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Modeling pollution and economic growth: the effect of a lethal threshold*

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Abstract

The accumulation of pollution negatively impacts human health. Extreme increases in pollution, in particular, may have lethal implications for human beings—and, indeed, all living organisms. This paper thus devises a new model of economic growth that takes into account these lethal effects of accumulated pollution via a pollution threshold to show two key results. First, if an abatement technology is relatively inefficient, there exists a stationary steady state in which consumption and pollution stop growing. Second, if the abatement technology is sufficiently efficient, there exists a path along which pollution decreases at an accelerating rate until finally reaching zero. In this case, consumption grows at a constant rate.

Keywords: Endogenous growth, Pollution disutility, Pollution abatement

JEL Classification Numbers: O44, Q52

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1 Introduction

According to the World Health Organization (2014), about 3.7 million people worldwide died as a result of outdoor air pollution in 2012. The most common sources of outdoor air pollution include particulates, ozone, nitrogen dioxide, and sulfur dioxide, which are highly prevalent in urban areas of developing countries. In Japan, corporate industrial waste during a period of rapid economic growth led to the so-called Big Pollution Diseases of Japan; Itai-itai disease, Minamata disease, Niigata Minamata disease and Yokkaichi Asthma. For example, Minamata disease, traced to the release of methyl mercury in industrial wastewater, was responsible for many human deaths. Many developing economies have recently begun to encounter similar problems. In addition to these direct harms, large volumes of greenhouse gas emissions contribute to global climate change, which also may have disastrous effects on human lives. Indeed, human health and lives may be strongly adversely impacted if pollution levels exceed a given set of thresholds. If economic growth proceeds with an unlimited accumulation of pollution, global society may find itself headed down a path to destruction.

In this paper, by using a simple endogenous growth model with AK technology, we consider a situation in which human beings cannot sustain their existence over a given pollution threshold. In other words, we incorporate an upper limit on pollution into the individual utility function. Therefore, the unlimited societal accumulation of pollution is analytically disallowed. Moreover, the existence of the pollution threshold introduces a strong nonlinearity into the model, complicating its dynamics. However, we show that there exists a unique optimal path, depending on the efficiency of the abatement technology available to reduce the pollution level. If the abatement technology is not sufficiently efficient—that is, it exhibits decreasing returns with respect to input—then sustained growth is not possible. The economy reaches a steady state with finite levels of output and consumption; the pollution level stays below the threshold and may even be zero. On the other hand, if the abatement technology is sufficiently efficient—that is, exhibits constant returns with respect to input—then sustained growth is possible while keeping the pollution level under the threshold.

Following new developments in economic growth theories in the 1990s, many studies have examined this problem.[1] For instance, Huang and Cai (1994) emphasize the possibility that the consumption level grows at a constant rate along the optimal path. Because they ignore any biological limitations on the accumulation of pollution, the level of pollution is also able to grow at a constant and positive rate on this "optimal" path. By focusing on the disutility of pollution, Michel and Rotillon (1995) show that unlimited economic growth and a continuous decrease in pollution are compatible. However, they neither explicitly examine the stability of the long-run equilibrium nor derive the transitional path. Furthermore, the above-mentioned lethal effects of pollution were not taken into consideration in any previous studies. In our analysis of the relationship between economic growth and environmental quality, the efficiency of the abatement technology is also an important factor. In their dynamic model, Brock and Taylor (2010) integrate technological progress not only in production but also in abatement, showing that achieving faster growth in the efficiency of abatement technology than in production is key for maintaining economic growth without causing additional damage to the ecosystem.

The remainder of the paper is organized as follows. The model is presented in Section 2, and Section 3 analyzes the equilibrium in which the marginal cost of abatement increases. Section 4 then considers the equilibrium in which the marginal cost of abatement remains constant. Section 5 concludes.

2 The model

Consider a closed economy with constant population normalized to one. The utility of the representative agent, U, depends on per capita consumption at time t, c_t, and on the level

[1] See Ricci (2007) for a survey.

of pollution at time t, p_t:

$$U = \int_0^\infty \exp(-\rho t)\{\ln c_t + \sigma \ln(\bar{p} - p_t)\}dt, \tag{1}$$

where $\rho > 0$ is the time preference rate, $\sigma < 1$ is a weight denoting the overall utility attached to environmental quality, and $\bar{p}$ is a biological threshold for pollution above which no humans can survive.

