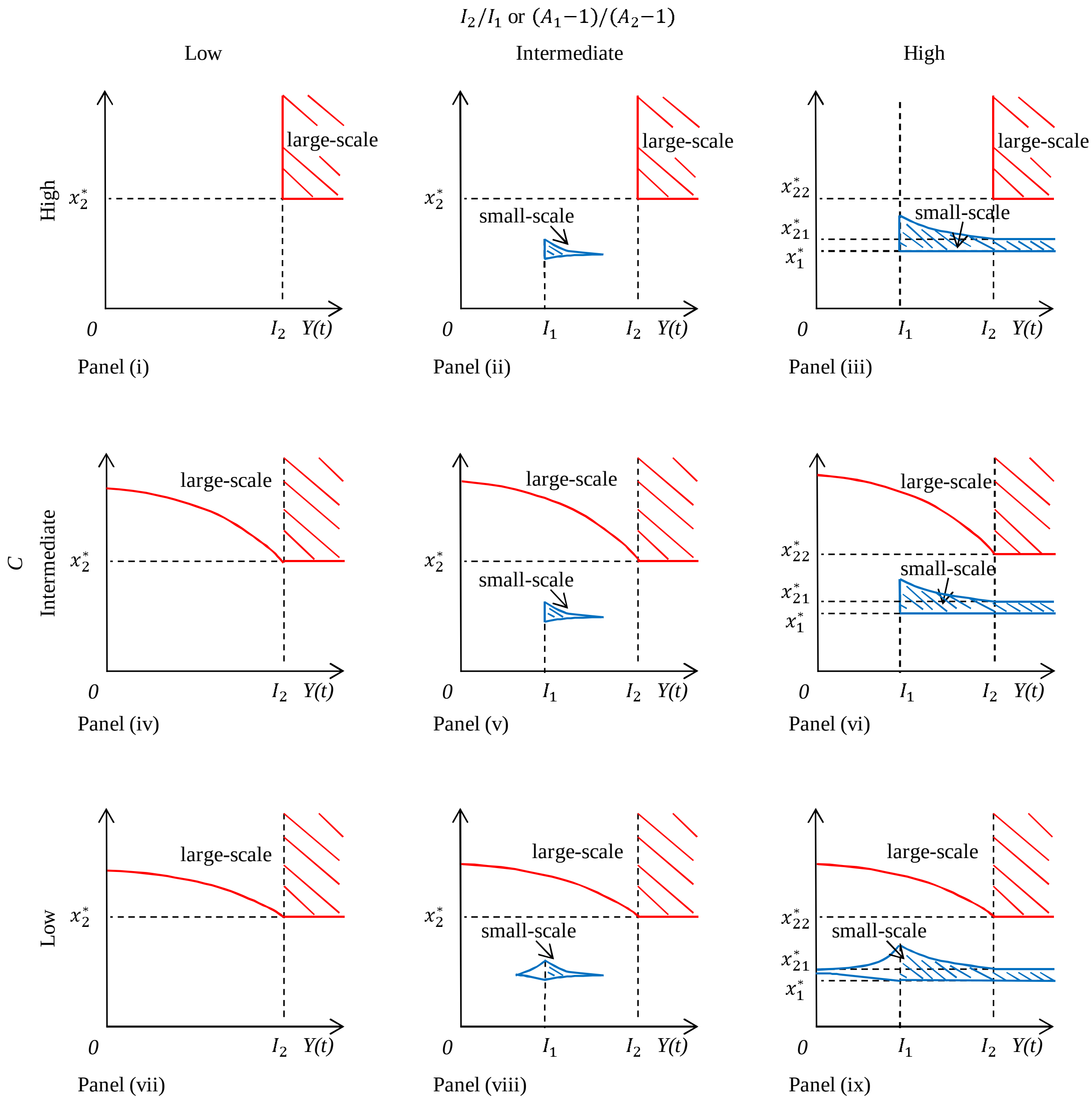

Figure 2: The exercise regions $S_{C,1,E}$, $S_{C,1,I}$, $S_{C,2,E}$, and $S_{C,2,I}$. This figure summarizes the properties proved in Propositions 3–6. The first, second, and third rows correspond to the cases with high, intermediate, and low cost of external financing, respectively. The first, second, and third columns correspond to the cases with high, intermediate, and low values of I_2/I_1 (or $(A_1-1)/(A_2-1)$). The hatched regions correspond to the entirely internal financing regions, $S_{C,1,I}$ and $S_{C,2,I}$, while the other regions correspond to the partially external financing regions, $S_{C,1,E}$ and $S_{C,2,E}$.

