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Sustainability of the public debt and wealth inequality in a general equilibrium model*

Noritaka Maebayashi[†] Kunihiro Konishi^{‡§}

Abstract

This study investigates the correlation between sustainability of the public debt and wealth inequality in an endogenous growth model with heterogeneous agents. We show that the threshold for the sustainability of public debt can be related to not only relative size of public debt but also wealth inequality. In addition, this study examines the effects of budget deficit and redistributive policies on the sustainability of the public debt and wealth inequality. We show that an increase in the deficit ratio or the redistributive tax makes the public debt less sustainable. If the economy falls into the unsustainable region as a result of the policy change, both public debt and wealth inequality continue to increase.

JEL classification: H62, H63

Keywords: Fiscal sustainability, Public debt, Wealth inequality, Redistributive policy,
Endogenous growth

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[†]Faculty of Economics and Business Administration, The University of Kitakyushu, 4-2-1 Kitagata, Kokura Minami-ku, Kitakyushu, Fukuoka 802-8577, JAPAN; e-mail: non818mn@kitakyu-u.ac.jp

[‡]Institute of Economic Research, Kyoto University, Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501, JAPAN; e-mail: k.konishi.econ@gmail.com

[§]Research Fellow of the Japan Society for the Promotion of Science (JSPS).

1 Introduction

Since the recent default risk on Greek government debt exposed by the 2008–2009 world crisis, the concern over whether a government deficit and debt are sustainable has been growing among countries whose public debt is very large. In addition to increased public debt, the expansion of wealth and income inequality is also observed in many developed countries. According to Cagetti and Nardi (2008), the concentration of wealth is more serious than that of income. Because the sustainability of the government debt depends on the amount of wealth in its country, the accumulation of the government debt and wealth inequality among households can be closely related to each other.¹ Therefore, investigating the relationship between the sustainability of the public debt and wealth inequality is an important issue.

Some recent studies examine fiscal sustainability using the overlapping-generations (OLG) models (life-cycle models) developed by Diamond (1965). In the OLG models, fiscal sustainability means that the ratio of public debt to GDP (or capital) converges to a stable level in the long run. Chalk (2000) shows that under a constant primary deficit rule, if the initial public debt is very large, public debt does not converge to a stable level but explodes, and thus, public debt is not sustainable. More recent studies (e.g., Bräuning 2005; Yakita 2008; Arai 2011; Teles and Mussolini 2014; Agénor and Yilmaz 2016) extend the analysis to the OLG model with endogenous growth structure and obtain a similar result to Chalk (2000).² Nevertheless, these studies ignore the relationship between the sustainability of the public debt and wealth inequality.

To our knowledge, there are some studies that investigate how the public debt affects inequality or wealth distribution (e.g., Mankiw 2000; Michel and Pestieau 2005; Pestieau and Thibaut 2012).³ However, these previous studies do not pay attention to the sustainability of the public debt but focus on the steady state in which the public debt converges

¹For example, as shown in Table 1 in Hoshi and Ito (2014), more than 90% of Japanese government debt is funded domestically, which ensures the Japanese government's solvency even though the debt is very large.

²In these endogenous growth models, a huge outstanding public debt can induce higher growth of public debt than growth of private capital and output and can make fiscal policy unsustainable.

³There is no consensus on how the public debt affects inequality or wealth distribution among these studies. Mankiw (2000) and Michel and Pestieau (2005) show that public debt increases steady-state inequality whereas Pestieau and Thibaut (2012) show that public debt redistributes wealth from the top wealthy to others.

to a stable level.

Accordingly, this study addresses the following open questions. (i) How are the sustainability of public debt and wealth inequality co-related? (ii) How does public deficit policy affect the sustainability of public debt and wealth inequality? (iii) How do redistributive policies aiming for a reduction in wealth inequality affect the sustainability of public debt? To tackle these problems, we develop a tractable OLG model in which *the transition path* of public debt and that of wealth inequality are jointly determined.

The present model is in line with Bräuninger (2005), which explores the sustainability of public deficit policy under a constant deficit/GDP rule in an AK model.⁴ We incorporate a mechanism generating endogenous transmission of wealth inequality into Bräuninger’s (2005) model. A key mechanism generating wealth inequality in this model is composed of (i) the heterogeneity in the agents’ subjective discount factors based on Becker (1980) and (ii) joy-of-giving bequest motives (e.g., Abel and Warshawsky 1988; Andreoni 1989, 1990). Our model splits the population into two classes, the rich and poor, and assumes that the rich have higher exogenous subjective discount factor than the poor (i.e., the rich are more patient than the poor).⁵ In addition, we assume that agents have joy-of-giving bequest motives independently of whether they belong to the rich or poor. Under these assumptions, the rich save more and bequeath more wealth to their children, which becomes the source of wealth inequality. Some empirical studies consider that such intergenerational linkages in saving behavior and wealth accumulation generate wealth inequality. For example, Dynan et al. (2004), Bozio et al. (2013), Alan et al. (2014), and Néstor (2015) show that the rich have a higher savings rate than do the poor. Other empirical evidence supports that bequests are one of the major causes of wealth inequality (e.g., Kotlikoff and Summers 1981; Gale and Scholz 1994).

Using this framework, we illustrate that the transition paths of both public debt and wealth inequality are determined in a two-dimensional phase diagram and we obtain the following results.

⁴The constant deficit/GDP rule employed by Bräuninger (2005), Yakita (2008), Arai (2011), and Teles and Mussolini (2014) follows the criterion of the Maastricht Treaty.

⁵As for such the classification of agents’ types, Borissov and Lambrecht (2009, pp.99) point out: “In the terminology of Mankiw (2000), this classes might be called savers and spenders.” Some empirical studies show that the rich are more likely to be patient than are the poor (see, e.g., Lawrence 1991; Harrison et al. 2002).

- (i) There is a threshold of public debt for each level of wealth inequality in order for the government to sustain the fiscal policy, and the threshold of public debt is increasing in wealth inequality. When the initial public debt is small, the economy can reach a stable equilibrium in which both public debt and wealth inequality converge to the stable level. When the initial public debt is very large, the economy with higher wealth inequality can converge to a stable equilibrium whereas the economy with low wealth inequality cannot converge to any stable equilibrium. If the economy is in the unsustainable region, both public debt and wealth inequality continue to increase, and the economy goes bankrupt in the long run.
- (ii) An increase in the public deficit ratio makes the public debt less sustainable. Therefore, if the economy with large public debt falls into the unsustainable region as a result of expanding the public deficit ratio, wealth inequality increases as public debt grows during the bankruptcy path.
- (iii) A redistributive policy that attempts to reduce wealth inequality affects the sustainability condition. If the government taxes bequests of the rich and redistributes the revenue to the poor, the economy is more likely to fall into the region in which the public debt is not sustainable and wealth inequality continues to increase. Thus, in the economy with large public debt, introducing such a redistributive policy might be very risky.
- (iv) The policy effects in the stable steady state are as follows. A rise in the public deficit ratio enlarges wealth inequality and decreases the growth rate. An increase in the redistributive tax reduces both wealth inequality and the growth rate, and hence, leads to the trade-off between equality and growth.

The remainder of this paper is organized as follows. Section 2 establishes the base model whose objective is to derive the relationship between sustainability of the public debt and wealth inequality. To achieve this as simply as possible, this base model excludes the redistributive public policy. Sections 3–5 explore this base model as follows. Section 3 derives the equilibrium condition and dynamic system of the economy. Section 4 derives the transition dynamics and the relationship between wealth inequality and sustainability

of public debt. Section 5 analyzes the effects of changes in the public debt finance ratio on the sustainability of the public debt and wealth inequality. Section 6 introduces the redistributive public policy into the base model and examines its effect on the sustainability of the public debt and wealth inequality. Section 7 concludes.

2 Model

2.1 Individuals

We consider a two-period OLG model following Diamond (1965). An individual lives for two periods and a cohort born in period t is called generation t . Therefore, two generations exist in period t ; that is, generation t (the young generation) and generation $t - 1$ (the old generation). In each period, the size of the newly born cohort is given by N . There are two groups of families, “rich” and “poor,” denoted by R and P , respectively. We assume that a constant fraction $\delta \in (0, 1)$ of individuals is the rich and a constant fraction $1 - \delta$ of individuals is the poor. Each individual supplies one unit of labor inelastically and earns labor income in their young period, and then the total labor supply is $L_t = N$. In the old period, they are retired, consume their savings, and leave bequests to their children. Individuals have perfect foresight.

Each individual $i \in \{R, P\}$ born at period t maximizes utility,

$$U_t^i = (1 - \alpha_i) \log c_t^{1i} + \alpha_i [(1 - \beta) \log c_{t+1}^{2i} + \beta \log b_{t+1}^i], \quad (1)$$

where c_t^{1i} is the consumption when young, c_{t+1}^{2i} is the consumption when old, and b_{t+1}^i is the bequest passed on to the child. Note that the utility depends on the amount b_{t+1}^i . This reflects a “joy of giving” savings motive. $\alpha_i \in (0, 1)$ is the intertemporal preference parameter and $\beta \in (0, 1)$ is the relative importance parameter of the consumption when old and the bequest. We assume β does not differ between the rich and poor following Bossmann et al. (2007). In addition, we assume that $\alpha_R > \alpha_P$ based on Becker (1980) and some empirical evidence (e.g., Lawrence 1991; Harrison et al. 2002). This assumption generates the mechanism by which the rich save a larger proportion of their income than do the poor, which is empirically supported by Dynan et al. (2004) for the US, Bozio et al.

(2013) for the UK, Alan et al. (2014) for Canada, and Néstor (2015) for Latin America. In addition, we assume that the wealth endowment of the rich old generation at initial period ($t = 0$) is larger than that of the poor old generation. As a result, the rich bequeath larger wealth than the poor do. Let s_t^i be savings in youth. The budget constraint of generation t can be written as follows:

$$c_t^{1i} = (1 - \tau_t)w_t - s_t^i + b_t^i, \quad (2)$$

$$c_{t+1}^{2i} = [1 + (1 - \tau_{t+1})r_{t+1}]s_t^i - b_{t+1}^i, \quad (3)$$

where w_t , r_{t+1} , and τ_t represent the wage rate, interest rate, and tax on wage and interest income. By solving the intertemporal utility maximization, we obtain the following optimal conditions:

$$c_t^{1i} = (1 - \alpha_i) [(1 - \tau_t)w_t + b_t^i], \quad (4a)$$

$$s_t^i = \alpha_i [(1 - \tau_t)w_t + b_t^i], \quad (4b)$$

$$c_{t+1}^{2i} = (1 - \beta) [1 + (1 - \tau_{t+1})r_{t+1}]s_t^i, \quad (4c)$$

$$b_{t+1}^i = \beta [1 + (1 - \tau_{t+1})r_{t+1}]s_t^i. \quad (4d)$$

From (4b) and (4d), savings are determined as follows:

$$s_t^i = \alpha_i \left[(1 - \tau_t)w_t + \underbrace{\beta \{1 + (1 - \tau_t)r_t\} s_{t-1}^i}_{\text{the bequest from parents}} \right]. \quad (5)$$

Equation (5) indicates the following. First, the savings of the current generation s_t^i are linked to the savings of parents s_{t-1}^i . This is because the bequest from parents depends on their wealth income $[1 + (1 - \tau_t)r_t]s_{t-1}^i$ from (4d). Second, from assumption $\alpha_R > \alpha_P$, the rich save more than the poor do, and leave more wealth to their offspring, who in turn, tend to do the same. This means that the rich tend to accumulate more wealth than the poor do.