The production function of final goods at time t, y_t, is given by:

$$y_t = Ak_t, \tag{2}$$

where k_t is capital and a constant parameter A represents productivity. We assume that the pollution level is a stock variable. The emission of environmental pollutants is the by-product of production and is proportional to output; that is, αAk_t, where α measures the degree of emissions emerging from the production sequence. We also assume that the stock of pollution decays at a fixed rate, δ. We further assume that the stock of pollution is reduced by using the abatement good, m_t. The level of pollution is assumed to be nonnegative in order to provide the traction needed to analyze the properties of the equilibrium.[2] Thus the level of the pollution stock is given by:

$$\dot{p}_t = \alpha Ak_t - \delta p_t - m_t. \tag{3}$$

The abatement good is produced using final goods, with the required input given by the following technological transformation:

$$c(m_t) = \theta m_t^\gamma, \tag{4}$$

[2] In our setting, the marginal rate of substitution between consumption and pollution is positive and finite when the level of pollution, p_t, is zero. Thus it would be preferable to reduce the pollution level to below zero. However, even if we allow the pollution level to take negative values, the equilibrium properties do not change.

where $\theta > 0$ and $\gamma \geq 1$. θ stands for the efficiency level of the abatement technology, and γ represents the degree of returns; when γ is greater than one, the abatement technology exhibits increasing marginal costs. On the other hand, when γ is equal to one, the abatement technology exhibits constant marginal costs.[3] Then capital accumulates according to the following equation:

$$\dot{k}_t = y_t - c_t - c(m_t). \tag{5}$$

Based on this analytical structure, we consider the social planner's problem in the next section.

3 Optimal path

The social planner maximizes the welfare level (1) by choosing consumption c_t and the abatement good m_t, subject to the production function (2) and the transition equations of the pollution (3) and capital stocks (5). The current value Hamiltonian of this problem is:

$$H_t = \ln c_t + \sigma \ln(\bar{p} - p_t) + \lambda_{kt}(Ak_t - c_t - \theta m_t^{\gamma}) + \lambda_{pt}(\alpha A k_t - \delta p_t - m_t) + \nu_t p_t,$$

where λ_{kt} and λ_{pt} denote the co-state variables associated with capital accumulation (5) and pollution accumulation (3), respectively. ν_t is the Lagrangian multiplier associated with the non-negative pollution constraint.

The first-order condition with respect to consumption c_t is:

$$\frac{1}{c_t} - \lambda_{kt} = 0. \tag{6}$$

The first-order conditions with respect to the abatement good and pollution stock are given by:

$$m_t(-\gamma\theta m_t^{\gamma-1}\lambda_{kt} - \lambda_{pt}) = 0,\ m_t \geq 0,\ -\gamma\theta m_t^{\gamma-1}\lambda_{kt} - \lambda_{pt} \leq 0, \tag{7}$$

[3] Michel and Rotillon (1995) examine only the case of $\gamma = 1$.

$$\nu_t p_t = 0,\ p_t \geq 0,\ -\nu_t \leq 0. \tag{8}$$

Condition (6) requires the marginal values of consumption to be equal to the shadow value of income, λ_{kt}. The complementary slackness condition, (7), indicates that if there is some positive input in abatement, the shadow cost of additional pollution reduction is equal to the shadow value of pollution reduction. If the shadow cost is above the shadow value, there is no input into abatement. Complementary slackness condition (8) ensures that the pollution stock remains positive. The dynamics of the co-state variables are given by:

$$\dot{\lambda_{kt}} = \rho\lambda_{kt} - (A\lambda_{kt} + \alpha A\lambda_{pt}), \tag{9}$$

$$\dot{\lambda_{pt}} = \rho\lambda_{pt} - \left(\frac{-\sigma}{\bar{p} - p_t} - \delta\lambda_{pt} + \nu_t\right). \tag{10}$$

Finally, the transversality conditions for this problem are:

$$\lim_{t\to\infty} \exp(-\rho t)\lambda_{kt}k_t = 0,\ \lim_{t\to\infty} \exp(-\rho t)\lambda_{pt}p_t = 0.$$

Properties of the solution depend crucially on the degree of returns, γ. We first consider the case of increasing marginal cost, $\gamma > 1$. We next examine the case of constant marginal cost, $\gamma = 1$. We assume that the following inequality holds throughout our analysis:

$$A - \rho - \alpha\theta A > 0. \tag{A1}$$

3.1 An abatement technology with increasing marginal cost

In this section, we consider the case in which the abatement technology is relatively inefficient and thus characterized by increasing marginal cost, $\gamma > 1$. In this case, the marginal cost of abatement would be zero if the input into abatement were zero; as this entails that it is always beneficial to use some final goods for abatement, therefore, m_t never becomes zero. Then equations (3)-(7), (9), and (10) constitute the following dynamic system for $k_t,\ p_t,\ c_t,\ m_t,$