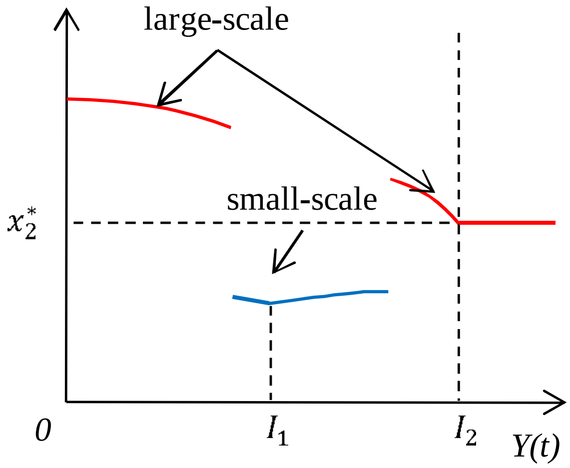

Figure 3: The minimum output price for which the firm invests. This figure corresponds to Panel (viii) in Figure 2.

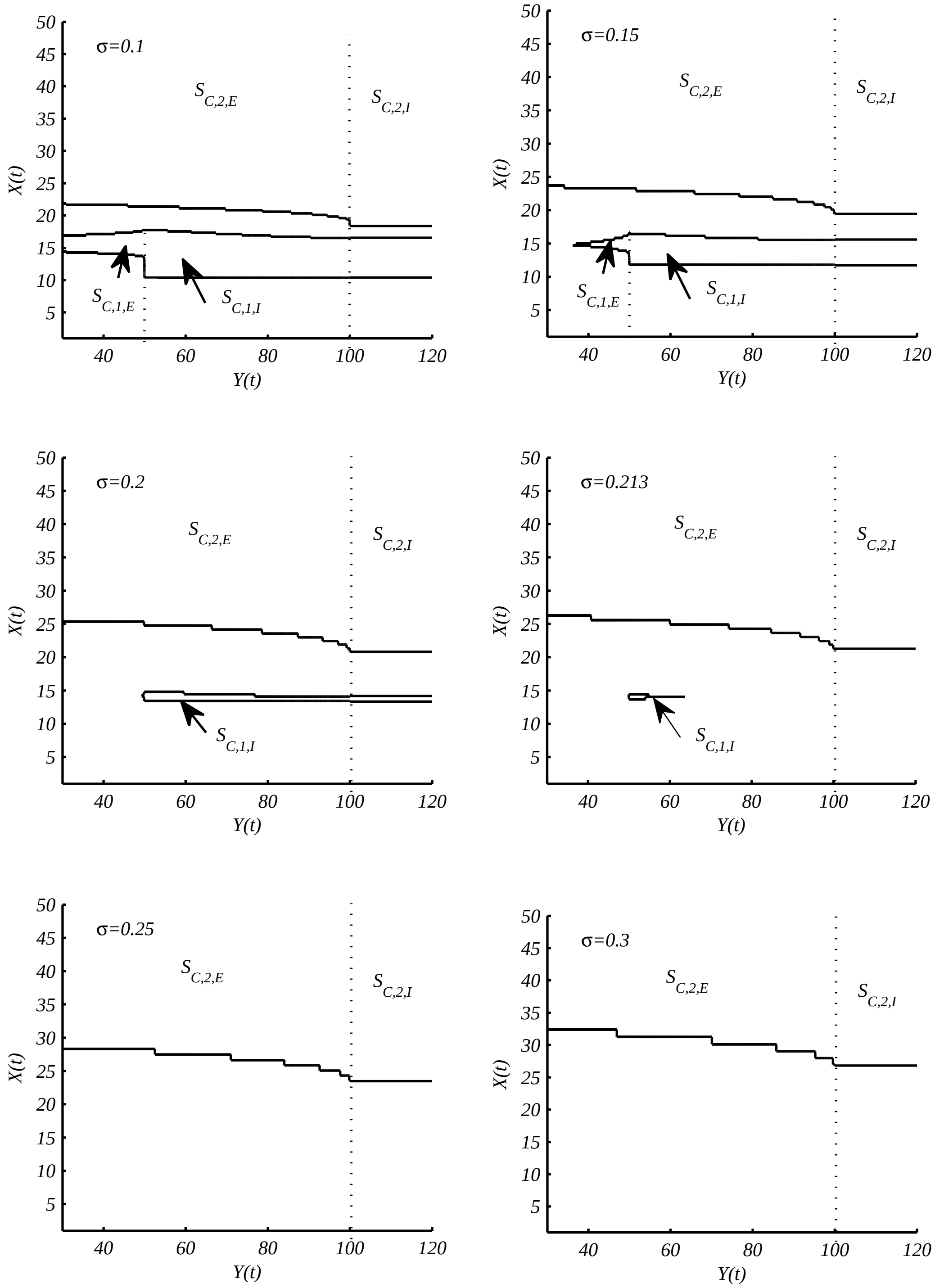

Figure 4: The comparative statics. The figure plots the exercise regions in the case with a proportional cost of external financing with varying levels of σ. The parameter values other than σ are set at the base case (13).