The total assets (savings) held by young agents in period t , $A_t \equiv \delta s_t^R N + (1 - \delta)s_t^P N$, are composed of public bonds, D_{t+1} , and private capital, K_{t+1} . Hence, the asset market

clears as follows:

$$K_{t+1} + D_{t+1} = A_t. \quad (6)$$

2.2 Production

There is a large number of identical firms denoted by j . Firm j produces a single final good using the production technology given by $Y_{j,t} = \Gamma K_{j,t}^\gamma (a_t L_{j,t})^{1-\gamma}$ ($0 < \gamma < 1$), where $Y_{j,t}$, $K_{j,t}$, and $L_{j,t}$ represent the output level, private capital, and labor input of firm j , respectively. a_t is the labor efficiency at time t . From profit maximization in competitive markets, factor prices become equal to the marginal products: $r_t = \partial Y_{j,t} / \partial K_{j,t} = \gamma \Gamma (K_{j,t} / L_{j,t})^{\gamma-1} a_t^{1-\gamma}$ and $w_t = \partial Y_{j,t} / \partial L_{j,t} = (1 - \gamma) \Gamma (K_{j,t} / L_{j,t})^\gamma a_t^{1-\gamma}$.

Following Romer (1986), we assume that the average capital per worker has positive external effects on labor productivity, and specify $a_t = K_t / L_t$, where K_t is the average stock of private capital and L_t is the average labor input in the economy. In equilibrium, $K_{j,t} = K_t$ and $L_{j,t} = L_t$ hold for all j , and thus, the factor prices and aggregate output, Y_t , in period t can be written as follows:

$$w_t = \Gamma(1 - \gamma) \frac{K_t}{L_t}, \quad (7a)$$

$$r_t = \Gamma\gamma, \quad (7b)$$

$$Y_t = \Gamma K_t. \quad (7c)$$

2.3 Government

The government in period t imposes a tax on income, $(w_t L_t + r_t A_{t-1})$, and issues bonds, $D_{t+1} - D_t$, to finance public spending, G_t and interest payment for the public debt, $r_t D_t$. The tax revenue in period t , $\tau_t(w_t L_t + r_t A_{t-1})$, is rewritten by $\tau_t(Y_t + r_t D_t)$ using (6), (7a), (7b), and (7c). Thus, the budget constraint of the government is

$$D_{t+1} - D_t + \tau_t(Y_t + r_t D_t) = G_t + r_t D_t, \quad (8)$$

Following Bräuninger (2005), we assume that a constant proportion, $g \in (0, 1)$, of national income, Y_t , is used for public expenditure: $G_t = gY_t$. In addition, the government borrows a constant proportion, $\lambda \in (0, 1)$, of GDP, that is, the government fixes the deficit ratio as follows:

$$D_{t+1} - D_t = \lambda Y_t. \quad (9)$$

As in Bräuninger (2005), Yakita (2008), Arai (2011), and Teles and Mussolini (2014), when g and λ are kept constant, the government must adjust the income tax rate, τ_t , so as to satisfy the budget constraint (8). By using (7b), (7c), (8), and (9), we obtain

$$\tau_t = 1 - \frac{1 + \lambda - g}{1 + \gamma x_t}, \quad (10)$$

where $x_t \equiv D_t/K_t$. A higher level of public debt means that a higher level of income taxation must be used to pay for the interest payments on the debt. Therefore, an increase in the ratio of public debt to private capital x raises income tax rate: $d\tau_t/dx_t > 0$ as in Bräuninger (2005), Yakita (2008), Arai (2011), and Teles and Mussolini (2014). We call this the *tax burden effect*.

3 Equilibrium condition and dynamic system

From individuals' budget constraints: (2), (3), distribution of output: $Y_t = r_t K_t + w_t L_t$, the government's budget constraint: (8), and the asset market-clearing condition: (6), the market equilibrium satisfies the resource constraint: $Y_t = \delta c_t^{1R} N + (1 - \delta) c_t^{1P} N + \delta c_t^{2R} N + (1 - \delta) c_t^{2P} N + K_{t+1} - K_t + G_t$.

We then characterize the equilibrium paths in this economy. Substituting (5) into $A_t = \delta s_t^R N + (1 - \delta) s_t^P N$, we obtain

$$A_t = \bar{\alpha}(1 - \tau_t)w_t N + \beta[1 + (1 - \tau_t)r_t] [\alpha_R s_{t-1}^R \delta N + \alpha_P s_{t-1}^P (1 - \delta)N], \quad (11)$$

where $\bar{\alpha} \equiv \delta\alpha_R + (1 - \delta)\alpha_P$. By using (6) and the definition of x_t , we obtain

$$K_t = (1 + x_t)^{-1}A_{t-1}. \quad (12)$$

This equation (12) indicates that an increase in x_t reduces investment in private capital because public bonds account for a larger proportion of aggregate assets, A_{t-1} . We call this the *crowding-out effect*.

Dividing (11) by A_{t-1} and substituting (7a), (7b), (10), and (12) into (11), we obtain the growth of aggregate savings, A , as follows:

$$\begin{aligned} \frac{A_t}{A_{t-1}} &= \frac{\bar{\alpha}(1 - \gamma)\mu_1}{(1 + x_t)(1 + \gamma x_t)} + \beta \left(1 + \frac{\gamma\mu_1}{1 + \gamma x_t}\right) [(\alpha_R - \alpha_P)\varphi_{t-1} + \alpha_P] \\ &\equiv G^A(x_t, \varphi_{t-1}), \end{aligned} \quad (13)$$

where $\varphi_t \equiv \frac{\delta s_t^R N}{A_t}$ and $\mu_1 \equiv \Gamma(1 + \lambda - g)$. Note that $1 - \varphi_t = \frac{(1 - \delta)s_t^P N}{A_t}$ holds from the definition of φ_t and A_t . In this study, because φ_t represents the ratio of total savings of the rich to aggregate savings, φ_t serves as a convenient measure of wealth inequality.

From (13), $G^A(x_t, \varphi_{t-1})$ satisfies (a) $\frac{\partial G^A(x_t, \varphi_{t-1})}{\partial x_t} < 0$ and (b) $\frac{\partial G^A(x_t, \varphi_{t-1})}{\partial \varphi_{t-1}} > 0$. The former, (a), comes from the following two channels. First, an increase in x_t reduces the wage rate through the *crowding-out effect* (see (12)). Second, an increase in x_t decreases disposable wage and bequest incomes through the *tax burden effect* (see (10)). Both have negative effects on the growth of aggregate savings. The latter, (b), indicates the following. The rich accumulate more wealth than the poor do and hold a larger proportion of aggregate wealth (see (5)). Then, wealth inequality φ driven by the rich contributes to the growth of aggregate savings $G^A(\cdot)$.

By using (5), (7a), (7b), (10), (12), and the definition of φ_t , we obtain the growth of the rich's savings as

$$\begin{aligned} \frac{\delta s_t^R N}{\delta s_{t-1}^R N} &= \frac{\alpha_R}{\varphi_{t-1}} \left[\frac{\delta(1 - \gamma)\mu_1}{(1 + x_t)(1 + \gamma x_t)} + \beta \left(1 + \frac{\gamma\mu_1}{1 + \gamma x_t}\right) \varphi_{t-1} \right] \\ &\equiv G^R(x_t, \varphi_{t-1}). \end{aligned} \quad (14)$$

Substituting (7c) into (9) and using $D_t/K_t \equiv x_t$ yields the growth of public debt as

$$\frac{D_{t+1}}{D_t} = 1 + \frac{\lambda\Gamma}{x_t} \equiv G^D(x_t). \quad (15)$$

From (6) and (9), we obtain $K_{t+1} + \lambda Y_t + D_t = A_t$. Dividing both sides of it by K_t and using (7c) yield $K_{t+1}/K_t = (A_t/A_{t-1})(A_{t-1}/K_t) - (x_t + \lambda\Gamma)$. Substituting (12) and (13) into it, we obtain

$$\frac{K_{t+1}}{K_t} = (1 + x_t)G^A(x_t, \varphi_{t-1}) - (x_t + \lambda\Gamma) \equiv G^K(x_t, \varphi_{t-1}). \quad (16)$$

The growth of private capital $G^K(\cdot)$ is linked positively with $G^A(\cdot)$ through the asset market-clearing condition (see (12)). From (13), (14), and the definition of φ_t , we obtain

$$\frac{\varphi_t}{\varphi_{t-1}} = \frac{\frac{\delta s_t^R N}{\delta s_{t-1}^R N}}{\frac{A_t}{A_{t-1}}} = \frac{G^R(x_t, \varphi_{t-1})}{G^A(x_t, \varphi_{t-1})}. \quad (17)$$

The growth of wealth inequality decreases (increases) when $G^A(\cdot)$ is relatively larger (smaller) than $G^R(\cdot)$. This is attributed to the definition of φ . From (15) and (16), we obtain

$$\frac{x_{t+1}}{x_t} = \frac{G^D(x_t)}{G^K(x_t, \varphi_{t-1})} = \frac{1 + \lambda\Gamma/x_t}{(1 + x_t)G^A(x_t, \varphi_{t-1}) - (x_t + \lambda\Gamma)}. \quad (18)$$

The above two difference equations (17) and (18) together with the initial values φ_{-1} and x_0 characterize the dynamics of the economy. Note that both x_t and φ_{t-1} in period t are predetermined variables.

4 Transition dynamics of wealth inequality and the public debt/private capital ratio

In this section, we derive global transition dynamics of the economy and investigate how the accumulation of public debt and wealth inequality correlate to each other.

We begin with the derivation of $\varphi_t = \varphi_{t-1}$ locus on the (x_t, φ_{t-1}) plane. Setting

$\varphi_t = \varphi_{t-1}$ in (17), that is, $1 = \frac{G^R(x_t, \varphi_{t-1})}{G^A(x_t, \varphi_{t-1})}$ yields

$$\beta(1+x_t) [\mu_1^{-1}(1+\gamma x_t) + \gamma] = \frac{1-\gamma}{(\alpha_R - \alpha_P)(1-\varphi_{t-1})} \left(\bar{\alpha} - \frac{\delta\alpha_R}{\varphi_{t-1}} \right). \quad (19)$$

Let us define the left- and right-hand side of (19) as $\varepsilon(x_t)$ and $\eta(\varphi_{t-1})$, respectively. Examining (19), we arrive at the following.

