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Fiscal sustainability under physical and human capital accumulation in an overlapping generations model *

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Abstract

We consider fiscal sustainability by using an overlapping generations model with human capital accumulation (private and public education) and public debt. Based on this model, we explicitly show (i) the parameter region in which the economy cannot be fiscally sustainable for any initial endowment, and (ii) the threshold of initial endowment over (under) which the economy diverges (converges) to the steady state. Importantly, the threshold is neutral to the level of initial human capital. Further, we show the existence and uniqueness of the growth-maximizing level of each policy variable (i.e., the tax rate and public education/production ratio).

Keywords: Human capital accumulation, Public education, Public debt, Fiscal sustainability.
JEL Classification: E62; H52; H63; I28

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1 Introduction

According to data from the International Monetary Fund,[1] the public debt/GDP ratios in 2015 varied across countries; for example, arranged in descending order, they were Japan (248%), Greece (177%), the United States (105%), United Kingdom (89%), Germany (71%), China (43%), and so on. Then, the following question can be asked. How much of the primary deficit can the government control? Or, in other words, how much public debt can the government issue? Such a problem is referred to as that of "fiscal sustainability."

As Tirole (1985) implied,[2][3] it has been known that in the overlapping generations (OLG) model developed by Diamond (1965) without any policy alternatives, the economy can be fiscally sustainable if the long-run growth rate without public debt is higher than the interest rate and is equivalent to the condition of dynamic inefficiency in his model, which we call the "debt-existing condition of Tirole" hereafter. Chalk (2000) showed that with a positive primary deficit in the OLG model, the economy can be fiscally sustainable when the primary deficit is sufficiently low in addition to dynamic inefficiency. In terms of endogenous growth, Bräuninger (2005) used the OLG model with *AK* production, as well as the constant government spending/GDP and deficit/GDP ratios, and showed that the economy becomes fiscally unsustainable when the deficit/GDP ratio is high. Arai (2011) considered fiscal sustainability with the OLG model in which the government invests into flow-government productive spending, following Barro (1990), thus resulting in the *AK* model. In his model, it was shown that the economy is fiscally sustainable when the government spending/production ratio is moderate. Further, Yakita (2008) investigated fiscal sustainability with public capital, such as did Futagami et al. (1993) with the constant budget deficit/production ratio in the OLG model, and showed that the threshold of initial public debt for the economy to be fiscally sustainable is increasing in initial public capital.

On the other hand, since the advent of the papers written by Uzawa (1965) and Lucas (1988), it has

[1]Source: International Monetary Fund, World Economic Outlook Database, October 2016 at https://www.imf.org/external/pubs/ft/weo/2016/02/weodata/index.aspx. The data in the text are equivalent to those of "General government gross debt and Percent of GDP" in the database.

[2]In fact, he considered the fiscal sustainability of asset bubbles instead of public debt by introducing useless assets in the canonical OLG model. However, the result can be applied to the case of public debt by regarding the control variable (useless asset) as the state variable (public debt).

[3]Grossman and Yanagawa (1993) showed the same debt-existing condition as that of Tirole (1985) by using the OLG model with *AK* structure.

been considered that both physical and human capital accumulation are crucial for the growth engine; hence, along the lines of the previous literature, it is important to consider fiscal sustainability by using a two-sector endogenous growth model in which both physical and human capital accumulation exist, that is, the Uzawa-Lucas model. By using this model, we can analyze the effect of human capital on fiscal sustainability. Further, in addition to private educational expenditure, it is important to consider the effect of public educational expenditure on economic performance, as studied in the previous literature, such as Glomm and Ravikumar (1992), Blankenau and Simpson (2004), Blankenau et al. (2007), and Agénor (2011). In fact, according to the report from the Organisation for Economic Co-operation and Development (OECD) (2016), the OECD-average public education/GDP ratio in 2013 was 4.8%, which is not a negligible value.[4] Since the government must prepare funds to finance public educational expenditure, it should affect fiscal sustainability and, by using our model, we can analyze the level of public education under which the economy can be fiscally sustainable besides the tax rate.

Therefore, we have extended the model developed by Boldrin and Montes (2005) by incorporating public education and public debt. Based on this model, we obtain following three results. Firstly, we explicitly show the parameter region in which the economy is fiscally unsustainable for any initial endowment. The condition is equivalent to that in which the economy has multiple steady states and, in line with the previous literature, an additional condition, besides the debt-existing condition of Tirole, is needed to obtain a positive steady-state value of public debt. Secondly, we explicitly obtain the threshold level of initial public debt over (under) which the economy diverges (converges) to the steady state. Counterintuitive as it may be, the threshold is neutral to the initial human capital; that is, whether the economy initially has high education or not has nothing to do with fiscal sustainability. Thirdly, we consider the effect of a marginal increase in the tax rate and public education/production ratio on the long-run growth rate. When the tax rate increases, private education decreases, since the future income decreases, which is a marginal cost of an increase in the tax rate; primary deficit improves; public debt decreases; and physical capital increases, which is a marginal benefit. When the public education/production ratio increases, human capital directly increases, which is a marginal

[4]See Table B4.2. on page 231 in OECD (2016). The public education/GDP ratio by country ranges from 3.3% in Hungary to 7.3% in Norway.

benefit, while the primary deficit worsens, public debt increases, and physical capital is crowded out, which is a marginal cost. Taking these into account, we can show the existence and uniqueness of the growth-maximizing tax rate and public education/production ratio, respectively.