and ν_t:

$$\dot{k}_t = Ak_t - c_t - \theta m_t^{\gamma}, \tag{11}$$

$$\dot{p}_t = \alpha Ak_t - \delta p_t - m_t, \tag{12}$$

$$\dot{c}_t = c_t(A - \rho - \alpha A\gamma\theta m_t^{\gamma-1}), \tag{13}$$

$$\dot{m}_t = \frac{m_t}{\gamma - 1}\left(A + \delta - \alpha A\gamma\theta m_t^{\gamma-1} - \frac{\sigma c_t m_t^{1-\gamma}}{(\bar{p} - p_t)\gamma\theta} + \frac{\nu_t c_t m_t^{1-\gamma}}{\gamma\theta}\right). \tag{14}$$

We can characterize the equilibrium path of the economy using the equations above along with complementary slackness condition (8) and the initial values of p_t and k_t. Then we prove the following proposition:

Proposition 1 *Suppose that inequality (A1) holds and $\gamma > 1$. Then there exists a unique stationary steady state in which all variables become constant. In particular, the level of pollution in the steady state becomes positive if $\bar{p} > \hat{p}$ and zero if $\bar{p} \leq \hat{p}$, where $\hat{p} \equiv \frac{\sigma}{(A-\rho)(\rho+\delta)}\left(\frac{A-\rho}{\alpha\gamma\theta A}\right)^{\frac{1}{\gamma-1}}\left(\frac{A(\gamma-1)}{\gamma} + \frac{\rho}{\gamma}\right)$. Furthermore, as long as $\gamma > 1$, the threshold level $\hat{p}$ is decreasing in the level of abatement technology γ.*

Proof. When the level of pollution is positive, equations (8), (11)-(14), and $\nu_t = 0$ constitute the following four-dimensional dynamic system for k_t, p_t, c_t, and m_t:

$$\dot{k}_t = Ak_t - c_t - \theta m_t^{\gamma}, \tag{15}$$

$$\dot{p}_t = \alpha Ak_t - \delta p_t - m_t, \tag{16}$$

$$\dot{c}_t = c_t(A - \rho - \alpha A\gamma\theta m_t^{\gamma-1}), \tag{17}$$

$$\dot{m}_t = \frac{m_t}{\gamma - 1}\left(A + \delta - \alpha A\gamma\theta m_t^{\gamma-1} - \frac{\sigma c_t m_t^{1-\gamma}}{(\bar{p} - p_t)\gamma\theta}\right). \tag{18}$$

Denote the long-run levels of capital, pollution, consumption, and the abatement good by k^*, p^*, c^*, and m^*, respectively. As $\dot{k} = \dot{p} = \dot{c} = \dot{m} = 0$ holds in the steady state, the steady state values are as follows:

$$m^* = \left(\frac{A-\rho}{\alpha\gamma\theta A}\right)^{\frac{1}{\gamma-1}}, \tag{19}$$

$$p^* = \frac{(A-\rho)(\rho+\delta)\bar{p} + \sigma A(\alpha\theta m^{*\gamma} - m^*)}{\sigma\delta A + (A-\rho)(\rho+\delta)}, \tag{20}$$

$$c^* = \frac{(A-\rho)(\rho+\delta)\{\bar{p}\delta - \alpha\theta m^{*\gamma} + m^*\}}{\alpha\{\sigma A\delta + (A-\rho)(\rho+\delta)\}}, \tag{21}$$

$$k^* = \frac{c^* + \theta m^{*\gamma}}{A}. \tag{22}$$

In equation (20), the necessary and sufficient condition for p^* to be positive is,

$$(A-\rho)(\rho+\delta)\bar{p} + \sigma A(\alpha\theta m^{*\gamma} - m^*) > 0. \tag{23}$$

By inserting equation (19) into condition (23) and solving with respect to $\bar{p}$, then the condition for p^* to be positive becomes $\bar{p} > \frac{\sigma}{(A-\rho)(\rho+\delta)}\left(\frac{A-\rho}{\alpha\gamma\theta A}\right)^{\frac{1}{\gamma-1}}\left(\frac{A(\gamma-1)}{\gamma} + \frac{\rho}{\gamma}\right)$. Here the left-hand side is what we define as $\hat{p}$.