Lemma 1

- (i) $\varphi_t = \varphi_{t-1}$ locus is an upward-sloping curve on the (x_t, φ_{t-1}) plane.
- (ii) $\varphi_t = \varphi_{t-1}$ locus has an asymptote $\varphi_{t-1} = 1$ when $x_t \rightarrow \infty$ and takes a lower limit $(0, \tilde{\varphi})$ on the (x_t, φ_{t-1}) plane. $\tilde{\varphi}$ is defined in Appendix A.

Proof: See Appendix A.

Next, we derive $x_{t+1} = x_t$ locus on the (x_t, φ_{t-1}) plane. Setting $x_{t+1} = x_t$ in (18), that is, $1 = \frac{1+\lambda\Gamma/x_t}{(1+x_t)G^A(x_t, \varphi_{t-1}) - (x_t + \lambda\Gamma)}$, leads to

$$\varphi_{t-1} = \frac{\zeta(x_t)}{\alpha_R - \alpha_P}, \quad (20)$$

where

$$\zeta(x_t) \equiv \frac{(1+x_t)(1+\gamma x_t) \left(1 + \frac{\lambda\Gamma}{x_t}\right) - \bar{\alpha}(1-\gamma)\mu_1}{\beta(1+x_t)[1+\gamma(x_t + \mu_1)]} - \alpha_P.$$

Examining (20), we arrive at the following.

Lemma 2 Suppose that $\gamma(1-g) > \lambda$.⁶

- (i) $x_{t+1} = x_t$ locus is a U-shaped curve on the (x_t, φ_{t-1}) plane.
- (ii) $x_{t+1} = x_t$ locus has asymptotes: $\lim_{x_t \rightarrow \infty} \varphi_{t-1} = \frac{\beta^{-1} - \alpha_P}{\alpha_R - \alpha_P} > 1$ and $\lim_{x_t \rightarrow 0} \varphi_{t-1} = +\infty$ on the (x_t, φ_{t-1}) plane.

Proof: See Appendix B.

⁶This is satisfied if λ is not so large and (γ, g) takes conventional parameters used in the literature, such as Bräuning (2005) and Michel et al. (2010). For example, taking a parameter set $(\gamma, g) = (0.2, 0.2)$, this condition is satisfied under $\lambda < 0.16$.

Hence, we can depict $\varphi_t = \varphi_{t-1}$ and $x_{t+1} = x_t$ loci on the (x_t, φ_{t-1}) plane in Figure 1.

Finally, we examine the regions in which $K_{t+1}/K_t \geq 0$, that is, $(1 + x_t)G^A(x_t, \varphi_{t-1}) - (x_t + \lambda\Gamma) \geq 0$ from (16) so far.⁷ This condition can be rewritten as

$$\varphi_{t-1} \geq \frac{1}{\alpha_R - \alpha_P} \left[\zeta(x_t) - \frac{(1 + \gamma x_t) \left(1 + \frac{\lambda\Gamma}{x_t}\right)}{\beta(1 + x_t)[1 + \gamma(x_t + \mu_1)]} \right] \equiv \frac{\Lambda(x_t)}{\alpha_R - \alpha_P}. \quad (21)$$

Thus, we can recognize that $K_{t+1}/K_t > 0$ is satisfied above $K_{t+1}/K_t = 0$ locus. Examining (21), we arrive at the following. From (20) and (21), $K_{t+1}/K_t = 0$ locus is always below $x_{t+1} = x_t$ locus. In addition, Appendix C shows that $K_{t+1}/K_t = 0$ locus is an upward-sloping curve if $\bar{\alpha}(1 - \gamma)(1 + \lambda - g) > \lambda$ and has an asymptote $\lim_{x_t \rightarrow \infty} \varphi_{t-1} = \frac{\beta^{-1} - \alpha_P}{\alpha_R - \alpha_P} > 1$ on the (x_t, φ_{t-1}) plane.⁸ Then, we can depict $K_{t+1}/K_t = 0$ locus as the broken curve in Figure 1.

[Figure 1]

Now we investigate the steady states of the economy wherein both x_t and φ_{t-1} are constant. In this study, we use an asterisk to represent variables in the steady states (i.e., $x_t = x_{t+1} = x^*$ and $\varphi_{t-1} = \varphi_t = \varphi^*$). As shown in Figure 1, the steady state values of (x^*, φ^*) are determined by the intersections of the curves, $\varphi_t = \varphi_{t-1}$ and $x_{t+1} = x_t$ loci, on the (x_t, φ_{t-1}) plane. From Lemmas 1 and 2, there are either two long-run equilibria or none. A case of two steady states is represented in Figure 1, and is obtained if **C1** is satisfied.⁹

C1: $\varepsilon(\bar{x}) > \eta(\bar{\varphi})$, where $\bar{\varphi} \equiv \zeta(\bar{x})$.

In addition, **C2** allows us to focus on the economy in which wealth of the rich is larger

⁷In the region where $K_{t+1}/K_t < 0$, the asset market-clearing condition (6) indicates that D_{t+1} becomes larger than A_t . In this situation, the public debt cannot be absorbed by aggregate savings and then no capital is installed in the production sector.

⁸Similarly to the assumption of $\gamma(1 - g) > \lambda$ in Lemma 2, $\bar{\alpha}(1 - \gamma)(1 + \lambda - g) > \lambda$ is satisfied if λ is not so large and $(\gamma, g, \bar{\alpha})$ takes conventional parameters used in the literature, such as Bräuning (2005) and Michel et al. (2010). For example, taking a parameter set $(\gamma, \bar{\alpha}, g) = (0.2, 0.35, 0.2)$, this condition is satisfied under $\lambda < 0.31$.

⁹A case of no steady state is realized and the fiscal policy is always unsustainable when the public debt finance ratio, λ , is very large, as in Bräuning (2005). We rule out this case because of the same discussion by Bräuning (2005).

than that of the poor (i.e., $\varphi_{t-1} > 0.5$ for all t).¹⁰

C2: $\tilde{\varphi} \equiv \eta^{-1}(\varepsilon(0)) > 0.5.$

Let us refer to (x_k^*, φ_k^*) as the steady state k ($k \in \{S, U\}$). We obtain $x_S^* < x_U^*$ and $\varphi_S^* < \varphi_U^*$ because $\varphi_t = \varphi_{t-1}$ locus is increasing in an upward-sloping curve. From (7c), (15), (16), and constant x_k^* , the steady state growth rate at each state is as follows:

$$\hat{Y}_k^* \equiv \left(\frac{Y_{t+1}}{Y_t}\right)^* = \left(\frac{K_{t+1}}{K_t}\right)^* = \left(\frac{A_t}{A_{t-1}}\right)^* = \left(\frac{D_{t+1}}{D_t}\right)^* = 1 + \frac{\lambda\Gamma}{x_k^*} \quad \text{for } k \in \{S, U\}. \quad (22)$$

Therefore, $x_S^* < x_U^*$ implies that the growth rate of the steady state S is higher than that of the steady state U . As shown in Appendix D, we can illustrate a phase diagram of this economy in Figure 1. As we can observe from the phase diagram, the steady state S is stable and the steady state U is saddle-point stable.¹¹ A dotted line HH in Figure 1 represents the stable arm converging to the steady state U . Because x_t and φ_{t-1} are predetermined variables at time t , as we mention before, and we need to note the following two points. First, the initial state of the economy is given by a point (x_0, φ_{-1}) on the (x_t, φ_{t-1}) plane. Second, the saddle arm, HH, is a knife-edge.

These facts lead to the following two cases. When an economy starts at the initial state (x_0, φ_{-1}) in the upper-left of the saddle arm: HH, it converges to the steady state S . At the steady state S , both x_t and φ_{t-1} are constant, and the government can run the constant budget deficit policy permanently because private capital grows at the constant rate \hat{Y}_S^* . By contrast, when (x_0, φ_{-1}) is in the lower-right of the saddle arm, HH, an economy will not converge to any steady states. In this case, the public debt grows more than private capital does: $G^D(\cdot) > G^K(\cdot)$; and finally the economy falls into $K_{t+1}/K_t < 0$ region in which the public debt/private capital ratio, x_t , becomes too large to sustain the investment in private capital. Therefore, the dotted line HH in Figure 1 represents the *threshold of public debt* for each level of wealth inequality in order for the government to

¹⁰ $\tilde{\varphi} \equiv \eta^{-1}(\varepsilon(0)) > 0.5$ holds if and only if

$$\varepsilon(0) > \eta(0.5) \Leftrightarrow \beta(\mu_1^{-1} + \gamma) > -\frac{2(1-\gamma)[\delta\alpha_R - (1-\delta)\alpha_P]}{\alpha_R - \alpha_P}.$$

¹¹In Appendix E, we consider the local stability at each steady state by using numerical examples.

sustain the fiscal policy. This threshold level of public debt has a positive relationship with wealth inequality. This is a noticeable result that departs from previous studies that do not include wealth inequality (e.g., Bräuning 2005; Yakita 2008; Arai 2011; Teles and Mussolini 2014).

In summary, we can state the following proposition:

Proposition 1. *Under a fiscal deficit policy, there is a threshold of public debt for each level of wealth inequality in order for the government to sustain its policy. The threshold of public debt is increasing in wealth inequality.*

In order to consider the intuition behind Proposition 1, let us begin with the case in which the initial public debt/private capital ratio, x , is sufficiently small. In this case, both the *tax burden effect* and the *crowding-out effect* are small from (10) and (12), and then wage and bequest incomes are relatively large. Under such an environment, aggregate savings can grow strongly, which avoids rapid growth of φ from (17) and drives private capital to grow strongly from (16). Therefore, we find that a sufficiently large growth of aggregate savings, $G^A(\cdot)$, helps both the public debt/private capital ratio, x , and wealth inequality, φ , converge to the steady state S .

Next, we move to the case when the initial x_t is large enough to be near the saddle arm, HH, for example, $x_0 = x^H$ in Figure 1. In this case, both the *tax burden effect* and the *crowding-out effect* are so large that wage and bequest incomes become very small. Thus, the growth rates of both aggregate savings, $G^A(\cdot)$, and private capital, $G^K(\cdot)$, are small. Even under such an environment, if the initial wealth of the rich is relatively larger than that of the poor, wealth accumulation of the rich can be strong enough to reinforce the growth of aggregate savings, $G^A(\cdot)$ (see (13)). Therefore, the economy with high wealth inequality as represented by Q_2 can sustain its public debt.