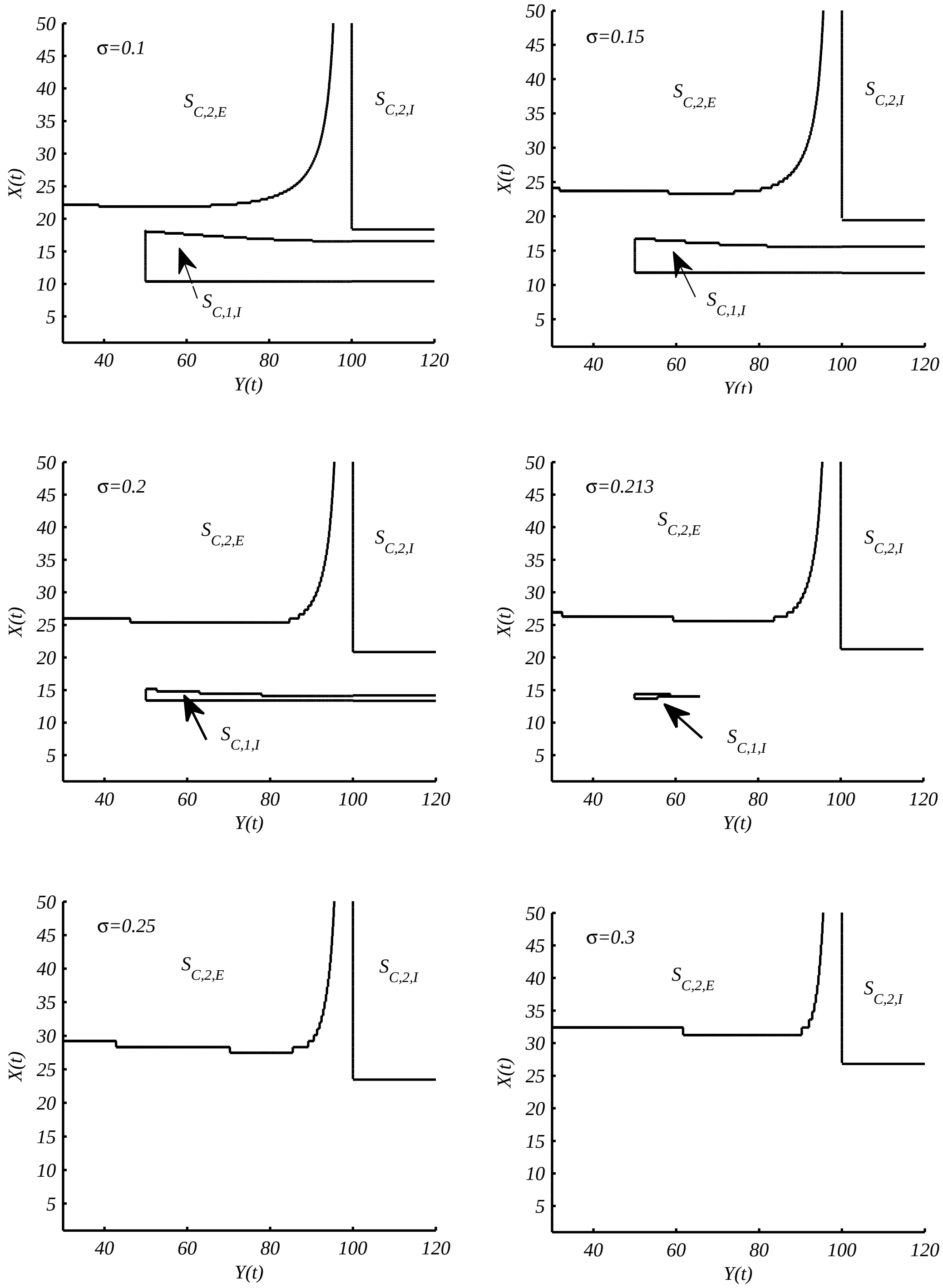

Figure 5: The comparative statics. The figure plots the exercise regions in the case with both fixed and proportional costs of external financing with varying levels of σ. The fixed cost is set at 1, and the other parameter values are set at the base case (13). Note that $S_{C,2,E} := \{(x,y) \in \mathbb{R}^2_+ \mid V(x,y) = (A_2-1)x/(r-\mu) - I_2 - C(I_2-y) - 1, y < I_2\}$ because of the fixed cost.

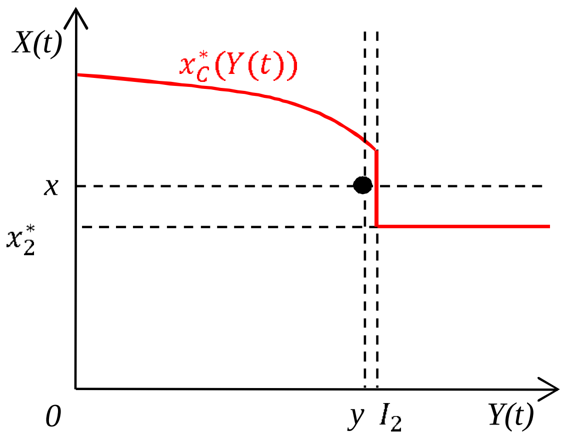

Figure 6: The assumption of $x_2^* < \lim_{y \uparrow I_2} x_C^*(y)$. The dot represents the initial point (x, y) satisfying $x_2^* < x < \lim_{y \uparrow I_2} x_C^*(y)$ and $y \approx I_2$.