On the other hand, when the level of pollution is zero, equations (8), (11)-(14), and $p_t = 0$ constitute the following dynamic system for $k_t,\ p_t,\ c_t$, and m_t:

$$p_t = 0, \tag{24}$$

$$m_t = \alpha A k_t, \tag{25}$$

$$\dot{k}_t = Ak_t - \theta(\alpha A k_t)^\gamma - c_t, \tag{26}$$

$$\dot{c}_t = c_t\left(A - \rho - \alpha\gamma\theta A(\alpha A k_t)^{\gamma-1}\right). \tag{27}$$

$\dot{k}_t = \dot{c}_t = 0$ must be satisfied in the steady state. Then the steady state values are derived as follows:

$$m^{**} = \left(\frac{A - \rho}{\alpha \gamma \theta A} \right)^{\frac{1}{\gamma - 1}}, \tag{28}$$

$$p^{**} = 0, \tag{29}$$

$$c^{**} = \frac{\gamma A + \rho - A}{\alpha \gamma A} \left(\frac{A - \rho}{\alpha \gamma \theta A} \right)^{\frac{1}{\gamma - 1}}, \tag{30}$$

$$k^{**} = \frac{1}{\alpha A} \left(\frac{A - \rho}{\alpha \gamma \theta A} \right)^{\frac{1}{\gamma - 1}}. \tag{31}$$

Next, to show that $\hat{p}$ is decreasing in γ, we differentiate $\hat{p}$ with respect to γ. Then we can show that:

$$\frac{\partial \hat{p}}{\partial \gamma} = - \frac{\sigma (A - \rho)^{\frac{2 - \gamma}{\gamma - 1}}}{(\rho + \delta) \gamma^{\frac{\gamma}{\gamma - 1}} (\alpha \theta A)^{\frac{1}{\gamma - 1}}} \left(\frac{\rho}{\gamma - 1} + \frac{A(\gamma - 1) + \rho}{(\gamma - 1)^2} \ln \gamma \right) < 0. \tag{32}$$

□

According to the household utility function (1), it is clear that the utility from consumption is independent of the upper bound on pollution, $\bar{p}$. By contrast, the disutility from pollution is negatively related to the upper bound, $\bar{p}$: the greater the marginal disutility from pollution, the lower the upper bound. Therefore, if $\bar{p} \leq \hat{p}$, then when $p = 0$ the marginal disutility is large enough to keep pollution at zero. If $\bar{p} > \hat{p}$, the marginal disutility from pollution when $p = 0$ is sufficiently small that it is optimal for the economy to maintain a positive level of pollution. In addition, if $\bar{p} > \hat{p}$, the levels of pollution, consumption, and capital in the steady state are increasing in $\bar{p}$, as a higher value for $\bar{p}$ entails a low marginal disutility of pollution. Considering pollution reduction efficiency, the marginal cost increases with γ so that it become more difficult to maintain pollution at a high level.

The following lemma defines the conditions for the acceptable amount of pollution, $\bar{p}$, under which the level of pollution can become zero in the transition.

Lemma 1 *Suppose inequality (A1) and* $\gamma > 1$*. If* $\bar{p} < \hat{p}$*, once the level of pollution becomes zero, it is optimal for the economy to keep it at zero along the equilibrium path. Otherwise, it is not optimal to maintain zero pollution, and it becomes positive in the long run.*

Proof. See Appendix A.1. □

We analyze the local stability of the economy in the following proposition.

Proposition 2 *The equilibrium path is uniquely determined both in the case of the steady state with a positive level of pollution and in the case of the steady state without pollution.*

Proof. Firstly, we analyze the case in which the level of pollution is positive in the steady state. Since the level of pollution is always positive, the transition dynamics are expressed by equations (15)-(18). We denote the Jacobi matrix of the linearized system around the steady state as $\mathbf{J_{p>0}}$. Then we have that:

$$\mathbf{J_{p>0}} = \begin{pmatrix} A & 0 & -1 & -\gamma\theta m^{*\gamma-1} \\ \alpha A & -\delta & 0 & -1 \\ 0 & 0 & 0 & -\alpha\gamma(\gamma-1)\theta A c^* m^{*\gamma-2} \\ 0 & \frac{\sigma c^* m^{*2-\gamma}}{\gamma(\gamma-1)\theta(\bar{p}-p^*)^2} & -\frac{\sigma m^{*2-\gamma}}{\gamma(\gamma-1)\theta(\bar{p}-p^*)} & -\alpha\gamma\theta A m^{*\gamma-1} + \frac{\sigma c^* m^{*(1-\gamma)}}{(\bar{p}-p^*)\gamma\theta} \end{pmatrix}. \tag{33}$$