By contrast, in the economy with low wealth inequality, as represented by Q_1 , the growth of aggregate savings, $G^A(\cdot)$, tends to remain small. In this case, public debt grows more than private capital does, and then, the economy cannot sustain its public debt. It is of great interest that wealth inequality increases as public debt grows during the bankruptcy path. An intuitive reason for this is as follows. As public debt grows, the growth of the wage rate keeps decreasing through both the *tax burden effect* and the

crowding-out effect. In this situation, bequest income plays a more important role in wealth accumulation than does the wage income. Because the rich leave more wealth to their offspring than do the poor, wealth inequality increases and absorption of larger public debt tends to rely more on wealth accumulation by the rich. However, this situation does not last long and the economy goes bankrupt in the long run.

5 Changes in public debt finance ratio λ

A few studies investigate the relationship between public debt and inequality (e.g., Mankiw 2000; Michel and Pestieau 2005) and show that increases in public debt finance raise wealth (or income) inequality. However, these studies focus only on the steady state, which corresponds to the stable steady state S in our study. Therefore, previous studies neglect the possibility of the unstable (unsustainable) paths. The main objective here is to study how increases in public debt finance affect wealth inequality and sustainability of the public debt.

We begin with the effects of an increase in public debt finance ratio λ on the $\varphi_t = \varphi_{t-1}$ locus. Taking the total differentials of (19) yields

$$\left. \frac{d\varphi_{t-1}}{d\lambda} \right|_{\varphi_t = \varphi_{t-1}} = - \frac{\beta\Gamma\mu_1^{-2}(\alpha_R - \alpha_P)(1+x_t)(1+\gamma x_t)(1-\varphi_{t-1})}{\eta(\varphi_{t-1})(\alpha_R - \alpha_P) + (1-\gamma)\delta\alpha_R\varphi_{t-1}^{-2}} < 0. \quad (23)$$

Note that $0 < \varphi_{t-1} < 1$ holds from the definition of φ_{t-1} . We next investigate the effect of changes in λ on $x_{t+1} = x_t$ locus. Differentiating (20) with respect to λ , we obtain

$$\left. \frac{d\varphi_{t-1}}{d\lambda} \right|_{x_{t+1} = x_t} = \frac{\Gamma(1+\gamma x_t) [(1+x_t^{-1})(1+\gamma\mu_1 - \gamma\lambda\Gamma) - (1-\gamma)\bar{\alpha}]}{(\alpha_R - \alpha_P)\beta(1+x_t)[1+\gamma(x_t + \mu_1)]^2}. \quad (24)$$

Because of $1+\gamma\mu_1 - \gamma\lambda\Gamma = 1+\gamma\Gamma(1-g) > 1$ (from the definition of μ_1) and $(1-\gamma)\bar{\alpha} \in (0, 1)$, $\left. \frac{d\varphi_{t-1}}{d\lambda} \right|_{x_{t+1} = x_t} > 0$ holds. Thus, we obtain the following.

Lemma 3

When the government increases the public debt finance ratio λ , (i) $\varphi_t = \varphi_{t-1}$ locus shifts downward from (23), and (ii) $x_{t+1} = x_t$ locus shifts upward from (24). These are represented in Figure 2.

[Figure 2]

From Lemma 3, we can observe that the steady state U shifts left-and-downward and therefore, the saddle arm, HH , also shifts left-and-downward, as depicted in Figure 2. That is, the threshold of public debt for each level of inequality in order for the government to sustain the fiscal policy becomes lower. This result implies that the range in which the deficit policy is sustainable is shrunk by an increase in λ . Thus, we obtain the following proposition.

Proposition 2. *An increase in the deficit ratio reduces the range of sustainable initial public debt.*

A larger budget deficit reinforces both the *tax burden effect* and the *crowding-out effect*, which leads to a decline in aggregate savings. Therefore, even if the initial level of wealth inequality is somewhat high, the economy is more likely to fall into an unsustainable path. This result leads to the following policy implication. In the economy with a large public debt, an increase in the public debt finance ratio not only makes public debt less sustainable but also can induce wealth inequality to increase persistently.

In the rest of this section, we shed light on the steady state S and investigate the effects of changes in λ on the public debt/private capital ratio, wealth inequality, and the long-run growth rate. Lemma 3 shows that at the steady state S , an increase in λ raises the public debt/private capital ratio, x_S^* , but the effect on wealth inequality, φ_S^* , is ambiguous. Moreover, the effect of λ on the growth rate at the steady state S is ambiguous. Then, we conduct a numerical analysis. We adopt the following benchmark parameters: $\gamma = 0.2$, $\Gamma = 12$, $g = 0.2$, $\delta = 0.5$, $\beta = 0.3$, $\alpha_R = 0.45$, and $\alpha_P = 0.25$.¹² Table 1 shows the steady state values for each deficit ratio and leads to the following result.

Result 1. *In the steady state S , an increase in λ (i) raises wealth inequality and (ii) reduces the long-run growth rate.*

¹²The values of γ , Γ , and g follow the methodology in Bräuninger (2005). $\gamma = 0.2$ and $\Gamma = 12$ represent $r = \gamma\Gamma = 2.4$. If we assume that the time period is about 30 years, $r = 2.4$ implies that the annual interest rate is 4.2%. We employ $\alpha_R = 0.45$ and $\alpha_P = 0.25$ so as to satisfy the conventional value of $\bar{\alpha} = \delta\alpha_R + (1 - \delta)\alpha_P = 0.35$. The value of δ follows the methodology in Galí et al. (2007). The results presented here are robust to other parameter values, δ . We conduct robustness checks in the technical appendix, which is available on request.

[Table 1]

The result (i) is similar to that of Mankiw (2000) and Michel and Pestieau (2005). As mentioned in the paragraph below Proposition 2, a higher budget deficit decreases aggregate savings, which reduces the growth of the wage rate because a decline in aggregate savings reduces the investment in private capital (see (7a) and (12)). In this study, from (5), the income of the young consists of the wage and bequest incomes. Lower wage income indicates that the bequest income becomes more important for the accumulation of wealth. The rich tend to hold a larger proportion of total wealth, and then wealth inequality increases. Furthermore, result (ii) is similar to that of Bräuninger (2005). A higher budget deficit implies that an increase in the level of public debt raises the interest payment of the government. To satisfy the government budget constraint, the income tax rate increases. This reduces total savings. In addition, a higher level of public debt crowds out the investment in private capital. As a result, the long-run growth rate declines.

6 Redistributive public policy

We have considered the relationship between the sustainability of public debt and wealth inequality. In many developed countries, policies aiming for a reduction in wealth inequality are implemented.¹³ Thus, we then wonder how a public redistributive policy affects the sustainability of public debt, wealth inequality, and economic growth. In order to tackle this problem, we introduce a public redistributive policy in the following simple way.

The government taxes bequests of the rich at rate τ^b and redistributes its revenue lump-sum to the poor in youth. This is the reduced form of the redistributive policy considered in Bossmann et al. (2007).¹⁴ The budget constraint for the rich is given by $c_t^{1R} = (1 - \tau_t)w_t - s_t^R + b_t^R$ and $c_{t+1}^{2R} = [1 + (1 - \tau_{t+1})r_{t+1}]s_t^R - (1 + \tau^b)b_{t+1}^R$ and that for the poor is given by $c_t^{1P} = (1 - \tau_t)w_t - s_t^P + b_t^P + T_t$ and $c_{t+1}^{2P} = [1 + (1 - \tau_{t+1})r_{t+1}]s_t^P - b_{t+1}^P$, where T_t is the uniform lump-sum transfer by the redistributive policy. The government redistributive policy is represented as $\delta N \tau^b b_t^R = (1 - \delta)NT_t$.

¹³For instance, France, Germany, Japan, the United States, and the United Kingdom have inheritance or estate taxes.

¹⁴In Bossmann et al. (2007), the government taxes bequests of both the rich and the poor and redistributes the revenue lump-sum to the young generation independently of whether the young belong to the rich or the poor.

Appendix F shows that the growth of aggregate savings and that of private capital are given by

$$G^A(x_t, \varphi_{t-1}; \tau^b) = \frac{\bar{\alpha}(1-\gamma)\mu_1}{(1+\gamma x_t)(1+x_t)} + \beta \left(1 + \frac{\gamma\mu_1}{1+\gamma x_t}\right) \left[\frac{\alpha_R - \alpha_P}{1+\tau^b}\varphi_{t-1} + \alpha_P\right], \quad (25)$$

and

$$G^K(x_t, \varphi_{t-1}; \tau^b) = (1+x_t)G^A(x_t, \varphi_{t-1}; \tau^b) - (x_t + \lambda\Gamma). \quad (26)$$

Furthermore, Appendix F shows that $\varphi_t = \varphi_{t-1}$ and $x_{t+1} = x_t$ loci are rewritten as

$$\varepsilon(x_t) = \frac{1+\tau^b}{1 - \frac{\tau^b\alpha_P}{(\alpha_R - \alpha_P)(1-\varphi_{t-1})}} \eta(\varphi_{t-1}) \equiv \tilde{\eta}(\varphi_{t-1}), \quad (27)$$

and

$$\varphi_{t-1} = \frac{1+\tau^b}{\alpha_R - \alpha_P} \zeta(x_t), \quad (28)$$

respectively. As shown in Appendix G, the introduction of redistributive policy does not affect the main properties of $\varphi_t = \varphi_{t-1}$ locus that we observe in Section 4 if we assume $\alpha_R > (1+\tau^b)\alpha_P$. Then, we easily recognize that (i) $\varphi_t = \varphi_{t-1}$ locus: (27) is an upward-sloping curve and $x_{t+1} = x_t$ locus: (28) is a U-shaped curve on the (x_t, φ_{t-1}) plane, and (ii) two steady states corresponding to S and U exists as depicted in Figure 3.

[Figure 3]

We next investigate the effects of a rise in the bequest tax rate. Taking the total differentials of (27) yields

$$\left. \frac{d\varphi_{t-1}}{d\tau^b} \right|_{\varphi_t = \varphi_{t-1}} = - \frac{\tilde{\eta}(\varphi_{t-1}) [\alpha_R(1-\varphi_{t-1}) + \alpha_P\varphi_{t-1}]}{(1+\tau^b) [\tilde{\eta}(\varphi_{t-1})(\alpha_R - \alpha_P) + (1+\tau^b)(1-\gamma)\delta\alpha_R\varphi_{t-1}^2]} < 0.$$

Thus, when the government increases τ^b , $\varphi_t = \varphi_{t-1}$ locus shifts downward. We then examine the effect of changes in τ^b on $x_{t+1} = x_t$ locus. Differentiating (28) with respect to τ^b , we obtain

$$\left. \frac{d\varphi_{t-1}}{d\tau^b} \right|_{x_{t+1} = x_t} = \frac{\zeta(x_t)}{\alpha_R - \alpha_P} > 0.$$

Therefore, when the government increases τ^b , $x_{t+1} = x_t$ locus shifts upward. These results imply that the effects of an increase in the bequest tax rate are qualitatively similar to those of a rise in the public debt finance ratio. We can state the following proposition in summary.

Proposition 3. *Taxing bequests of the rich and redistributing it to the poor (or a rise in the bequest tax) reduces the range of sustainable initial public debt and makes fiscal policy less sustainable.*

An intuitive explanation is as follows. The redistributive policy increases income and savings of the poor whereas it decreases income and savings of the rich. We find that the latter dominates the former and then the growth of aggregate savings declines by an increase in τ^b as follows:

$$\frac{dG^A(x_t, \varphi_{t-1}; \tau^b)}{d\tau^b} = -\beta \left(1 + \frac{\gamma\mu_1}{1 + \gamma x_t} \right) \frac{\alpha_R - \alpha_P}{(1 + \tau^b)^2} \varphi_{t-1} < 0. \quad (29)$$

This is because the rich are more patient than the poor. If the initial public debt/private capital ratio, x , is sufficiently large, the growth of private capital declines sufficiently to be below that of public debt. Thus, the redistributive policy makes the government budget deficit policy less sustainable. If the economy falls into a region where the public debt is unsustainable, then wealth inequality increases during the bankruptcy path. Hence, it is noticeable that the redistributive policy can result in widening wealth inequality.

We investigate the effects of changes in the redistributive policy at the steady state S . When the government raises τ^b , the steady state S shifts rightward. The public debt/private capital ratio at the steady state S increases (i.e., $dx_S^*/d\tau^b > 0$). From (22), it is obvious that an increase in τ^b reduces the growth rate at the steady state S . The result of the declining growth rate is brought about by the reduction in the growth of aggregate savings, as in (29). On the other hand, the effect of τ^b on wealth inequality at the steady state S is ambiguous because it is not certain whether the steady state S shifts upward or downward. To clarify this ambiguity, we conduct a numerical analysis with the following parameters: $\gamma = 0.2$, $\Gamma = 12$, $g = 0.2$, $\delta = 0.5$, $\beta = 0.3$, $\alpha_R = 0.45$, and $\alpha_P = 0.25$. Let us denote the steady state values after the policy change as x_k^{**} , φ_k^{**} , and \hat{Y}_k^{**} . The effects of the policy change are measured by $\Delta x_k^* \equiv (x_k^{**} - x_k^*)/x_k^*$, $\Delta \varphi_k^* \equiv (\varphi_k^{**} - \varphi_k^*)/\varphi_k^*$, and

$$\Delta \hat{Y}_k^* \equiv (\hat{Y}_k^{**} - \hat{Y}_k^*) / \hat{Y}_k^* \quad (k = \{S, U\}).$$

[Table 2]

Table 2 represents percentage changes in the steady state values when the government increases the bequest tax rate, τ^b , from 0 to 0.3. This numerical analysis shows the following. An increase in τ^b reduces wealth inequality. However, the effect of decreasing wealth inequality is relatively small (only about 2% or 3% changes in φ_S^* even by a 30% increase in τ^b). This is attributed to the adverse effect of τ^b on wealth inequality. As mentioned in the paragraph below Proposition 3, a rise in τ^b not only redistributes income from the rich to the poor but also decreases the long-run growth rate and the investment in private capital. The latter leads to a decrease in the wage income (from (7a)), and the bequest income plays a more important role in savings. Because the rich receive more bequest, wealth inequality increases. This mitigates the effect of τ^b on wealth inequality. We summarize the redistributive policy effect in the steady state S .

Result 2. *An increase in τ^b reduces wealth inequality and the growth rate at the steady state S . The effect of decreasing wealth inequality is relatively small.*

Our investigation throughout this section leads to the following policy implication. A redistributive policy aimed at reducing wealth inequality faces a trade-off between equality and growth in the steady state S . Moreover, in the economy with a large public debt/private capital ratio, implementing such a redistributive policy might be very risky because it can take the economy on an unstable path from which wealth inequality and public debt continue to increase.

7 Conclusion

This study constructs an endogenous growth model with heterogeneous agents to examine the correlation between the sustainability of the public debt and wealth inequality. We show there is a threshold of public debt for each level of wealth inequality in order for government to sustain the fiscal policy and the threshold of public debt is increasing in wealth inequality. In addition, we investigate the effects of budget deficit and redistributive

policies on the sustainability of the public debt. We show that an increase in the deficit ratio or the redistributive tax reduces the range of sustainable public debt. That is, in the economy with large public debt, such a policy change makes the economy fall into the unsustainable region in which both public debt and wealth inequality continue to increase.

There are several interesting directions for future research. First, this study used an OLG model with the AK production structure and assumed that the government expenditure is public consumption. However, it is well known that productive government spending is one of the important factors of economic growth. In future research, it would be interesting to consider Barro's (1990) model. Second, we do not consider the heterogeneity of wage income. Incorporating human capital accumulation generates an endogenous disparity of wage income and could allow us to study how the growth-income inequality relationship affects the sustainability of public debt.

Appendix

A Proof of Lemma 1

As shown in Figure 4, the intersection of $\varepsilon(x_t)$ and $\eta(\varphi_{t-1})$ determines the value of x_t , which satisfies (19) for a given φ_{t-1} . When φ_{t-1} increases, the function $\eta(\varphi_{t-1})$ increases, as depicted by the broken line in Figure 4. Thus, x_t rises correspondingly. As a result, $\varphi_t = \varphi_{t-1}$ locus can be depicted as an upward-sloping curve on the (x_t, φ_{t-1}) plane in Figure 1. Note that the definition of φ_t implies $\varphi_{t-1} \in (0, 1)$. Let us define $\tilde{\varphi}$ as $\tilde{\varphi} \equiv \eta^{-1}(\varepsilon(0))$. Since $\lim_{\varphi_{t-1} \rightarrow 1} \eta(\varphi_{t-1}) = +\infty$ and $\lim_{\varphi_{t-1} \rightarrow 0} \eta(\varphi_{t-1}) = -\infty$, $\varphi_t = \varphi_{t-1}$ locus has an asymptote $\varphi_{t-1} = 1$ when $x_t \rightarrow \infty$ and φ_{t-1} has a lower limit $\tilde{\varphi}$ when $x_t = 0$.¹⁵

[Figure 4]

¹⁵Taking the total differentials of (19) yields

$$\left. \frac{d\varphi_{t-1}}{dx_t} \right|_{\varphi_t = \varphi_{t-1}} = \frac{\beta(\alpha_R - \alpha_P)(1 - \varphi_{t-1})[\mu_1^{-1}(1 + \gamma + 2\gamma x_t) + \gamma]}{(1 - \gamma)\delta\alpha_R\varphi_{t-1}^{-2} + \beta(\alpha_R - \alpha_P)(1 + x_t)[\mu_1^{-1}(1 + \gamma x_t) + \gamma]} > 0.$$

Therefore, we can observe that $\varphi_t = \varphi_{t-1}$ locus is an upward-sloping curve and has an asymptote $\varphi_{t-1} = 1$ on the (x_t, φ_{t-1}) plane.

B Proof of Lemma 2

We derive $x_{t+1} = x_t$ locus. Differentiating $\zeta(x_t)$ with respect to x_t yields

$$\zeta'(x_t) = \frac{\left[\gamma(1+x_t) \left(1 + \frac{\lambda\Gamma}{x_t}\right) + (1+\gamma x_t) \left(1 + \frac{\lambda\Gamma}{x_t}\right) - (1+\gamma x_t)(1+x_t) \frac{\lambda\Gamma}{x_t^2} \right]}{\beta(1+x_t)[1+\gamma(x_t+\mu_1)]} - \frac{\left[(1+\gamma x_t)(1+x_t) \left(1 + \frac{\lambda\Gamma}{x_t}\right) - \bar{\alpha}(1-\gamma)\mu_1 \right] [1+\gamma(1+2x_t+\mu_1)]}{\beta(1+x_t)^2 [1+\gamma(x_t+\mu_1)]^2}. \quad (\text{B.1})$$

We rearrange (B.1) as follows:

$$\begin{aligned} \zeta'(x_t) &= \frac{(1+x_t)^2 \left[\gamma^2 \mu_1 \left(1 + \frac{\lambda\Gamma}{x_t}\right) - (1+\gamma x_t)(1+\gamma x_t + \gamma\mu_1) \frac{\lambda\Gamma}{x_t^2} \right] + \bar{\alpha}(1-\gamma)\mu_1 [1+\gamma(1+2x_t+\mu_1)]}{\beta(1+x_t)^2 [1+\gamma(x_t+\mu_1)]^2}, \\ &= \frac{\Gamma(1+x_t)^2 [\gamma^2(1-g)x_t^2 - \lambda\{1+\gamma(2x_t+\mu_1)\}] + \bar{\alpha}(1-\gamma)\mu_1 x_t^2 [1+\gamma(1+2x_t+\mu_1)]}{\beta x_t^2 (1+x_t)^2 [1+\gamma(x_t+\mu_1)]^2}, \\ &= \frac{\sigma(x_t)}{\beta x_t^2 (1+x_t)^2 [1+\gamma(x_t+\mu_1)]^2}. \end{aligned}$$

where

$$\sigma(x_t) \equiv \Gamma(1+x_t)^2 [\gamma^2(1-g)x_t^2 - \lambda\{1+\gamma(2x_t+\mu_1)\}] + \bar{\alpha}(1-\gamma)\mu_1 x_t^2 [1+\gamma(1+2x_t+\mu_1)].$$

We then differentiate $\sigma(x_t)$ with respect to x_t as follows:

$$\sigma'(x_t) = \mu_2 x_t^3 + \mu_3 x_t^2 + \mu_4 x_t + \mu_5,$$

where

$$\begin{aligned} \mu_2 &\equiv 4\Gamma\gamma^2(1-g) + 2\gamma\bar{\alpha}(1-\gamma)\mu_1 > 0, \\ \mu_3 &\equiv 6\Gamma\gamma\{\gamma(1-g) - \lambda\} + 2\bar{\alpha}(1-\gamma)\mu_1(1+2\gamma+\gamma\mu_1), \\ \mu_4 &\equiv 2\Gamma\gamma^2(1-g) + 2\gamma\bar{\alpha}(1-\gamma)\mu_1 - 8\lambda\Gamma\gamma - 2\lambda\Gamma(1+\gamma\mu_1), \\ \mu_5 &\equiv -2\lambda\Gamma(1+\gamma+\gamma\mu_1) < 0. \end{aligned}$$

Assuming that $\gamma(1-g) > \lambda$, we obtain $\mu_4 > 0$. As a result, there is a unique $\bar{x} > 0$ that satisfies $\sigma'(\bar{x}) = 0$. $\sigma'(x_t) < 0$ holds when $0 < x_t < \bar{x}$ and $\sigma'(x_t) > 0$ holds when $x_t > \bar{x}$.