Thus the characteristic equation becomes:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0. \tag{34}$$

where the coefficients are

$$a_1 = -2\rho, \tag{35}$$

$$a_2 = (A-\delta)(2\rho+\delta-A) - A\delta - \frac{(\rho+\delta)m^*}{(\gamma-1)(\bar{p}-p^*)} - \frac{\alpha\sigma A c^*}{\bar{p}-p^*}, \tag{36}$$

$$a_3 = A\delta(2\rho+\delta-A) - \frac{\rho(\rho+\delta)m^*}{(\gamma-1)(\bar{p}-p^*)} + (A-\delta)\frac{\alpha\sigma A c^*}{\bar{p}-p^*}, \tag{37}$$

$$a_4 = \frac{\alpha A c^*}{\bar{p}-p^*}(\sigma\delta A - (A-\rho)(\rho+\delta)). \tag{38}$$

According to the Routh-Hurwitz theorem,[4] the number of roots of equation (34) with positive real parts is equal to the number of variations of sign in the following scheme:

$$1, \quad a_1, \quad \frac{a_1a_2 - a_3}{a_1}, \quad a_3 - \frac{a_1^2a_4}{a_1a_2 - a_3}, \quad a_4. \tag{39}$$

We know that $a_1 < 0$, $a_3 > 0$, and $a_4 > 0$ under assumption (A1) and the signs of $\frac{a_1a_2-a_3}{a_1}$ and $a_3 - \frac{a_1^2a_4}{a_1a_2-a_3}$ are ambiguous. Considering the following four cases in Table 1 is enough.

Table 1: Summary of signs in (39)

	1	a_1	$\frac{a_1a_2-a_3}{a_1}$	$a_3 - \frac{a_1^2a_4}{a_1a_2-a_3}$	a_4
Case 1	+	−	+	+	+
Case 2	+	−	+	−	+
Case 3	+	−	−	+	+
Case 4	+	−	−	−	+

If $\frac{a_1a_2-a_3}{a_1} > 0$, then $a_1a_2 - a_3 < 0$ because a_1 is negative. Thus, the sign of $a_3 - \frac{a_1^2a_4}{a_1a_2-a_3}$ becomes positive. Therefore case 2 in Table 1 does not arise. We find that the number of variations of sign in (39) is two in the other cases listed in Table 1. The characteristic equation (34) has only two roots with positive real parts. Because the number of predetermined variables is two, the equilibrium path is uniquely determined.

Second, we analyze the case in which the level of pollution is zero in the steady state. In this case, the dynamic system around the steady state is expressed by equations (26) and (27). We denote the Jacobi matrix of the linearized system around the steady state as $\mathbf{J_{p=0}}$. Then we have that

$$\mathbf{J_{p=0}} = \begin{pmatrix} \rho & -1 \\ -\alpha\gamma\theta A(\gamma-1)(\alpha A)^{\gamma-1}k^{**\gamma-2} & 0 \end{pmatrix}. \tag{40}$$

[4] See Gantmacher (1960) p.194.

The determinant of $\mathbf{J_{p=0}}$ is given by $|\mathbf{J_{p=0}}| = -\alpha\gamma\theta A(\gamma-1)(\alpha A)^{\gamma-1}k^{*\gamma-2} < 0$. Because the number of predetermined variables is one in this case, the equilibrium path is uniquely determined. □

We have now established that if $\bar{p} < \hat{p}$, any pollution must be fully abated within a finite period. The dynamic system follows equations (15)-(18) until the level of pollution becomes zero. After this point, it is optimal for the economy to keep pollution at zero, creating a final steady state with no pollution. If $\bar{p} > \hat{p}$, the level of pollution never becomes zero and converges to a steady state with a positive level of pollution.

4 An abatement technology with constant marginal cost

In this section, we consider the case in which the abatement technology is characterized by constant marginal cost. In contrast to the case of $\gamma > 1$, the marginal cost of abatement is positive and finite when the input into abatement is zero. Thus if the marginal cost of additional pollution reduction is larger than the marginal value when input into abatement is zero, it is not beneficial to devote any final goods to abatement—that is, $m_t = 0$ becomes optimal. However, for analytical simplicity, we assume the following condition, which ensures that it is beneficial to reduce pollution at the initial time:

$$\delta p_0 \leq \alpha A k_0. \tag{A2}$$

This states that the pollution level emitted is greater than that absorbed at the initial time.