Moreover, since $\sigma(0) = -\lambda\Gamma(1 + \gamma\mu_1) < 0$ and $\lim_{x_t \rightarrow \infty} \sigma(x_t) = \infty$, there is a unique $\bar{x} > 0$ that satisfies $\sigma(\bar{x}) = 0$. $\sigma(x_t) < 0$ holds when $0 < x_t < \bar{x}$ and $\sigma(x_t) > 0$ holds when $x_t > \bar{x}$. That is, we obtain

$$\begin{aligned}\zeta'(x_t) &< 0 & \text{if } 0 < x_t < \bar{x}, \\ \zeta'(x_t) &> 0 & \text{if } x_t > \bar{x}.\end{aligned}$$

In addition, we obtain the following results:

$$\begin{aligned}\lim_{x_t \rightarrow 0} \zeta(x_t) &= \infty, \\ \lim_{x_t \rightarrow \infty} \zeta(x_t) &= \lim_{x_t \rightarrow \infty} \left[\frac{\left(\frac{1}{x_t} + \gamma\right) \left(\frac{1}{x_t} + 1\right) \left(1 + \frac{\lambda\Gamma}{x_t}\right) - \frac{\bar{\alpha}(1-\gamma)\mu_1}{x_t^2}}{\beta \left(\frac{1}{x_t} + 1\right) \left(\frac{1+\gamma\mu_1}{x_t} + \gamma\right)} - \alpha_P \right] = \frac{1}{\beta} - \alpha_P.\end{aligned}$$

By using these results, we obtain a curve, $x_{t+1} = x_t$ locus, which is U-shaped and has asymptotes $\varphi_{t-1} = \frac{1}{\alpha_R - \alpha_P} (\beta^{-1} - \alpha_P) > 1$ when $x_t \rightarrow \infty$ and $\varphi_{t-1} \rightarrow \infty$ when $x_t \rightarrow 0$. $x_{t+1} = x_t$ locus is depicted in Figure 1.

C Properties of $K_{t+1}/K_t = 0$ locus

Differentiating $\Lambda(x_t)$ with respect to x_t , we obtain

$$\begin{aligned}\Lambda'(x_t) &= \frac{\left[(1 + \gamma x_t) \left(1 + \frac{\lambda\Gamma}{x_t}\right) + \gamma x_t \left(1 + \frac{\lambda\Gamma}{x_t}\right) - (1 + \gamma x_t) \frac{\lambda\Gamma}{x_t} \right]}{\beta(1 + x_t)[1 + \gamma(x_t + \mu_1)]} \\ &\quad - \frac{\left[x_t(1 + \gamma x_t) \left(1 + \frac{\lambda\Gamma}{x_t}\right) - \bar{\alpha}(1 - \gamma)\mu_1 \right] [1 + \gamma(1 + 2x_t + \mu_1)]}{\beta(1 + x_t)^2 [1 + \gamma(x_t + \mu_1)]^2}, \\ &= \frac{(1 + 2\gamma x_t + \gamma\lambda\Gamma)(1 + x_t)[1 + \gamma(x_t + \mu_1)] - (1 + \gamma x_t)(x_t + \lambda\Gamma)[1 + \gamma(1 + 2x_t + \mu_1)]}{\beta(1 + x_t)^2 [1 + \gamma(x_t + \mu_1)]^2} \\ &\quad + \frac{\bar{\alpha}(1 - \gamma)\mu_1 [1 + \gamma(1 + 2x_t + \mu_1)]}{\beta(1 + x_t)^2 [1 + \gamma(x_t + \mu_1)]^2}, \\ &= \frac{(1 - \lambda\Gamma)(1 + \gamma x_t)^2 + \gamma\mu_1(\gamma x_t^2 + 2\gamma x_t + 1 - \lambda\Gamma + \gamma\lambda\Gamma) + \bar{\alpha}(1 - \gamma)\mu_1 [1 + \gamma(1 + 2x_t + \mu_1)]}{\beta(1 + x_t)^2 [1 + \gamma(x_t + \mu_1)]^2}.\end{aligned}\tag{C.1}$$

Using $\mu_1 \equiv \Gamma(1 + \lambda - g)$, we can rewrite the numerator of (C.1) as

$$\nu_1 x_t^2 + \nu_2 x_t + \nu_3,$$

where

$$\nu_1 \equiv \gamma^2 [1 + (1 - g)\Gamma],$$

$$\nu_2 \equiv 2\gamma \{1 + [\gamma\lambda + (1 - g)\gamma - \lambda]\Gamma + (1 - \gamma)\bar{\alpha}\mu_1\},$$

$$\nu_3 \equiv (1 + \gamma\mu_1) \{1 + [\bar{\alpha}(1 - \gamma)(1 + \lambda - g) - \lambda]\Gamma\} + \gamma^2 \mu_1 \lambda \Gamma + \bar{\alpha}(1 - \gamma)\gamma\mu_1.$$

Assuming that $\bar{\alpha}(1 - \gamma)(1 + \lambda - g) > \lambda$ and using the assumption $\gamma(1 - g) > \lambda$ in Lemma 2, $\nu_1 > 0$, $\nu_2 > 0$, and $\nu_3 > 0$ hold, and hence, $\Lambda'(x_t) > 0$ holds for all $x_t \geq 0$. In addition, we obtain the following property for $\Lambda(x_t)$:

$$\lim_{x_t \rightarrow \infty} \Lambda(x_t) = \lim_{x_t \rightarrow \infty} \left[\frac{\left(\frac{1}{x_t} + \gamma\right) \left(1 + \frac{\lambda\Gamma}{x_t}\right) - \frac{\bar{\alpha}(1-\gamma)\mu_1}{x_t^2}}{\beta \left(\frac{1}{x_t} + 1\right) \left(\frac{1+\gamma\mu_1}{x_t} + \gamma\right)} - \alpha_P \right] = \frac{1}{\beta} - \alpha_P.$$

Thus, $K_{t+1}/K_t = 0$ locus is upward sloping if $\bar{\alpha}(1 - \gamma)(1 + \lambda - g) > \lambda$ and has an asymptote $\varphi_{t-1} = \frac{1}{\alpha_R - \alpha_P} (\beta^{-1} - \alpha_P) > 1$ when $x_t \rightarrow \infty$ on the (x_t, φ_{t-1}) plane.

D Phase diagram

First, we examine whether $\varphi_t > \varphi_{t-1}$ or $\varphi_t < \varphi_{t-1}$ at each point of the (x_t, φ_{t-1}) plane. By using (9), we obtain

$$\varphi_t \begin{matrix} \geq \\ \leq \end{matrix} \varphi_{t-1} \Leftrightarrow \varepsilon(x_t) \begin{matrix} \geq \\ \leq \end{matrix} \eta(\varphi_{t-1}).$$

Suppose that $(\underline{x}, \underline{\varphi})$ is a combination that satisfies (19); that is, $\varepsilon(\underline{x}) = \eta(\underline{\varphi})$ holds. Moreover, let us define \underline{x} by $x > \underline{x}$. Since $\varepsilon(x_t)$ is increasing in x_t , $\varepsilon(\underline{x}) < \eta(\underline{\varphi})$ holds. As a result, we obtain $\varphi_t < \varphi_{t-1}$ on the left of $\varphi_t = \varphi_{t-1}$ locus. Similarly, we obtain $\varphi_t > \varphi_{t-1}$ on the right of $\varphi_t = \varphi_{t-1}$ locus.

Next, we investigate whether $x_{t+1} > x_t$ or $x_{t+1} < x_t$ at each point of the (x_t, φ_{t-1})

plane. From (6), we obtain

$$x_{t+1} \begin{matrix} \geq \\ \leq \end{matrix} x_t \Leftrightarrow \varphi_{t-1} \begin{matrix} \leq \\ \geq \end{matrix} \zeta(x_t).$$

Therefore, we obtain $x_{t+1} > x_t$ below $x_{t+1} = x_t$ locus and $x_{t+1} < x_t$ above $x_{t+1} = x_t$ locus. By using these results, we can depict a phase diagram, as in Figure 1.

E Local stability around the steady states

Approximating (17) and (18) linearly around the steady state k ($k \in \{S, U\}$), we obtain

$$\begin{pmatrix} \varphi_t - \varphi_k^* \\ x_{t+1} - x_k^* \end{pmatrix} = \begin{pmatrix} J_{\varphi\varphi}^k & J_{\varphi x}^k \\ J_{xx}^k & J_{x\varphi}^k \end{pmatrix} \begin{pmatrix} \varphi_{t-1} - \varphi_k^* \\ x_t - x_k^* \end{pmatrix}, \quad (\text{E.1})$$

where

$$\begin{aligned} J_{\varphi\varphi}^k &= \frac{\varphi_k^* G_\varphi^R(x_k^*, \varphi_k^*) + G^R(x_k^*, \varphi_k^*) - \varphi_k^* G_\varphi^A(x_k^*, \varphi_k^*)}{G^A(x_k^*, \varphi_k^*)}, \\ J_{\varphi x}^k &= \frac{\varphi_k^* G_x^R(x_k^*, \varphi_k^*) - \varphi_k^* G_x^A(x_k^*, \varphi_k^*)}{G^A(x_k^*, \varphi_k^*)}, \\ J_{xx}^i &= \frac{G^A(x_k^*, \varphi_k^*)(1 - \lambda\Gamma) - (1 + x_k^*)(x_k^* + \lambda\Gamma)G_x^A(x_k^*, \varphi_k^*)}{[(1 + x_k^*)G^A(x_k^*, \varphi_k^*) - (x_k^* + \lambda\Gamma)]^2}, \\ J_{x\varphi}^k &= -\frac{(x_k^* + \lambda\Gamma)(1 + x_k^*)G_\varphi^A(x_k^*, \varphi_k^*)}{[(1 + x_k^*)G^A(x_k^*, \varphi_k^*) - (x_k^* + \lambda\Gamma)]^2}, \end{aligned}$$

where $G_z^R(x_k^*, \varphi_k^*) \equiv \frac{\partial G^R(x_t, \varphi_{t-1})}{\partial z} \Big|_{(x_t, \varphi_{t-1})=(x_k^*, \varphi_k^*)}$ and $G_z^A(x_k^*, \varphi_k^*) \equiv \frac{\partial G^A(x_t, \varphi_{t-1})}{\partial z} \Big|_{(x_t, \varphi_{t-1})=(x_k^*, \varphi_k^*)}$ for $z \in \{x_t, \varphi_{t-1}\}$. Note that $G^A(x_k^*, \varphi_k^*) = G^R(x_k^*, \varphi_k^*)$ holds. Let us denote the two eigenvalues of the Jacobian matrix of the linearized system as e_1^k and e_2^k . These eigenvalues are the roots of the characteristic polynomial: $P(e) = e^2 - (J_{\varphi\varphi}^k + J_{xx}^k)e + (J_{xx}^k J_{\varphi\varphi}^k - J_{\varphi x}^k J_{x\varphi}^k)$. To consider this, we conduct a numerical analysis. We adopt the following benchmark parameters: $\gamma = 0.2$, $\Gamma = 12$, $g = 0.2$, $\delta = 0.5$, $\beta = 0.3$, $\alpha_R = 0.45$, and $\alpha_P = 0.25$. Table 3 shows the two eigenvalues for each steady state.