By substituting $\gamma = 1$ and $m_t > 0$ into equations (4) and (7), we can rewrite the necessary conditions (8), (11)-(14) as follows:

$$\dot{k}_t = Ak_t - c_t - \theta m_t, \tag{41}$$

$$\dot{p}_t = \alpha Ak_t - \delta p_t - m_t, \tag{42}$$

$$\dot{c}_t = (A - \rho - \alpha\theta A)c_t, \tag{43}$$

$$p_t = \bar{p} - \frac{\sigma c_t}{(A + \delta - \alpha\theta A)\theta + \nu_t c_t}. \tag{44}$$

Then we can describe the equilibrium path of the economy using the above equations together with the complementary slackness condition (8) and the initial values of p_t and k_t.

Lemma 2 *When $\gamma = 1$, under assumption (A1), the level of pollution must be zero in the long run regardless of the level of upper bound $\bar{p}$. In addition, once the level of pollution becomes zero, it is optimal to keep its level at zero.*

Proof. See Appendix A.2. □

Lemma 2 indicates that there exists a unique time at which the level of pollution becomes zero. Let us define this time as T. Since the dynamics depend on the level of pollution, we separate the analyses into two regimes: one corresponds to the case in which the level of pollution is still positive ($t < T$), and the other corresponds to the case in which the level of pollution is zero ($T \leq t$).

4.1 Dynamics without pollution

First, we analyze the equilibrium path when $T \leq t$. We use the normalization $\tilde{c}_t = c_t/k_t$, $\tilde{p}_t = p_t/k_t$, $\tilde{m}_t = m_t/k_t$ and define the growth rate of consumption as $g \equiv A - \rho - \alpha\theta A$ (see equation (43)). Since pollution is always completely eliminated, by substituting $p_t = 0$ and $\dot{p}_t = 0$ into equations (42) and by using the normalization above, we can rewrite equations

(41) and (42) as follows:

$$\dot{\tilde{c}}_t = \tilde{c}_t(\tilde{c}_t - \rho), \tag{45}$$

$$\tilde{m}_t = \alpha A. \tag{46}$$

Given the ratio of consumption and capital at time T, $\tilde{c}_T$, the path can be uniquely determined. If $\tilde{c}_T > \rho$, the growth rate of consumption per unit of capital accelerates so that it violates the TVC. If $\tilde{c}_T < \rho$, then $\tilde{c}_T$ shrinks and finally approaches zero. The only path that is optimal is realized by $\tilde{c}_T = \rho$. Along this optimal path, c_t, k_t, and m_t grow at a constant rate, g. This entails that the economy remains on the balanced growth path only when the ratio of consumption to capital at time T, $\tilde{c}_T$, becomes ρ. The ratio of abatement good to capital at time T, $\tilde{m}_T$, simultaneously becomes αA. Otherwise, the economy embarks on a destructive path along which the level of consumption shrinks over time or the level of capital converges to zero.

4.2 Dynamics with a positive level of pollution

Next, let us characterize the dynamics when the level of pollution is still positive. Since $p_t > 0$ as long as $t < T$, by substituting $\nu_t = 0$ into equations (44), we can rewrite equations (41)-(44) as follows:

$$\dot{\tilde{c}}_t = \left\{ \left(1 + \frac{\sigma g}{\rho + \delta + g}\right) \tilde{c}_t - \delta\theta\tilde{p}_t - \rho \right\} \tilde{c}_t, \tag{47}$$

$$\dot{\tilde{p}}_t = \left\{ \left(1 + \frac{\sigma g}{\rho + \delta + g}\right) \tilde{p}_t - \frac{\sigma g}{\theta(\rho + \delta + g)} \right\} \tilde{c}_t - \{\delta\theta\tilde{p}_t + (1 - \alpha\theta)A\} \tilde{p}_t, \tag{48}$$

$$\tilde{m}_t = \alpha A - \delta\tilde{p}_t + \frac{\sigma g \tilde{c}_t}{\theta(\rho + \delta + g)}, \tag{49}$$

where $\tilde{m}_t > 0$.[5] Thus, we can draw the dynamics of $\tilde{p}_t$ and $\tilde{c}_t$ on the $\tilde{p} - \tilde{c}$ plane. From equation (47), we can derive the relation between $\tilde{c}_t$ and $\tilde{p}_t$ that satisfies $\dot{\tilde{c}}_t = 0$. As long as $\tilde{c}_t > 0$, the $\dot{\tilde{c}}_t = 0$ locus is expressed as follows:

$$\tilde{c}_t = \frac{\delta\theta(\rho+\delta+g)}{\rho+\delta+(1+\sigma)g}\tilde{p}_t + \frac{\rho(\rho+\delta+g)}{\rho+\delta+(1+\sigma)g}. \tag{50}$$

Since the coefficient on $\tilde{p}_t$ in equation (50) is positive and constant, the $\dot{\tilde{c}}_t = 0$ locus is a straight line and increasing in $\tilde{p}_t$. Moreover, we can find that the coefficient on $\tilde{c}_t$ in brackets in equation (47) is positive (namely, $\dot{\tilde{c}}_t/\tilde{c}_t$ is increasing in $\tilde{c}_t$). Therefore, $\tilde{c}_t$ increases over time above the $\dot{\tilde{c}}_t = 0$ locus, whereas it decreases below the $\dot{\tilde{c}}_t = 0$ locus.

In order to analyze the dynamics of the pollution level, we make the following assumption regarding the initial condition:

$$\tilde{p}_0 < \frac{\sigma g}{\theta(\rho+\delta+g+\sigma g)}. \tag{A3}$$

Under assumption (A3), since the coefficient on $\tilde{c}_t$ in the first bracket is negative at the initial time, then $\tilde{p}_t$ decreases. It follows that $\tilde{p}_t$ always decreases over time because the first bracket of equation (48) is also decreasing in $\tilde{p}_t$. Hence, the phase diagram in $(\tilde{p}_t, \tilde{c}_t)$ space can be drawn as in Figure 1.

4.3 Optimal path for the economy

As mentioned above, in order to reach the unique balanced growth path, consumption per capital $\tilde{c}_t$ should reach ρ exactly when pollution is eliminated. To analyze the optimal path, we run dynamics (47) and (48) backwards in time. Inserting $t = T$ and $\tilde{p}_T = 0$ in equation (48), we find that $\dot{\tilde{p}}_T = -\frac{\sigma\rho g}{\theta(\rho+\delta+g)} < 0$. It follows that the level of pollution $\tilde{p}_t$ becomes zero

[5]The derivation of equation (49) is as follows. Since equation (44) and $\nu_t = 0$ yield $p_t = \bar{p} - \frac{\sigma c_t}{(A+\delta-\alpha\theta A)\theta}$, it follows that $\dot{p}_t = \frac{\sigma \dot{c}_t}{(A+\delta-\alpha\theta A)\theta} = \frac{\sigma g c_t}{(A+\delta-\alpha\theta A)\theta}$. Then by substituting this equation into equation (42), we obtain $m_t = \alpha A k_t - \delta p_t + \frac{\sigma g c_t}{\theta(\rho+\delta+g)}$, which leads to $\tilde{m}_t = \alpha A - \delta\tilde{p}_t + \frac{\sigma g \tilde{c}_t}{\theta(\rho+\delta+g)}$.

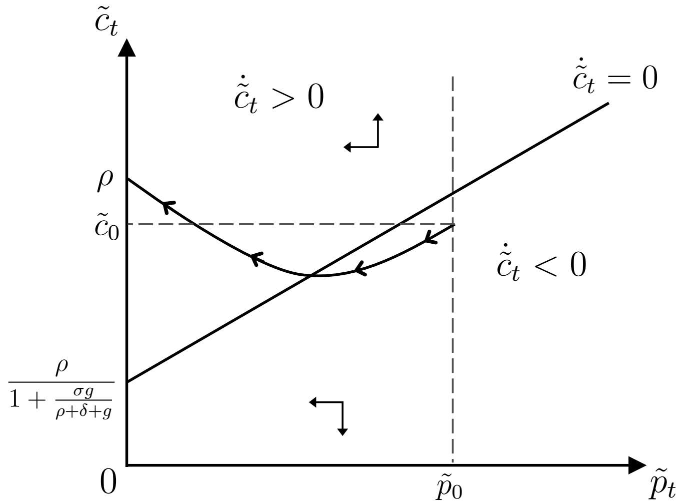

Figure 1: Phase diagram in $(\tilde{p}_t,\ \tilde{c}_t)$ space

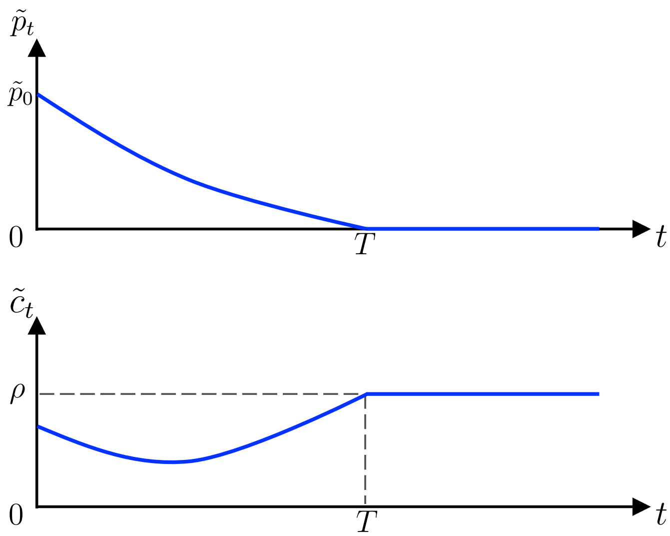

Figure 2: Dynamics of $\tilde{p}_t$ and $\tilde{c}_t$

in a finite amount of time. Then we can find a unique path toward a balanced growth path. As a result, there exists a unique $\tilde{c}_0$ that allows the economy to reach that unique path.