[Table 3]

From Table 3, we find that both e_1^S and e_2^S take real positive values and satisfy $0 < e_1^S < e_2^S < 1$, and then the steady state S is a sink. We further find that both e_1^U and e_2^U

take real positive values and satisfy $0 < e_1^U < 1 < e_2^U$, and then the steady state U is a saddle point.

F Introduction of a redistributive public policy

Under the redistributive public policy, we rewrite the following optimal conditions of individuals:

$$b_{t+1}^R = \frac{\beta [1 + (1 - \tau_{t+1})r_{t+1}] s_t^R}{1 + \tau^b}, \quad (\text{F.1})$$

$$c_t^{1P} = (1 - \alpha_P) [(1 - \tau_t)w_t + b_t^P + T_t], \quad (\text{F.2})$$

$$s_t^P = \alpha_P [(1 - \tau_t)w_t + b_t^P + T_t], \quad (\text{F.3})$$

Other optimal conditions are the same as (4a)-(4d). In addition, (5) becomes

$$s_t^R = \alpha_R \left[(1 - \tau_t)w_t + \frac{\beta \{1 + (1 - \tau_t)r_t\}}{1 + \tau^b} s_{t-1}^R \right], \quad (\text{F.4})$$

$$s_t^P = \alpha_P [(1 - \tau_t)w_t + T_t + \beta \{1 + (1 - \tau_t)r_t\} s_{t-1}^P]. \quad (\text{F.5})$$

Substituting (F.1) into the government redistributive policy $\delta N \tau^b b_t^R = (1 - \delta) N T_t$, we obtain

$$T_t = \left(\frac{\delta}{1 - \delta} \right) \left(\frac{\tau^b}{1 + \tau^b} \right) \beta [1 + (1 - \tau_t)r_t] s_{t-1}^R. \quad (\text{F.6})$$

The asset market-clearing condition (6), equations determined in the production sector ((7a), (7b) and (7c)), and those in the public sector ((8), (9) and (10)) remain unchanged. Furthermore, derivations of the other equations are conducted in the same manner as Section 3. Some algebra rewrite $G^A(x_t, \varphi_{t-1})$ (in (13), (16), (17), and (18)), $G^K(x_t, \varphi_{t-1})$ (in (16) and (18)), and $G^R(x_t, \varphi_{t-1})$ (in (14) and (17)) into

$$G^A(x_t, \varphi_{t-1}; \tau^b) = \frac{\bar{\alpha}(1 - \gamma)\mu_1}{(1 + \gamma x_t)(1 + x_t)} + \beta \left(1 + \frac{\gamma\mu_1}{1 + \gamma x_t} \right) \left[\frac{\alpha_R - \alpha_P}{1 + \tau^b} \varphi_{t-1} + \alpha_P \right], \quad (\text{F.7})$$

$$G^K(x_t, \varphi_{t-1}; \tau^b) = (1 + x_t)G^A(x_t, \varphi_{t-1}; \tau^b) - (x_t + \lambda\Gamma), \quad (\text{F.8})$$

and

$$G^R(x_t, \varphi_{t-1}; \tau^b) = \frac{\alpha_R}{\varphi_{t-1}} \left[\frac{\delta(1-\gamma)\mu_1}{(1+\gamma x_t)(1+x_t)} + \frac{\beta}{1+\tau^b} \left(1 + \frac{\gamma\mu_1}{1+\gamma x_t} \right) \varphi_{t-1} \right], \quad (\text{F.9})$$

respectively. Translating these into (17), and (18) transforms the dynamic systems as

$$\begin{aligned} \frac{\varphi_t}{\varphi_{t-1}} &= \frac{G^R(x_t, \varphi_{t-1}; \tau^b)}{G^A(x_t, \varphi_{t-1}; \tau^b)} \\ \frac{x_{t+1}}{x_t} &= \frac{1 + \lambda\Gamma/x_t}{(1+x_t)G^A(x_t, \varphi_{t-1}; \tau^b) - (x_t + \lambda\Gamma)}. \end{aligned} \quad (\text{F.10})$$

Setting $\varphi_t = \varphi_{t-1}$ and $x_t = x_{t+1}$ in (F.10) yields (27) and (28).

G $\varphi_t = \varphi_{t-1}$ locus under the redistributive public policy

We rearrange (27) as follows:

$$\tilde{\eta}(\varphi_{t-1}) = \frac{(1+\tau^b)(1-\gamma)}{[\alpha_R - (1+\tau^b)\alpha_P] - (\alpha_R - \alpha_P)\varphi_{t-1}} \left(\bar{\alpha} - \frac{\delta\alpha_R}{\varphi_{t-1}} \right).$$

Here, we assume $\alpha_R > (1+\tau^b)\alpha_P$. Under this assumption, when φ_{t-1} increases, the function $\tilde{\eta}(\varphi_{t-1})$ increases. Similarly to the analysis in Appendix A, $\varphi_t = \varphi_{t-1}$ locus can be depicted as an upward-sloping curve on the (x_t, φ_{t-1}) plane. Moreover, we define $\hat{\varphi}$ and $\tilde{\varphi}(\tau^b)$ as $\hat{\varphi} \equiv \frac{1+\tau^b}{\alpha_R - \alpha_P} \left(\frac{\alpha_R}{1+\tau^b} - \alpha_P \right) < 1$ and $\tilde{\varphi}(\tau^b) \equiv \tilde{\eta}^{-1}(\varepsilon(0))$, respectively. Since $\lim_{\varphi_{t-1} \rightarrow \hat{\varphi}} \tilde{\eta}(\varphi_{t-1}) = +\infty$ and $\lim_{\varphi_{t-1} \rightarrow 0} \tilde{\eta}(\varphi_{t-1}) = -\infty$, $\varphi_t = \varphi_{t-1}$ locus has an asymptote $\varphi_{t-1} = \hat{\varphi}$ when $x_t \rightarrow \infty$ and φ_{t-1} has a lower limit $\tilde{\varphi}(\tau^b)$ when $x_t = 0$.¹⁶

¹⁶Taking the total differentials of (27) yields

$$\left. \frac{d\varphi_{t-1}}{dx_t} \right|_{\varphi_t = \varphi_{t-1}} = \frac{\beta \left[\left(\frac{\alpha_R}{1+\tau^b} - \alpha_P \right) (1 - \varphi_{t-1}) - \frac{\alpha_P \tau^b}{1+\tau^b} \varphi_{t-1} \right] [\mu_1^{-1}(1 + \gamma + 2\gamma x_t) + \gamma]}{(1-\gamma)\delta\alpha_R\varphi_{t-1}^{-2} + \beta \left(\frac{\alpha_R}{1+\tau^b} - \alpha_P + \frac{\alpha_P \tau^b}{1+\tau^b} \right) (1+x_t) [\mu_1^{-1}(1 + \gamma x_t) + \gamma]}.$$

From the definition of $\hat{\varphi}$, $\left. \frac{d\varphi_{t-1}}{dx_t} \right|_{\varphi_t = \varphi_{t-1}} > 0$ holds when $0 < \varphi_{t-1} < \hat{\varphi}$. Therefore, we can also observe that $\varphi_t = \varphi_{t-1}$ locus is an upward-sloping curve and has an asymptote $\varphi_{t-1} = \hat{\varphi}$ on the (x_t, φ_{t-1}) plane.

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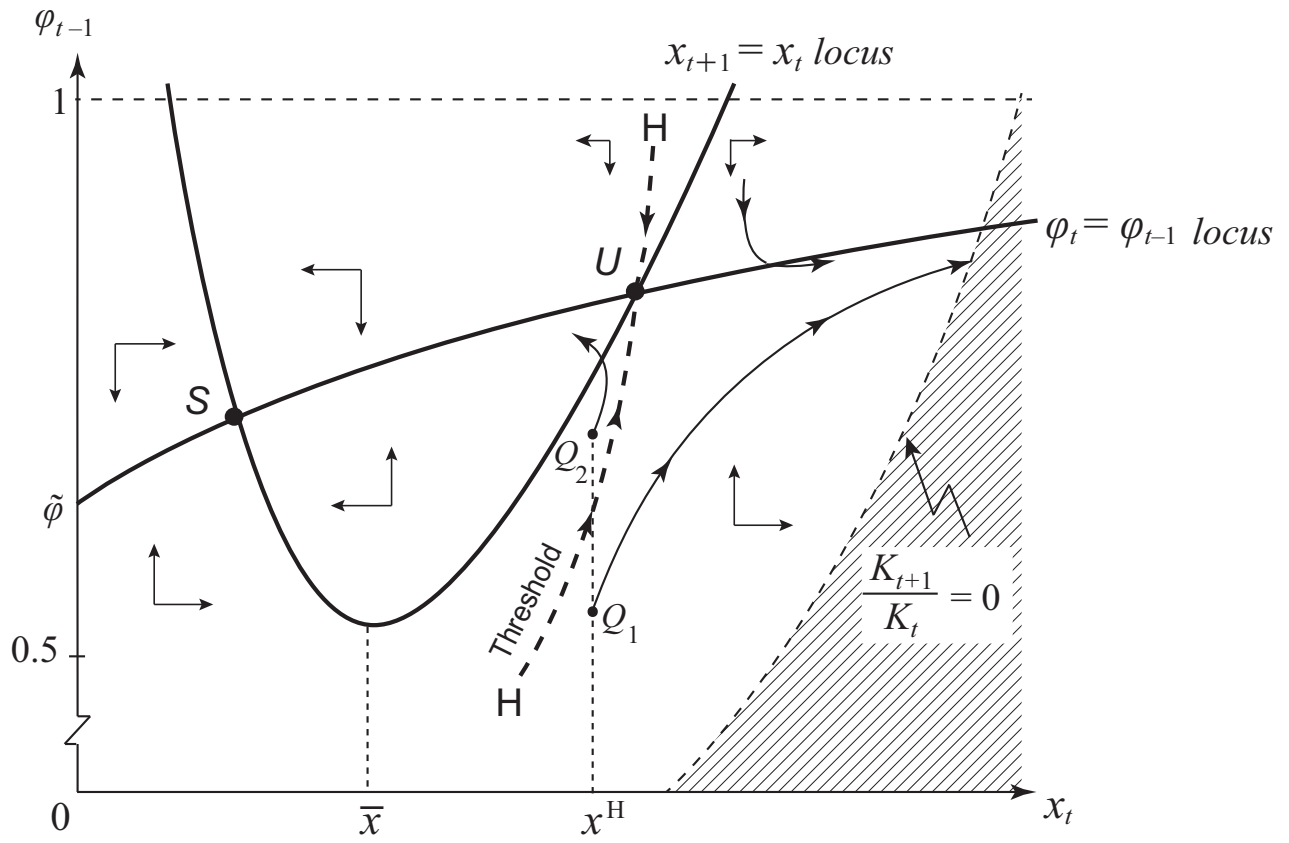


Figure 1: Phase Diagram on the (x_t, φ_{t-1}) plane.

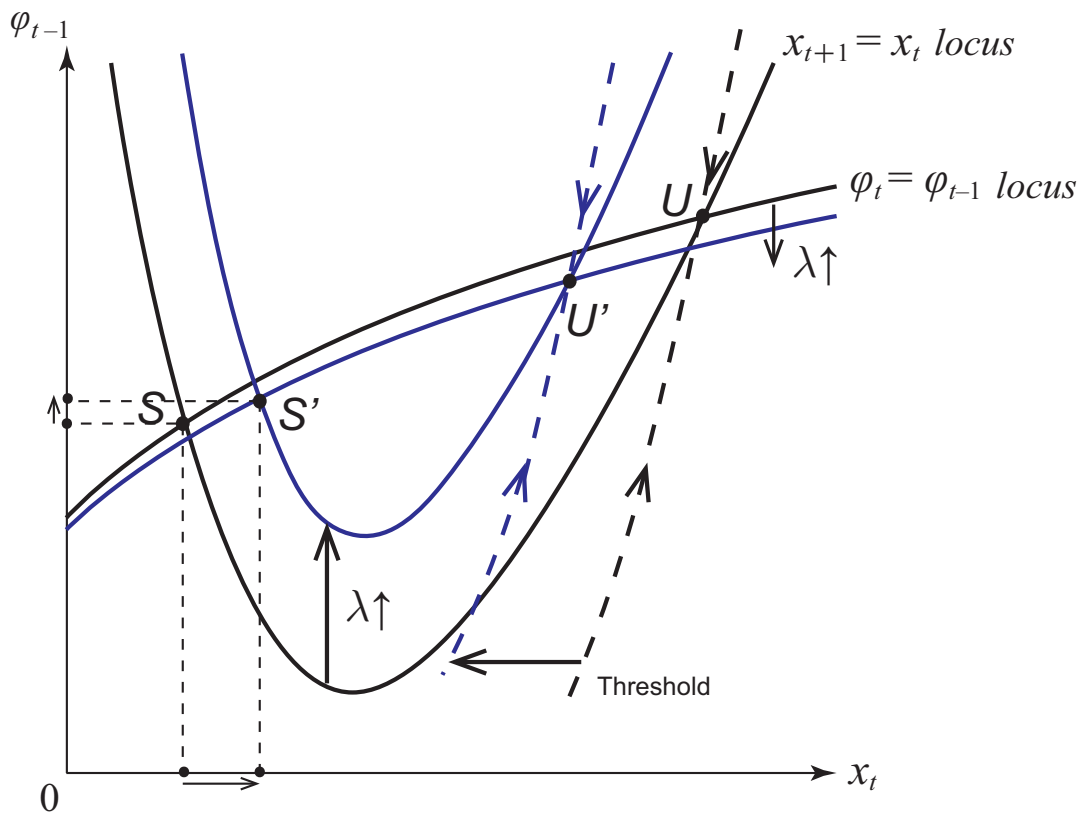


Figure 2: Effects of an increase in λ .

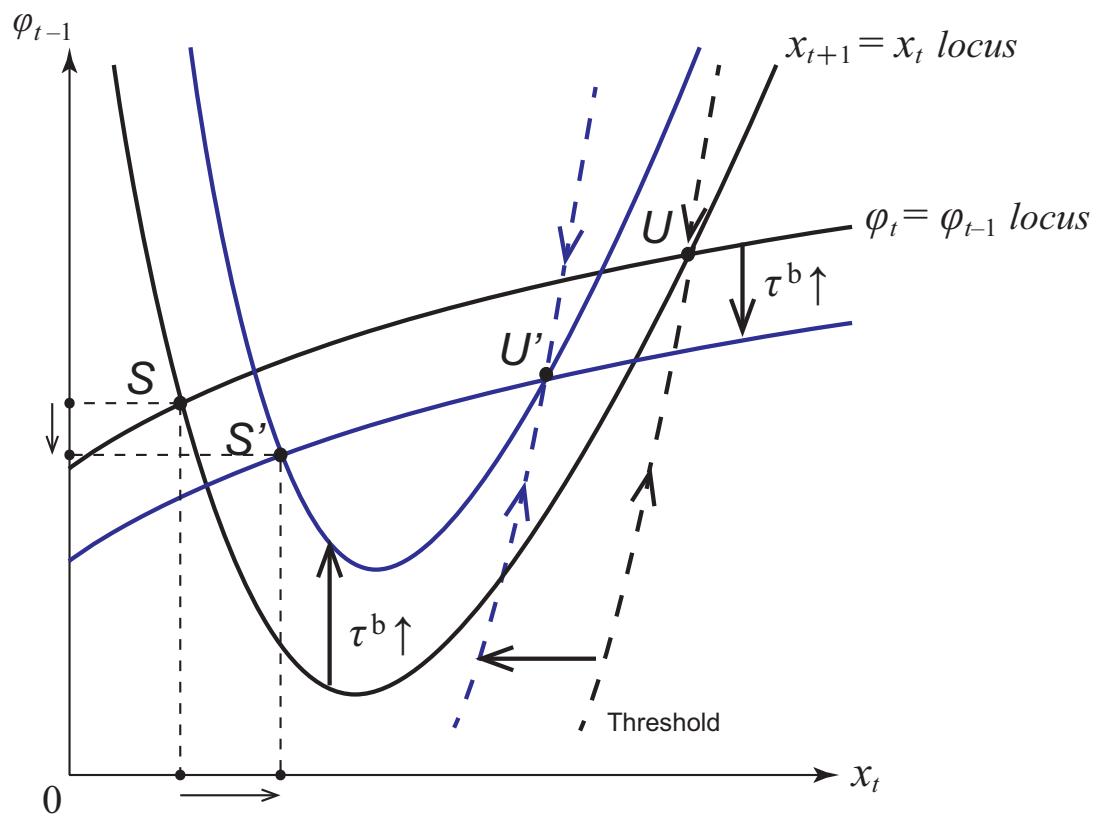


Figure 3: Effects of an increase in τ^b .

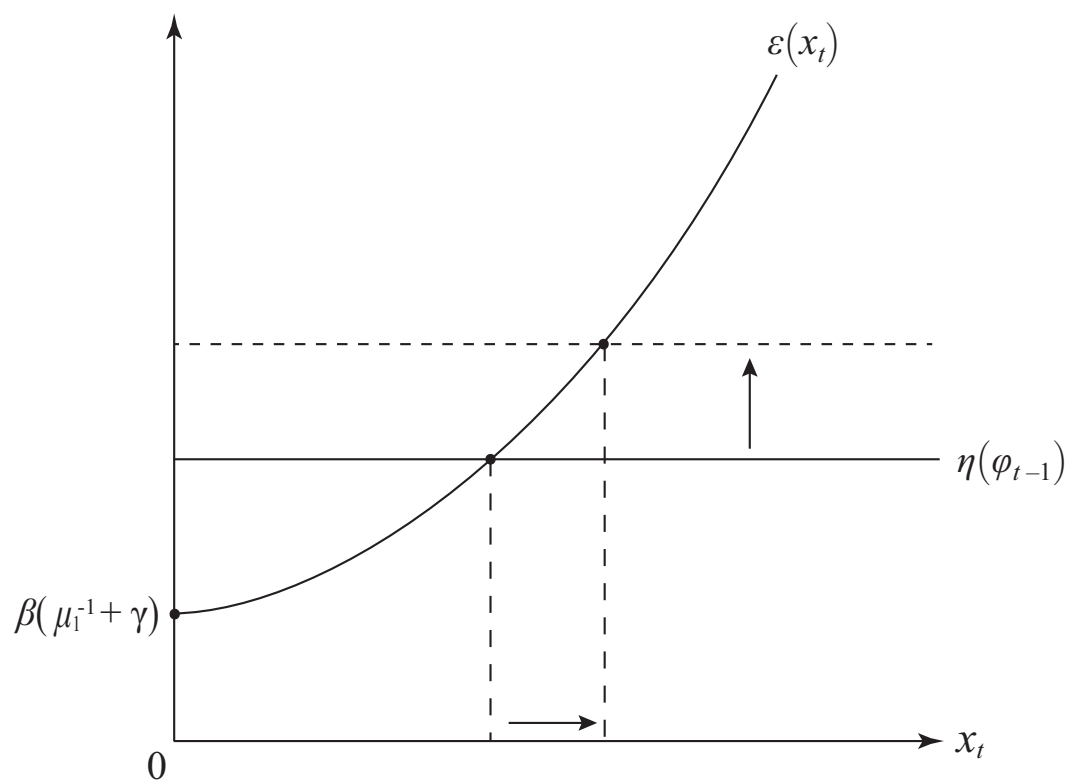


Figure 4: Derivation of $x_{t+1} = x_t$ locus.

λ	x_S^*	φ_S^*	\hat{Y}_S^*	x_U^*	φ_U^*	\hat{Y}_U^*
0.01	0.0646	0.6585	2.8582	1.6121	0.6835	1.0744
0.02	0.1431	0.6596	2.6770	1.4470	0.6806	1.1659
0.03	0.2454	0.6611	2.4669	1.2588	0.6773	1.2860
0.04	0.4033	0.6635	2.1901	1.0156	0.6732	1.4726

Table 1: Deficit ratio and steady state values.

λ	Δx_S^*	$\Delta \varphi_S^*$	$\Delta \hat{Y}_S^*$	Δx_U^*	$\Delta \varphi_U^*$	$\Delta \hat{Y}_U^*$
0.01	1.76%	-2.46%	-1.12%	-4.28%	-6.17%	0.31%
0.02	2.27%	-2.64%	-1.39%	-4.75%	-5.75%	0.71%
0.03	3.25%	-2.87%	-1.87%	-5.65%	-5.27%	1.33%
0.04	6.61%	-3.23%	-3.37%	-8.62%	-4.67%	3.03%

Table 2: Effects of an increase in bequest tax rate.

Note: The changes in the steady state values are expressed in percentage points.

λ	(e_1^S, e_2^S)	(e_1^U, e_2^U)
0.01	(0.098, 0.378)	(0.219, 2.566)
0.02	(0.104, 0.439)	(0.207, 2.227)
0.03	(0.113, 0.527)	(0.193, 1.874)
0.04	(0.126, 0.678)	(0.174, 1.467)

Table 3: Eigenvalues at the steady states S and U .