We have now completed our analysis of an economy when the abatement technology is characterized by constant marginal cost. The optimal path of each variable $(\tilde{p}_t, \tilde{c}_t)$ is drawn in Figure 2. Unlike in the case of an abatement technology characterized by increasing marginal cost, economic growth is compatible with a high-quality environment in the long run.

5 Conclusion

By setting an upper limit on pollution in the utility function, we have shown that economic growth is compatible with environmental conservation only when the abatement technology is sufficiently efficient. Otherwise, the economy will stagnate even if abatement is conducted efficiently. To briefly consider policy implications, our results imply that highly polluted countries with high abatement costs (e.g., developing countries) must invest not only in undertaking abatement activities but also in developing better abatement technologies.

A Appendix

A.1 Proof of Lemma 1

Assume that the level of pollution becomes zero in the long run when $\bar{p} > \hat{p}$. Then there exists a time point after which the level of pollution remains at zero. Define this time point as s. As long as $s \leq t$, the economic variables follow equations (11)-(14) with $\dot{p}_t = p_t = 0$. Thus, the dynamic system describing the economy consists of k_t and c_t and becomes as follows:

$$\dot{k}_t = Ak_t - \theta(\alpha A k_t)^{\gamma} - c_t, \tag{A.1}$$

$$\dot{c}_t = c_t \left(A - \rho - \alpha\gamma\theta A(\alpha A k_t)^{\gamma-1} \right). \tag{A.2}$$

It can be immediately shown that there exists a unique steady state where $k_t = \frac{1}{\alpha A}\left(\frac{A-\rho}{\alpha\gamma\theta A}\right)^{\frac{1}{\gamma-1}}$ and $c_t = \frac{\gamma A + \rho - A}{\alpha\gamma A}\left(\frac{A-\rho}{\alpha\gamma\theta A}\right)^{\frac{1}{\gamma-1}}$. By substituting these variables into equation (14), we can derive the value of ν_t in the long run as $\frac{\sigma(\hat{p}-\bar{p})}{\bar{p}\hat{p}}$. Since we assume $\bar{p} > \hat{p}$, it follows that the long-run value of ν_t should be negative. However, due to equation (8), ν_t must be nonnegative when $p_t = 0$, giving rise to a contradiction. Hence, it is not optimal for the economy to maintain zero pollution in the long run when $\bar{p} > \hat{p}$. On the other hand, when $\bar{p} < \hat{p}$, the value of ν_t

in the long run is nonnegative. Therefore, in this case the necessary conditions are optimal and also sufficient for the level of pollution to become zero. □

A.2 Proof of Lemma 2

Assume that the level of pollution again becomes positive after it becomes zero. That is, there are at least two time points at which the level of pollution decreases to zero. Define the time point T_1 as the earlier time at which pollution is completely eliminated and $T_2 (\neq T_1)$ as the later one. As long as $p_t > 0$ the dynamics of the economy are summarized by equations (41)-(44) with $\nu_t = 0$. That is,

$$\dot{k}_t = Ak_t - c_t - \theta m_t, \tag{A.3}$$

$$\dot{p}_t = \alpha A k_t - m_t, \tag{A.4}$$

$$\dot{c}_t = g c_t, \tag{A.5}$$

$$p_t = \bar{p} - \frac{\sigma}{(\rho + \delta + g)\theta} c_t. \tag{A.6}$$

These equations should be satisfied when $t = T_1$ as well as when $t = T_2$. Therefore, by substituting $p_{T_1} = 0$ and $p_{T_2} = 0$ into equation (A.6), we obtain:

$$c_{T_1} = c_{T_2} = \frac{(\rho + \delta + g)\theta \bar{p}}{\sigma}. \tag{A.7}$$

On the other hand, due to (A.5), c_t grows at a constant rate g except when $c_t = 0$. Since $T_1 < T_2$, the following inequality must be satisfied:

$$c_{T_1} < c_{T_2}. \tag{A.8}$$

This implies that inequality (A.8) contradicts equation (A.7). □

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