Some literature exists that simultaneously treats public education and public debt in the growth model. Zhang (2003) considered public debt and subsidy for education with a constant public debt/production ratio in the OLG model with altruism, and obtained the optimal policy and indirect utility functions. Greiner (2008) considered the relationship between economic growth and public debt with public education in the Ramsey model, and showed the steady states and their stability through a calibration. Gottardi et al. (2015) considered the Ramsey model in which uninsurable idiosyncratic shocks to human capital exist and the government can issue public debt, showing the existence of a positive long-run, welfare-maximizing capital tax rate. Further, Ono and Uchida (2016) simultaneously considered public education and public debt in the OLG model; they compared the welfare of the only-tax regime as well as both tax and public debt regimes in terms of competitive and political (middle-aged, welfare-maximizing policy) equilibriums. Nevertheless, these studies did not focus on the fiscal sustainability problem, and, to my best knowledge, this is the first study that considers fiscal sustainability under the model with both public debt and human capital accumulation.

The remainder of this paper is organized as follows. In Section 2, we describe the behavior of a household, firm, and government, and obtain simultaneous difference equations that describe the model's equilibrium. In Section 3.1, we discuss the condition of fiscal sustainability; in Section 3.2, we conduct comparative statics of the stable steady-state value with respect to each policy variable; and in Section 3.3, the existence and uniqueness of the long-run, growth-maximizing tax rate and public education/production ratio are shown. Finally, Section 4 summarizes the study.

2 Model

2.1 Model setting

This model is a two-sector growth model consisting of final goods and human capital production. We consider the three-period OLG model, in which three types of households coexist in each period (i.e., the young, middle-aged, and old).

Firstly, we consider the representative firm's profit-maximizing problem. This firm produces final goods by using physical and human capital, (k_t, h_t) with Cobb-Douglas production technology:

$$Y_t = F(k_t, h_t) = Ak_t^{\alpha} h_t^{1-\alpha}, \quad A > 0, \quad \alpha \in (0, 1).$$

We assume that physical capital is fully depreciated after use;[5] then, profit-maximizing conditions become:

$$\begin{aligned} 1 + r_t &= A\alpha k_t^{\alpha-1} h_t^{1-\alpha}, \\ w_t &= A(1-\alpha) k_t^{\alpha} h_t^{-\alpha}, \end{aligned} \tag{1}$$

where r_t denotes the interest rate and w_t denotes the wage rate per human capital.

Next, we turn to the household's utility-maximizing problem. A $t-1$ generation, born in period $t-1$, behaves as follows. In period $t-1$, s/he is young and borrows e_{t-1} from the middle-aged, and invests it into education. Then, s/he obtains human capital described as follows:[6]

$$h_t = Be_{t-1}^{\beta_1} E_{t-1}^{\beta_2} h_{t-1}^{1-\beta_1-\beta_2}, \quad B > 0, \; \beta_1, \beta_2, 1-\beta_1-\beta_2 \in (0, 1). \tag{2}$$

In (2), E_{t-1} denotes the level of public education in period $t-1$ and h_t is positively affected by the previous level of human capital h_{t-1}, which captures the externality of human capital. Further, β_1 (β_2) measures the productivity of private (public) educational expenditure. In period t, the $t-1$ generation becomes middle-aged and earns after-tax income $(1-\tau)w_t h_t$, where τ denotes the constant income tax rate. S/he distributes her/his after-tax income into middle-age consumption c_t^m, savings s_t, and repayment of educational expenses to the old $(1+r_t)e_{t-1}$. In period $t+1$, the $t-1$ generation becomes old and consumes all of its income $(1+r_{t+1})s_t$. The intertemporal utility function of the $t-1$ generation is given by $U_{t-1} = \ln c_t^m + \delta \ln c_{t+1}^o$, where $\delta \in (0, 1)$ is a discount factor and c_{t+1}^o is the consumption level at old age. The $t-1$ generation's budget constraints are $c_t^m + s_t + (1+r_t)e_{t-1} = (1-\tau)w_t h_t$ and $c_{t+1}^o = (1+r_{t+1})s_t$. Hence, the $t-1$ generation's utility-maximization problem is given as:

$$\max_{s_t, e_{t-1}} \ln\left[(1-\tau)w_t Be_{t-1}^{\beta_1} E_{t-1}^{\beta_2} h_{t-1}^{1-\beta_1-\beta_2} - s_t - (1+r_t)e_{t-1}\right] + \delta \ln\left[(1+r_{t+1})s_t\right].$$

[5] If the yearly depreciation rate is 7%, after one period, say 25 years, about 84% of physical capital is depreciated, which we can equate with full depreciation.

[6] The functional form (2) is assumed in Blankenau and Simpson (2004).

The utility-maximizing conditions are given by:

$$e_{t-1} = (1-\tau)\beta_1 \frac{w_t h_t}{1+r_t},$$
$$s_t = \frac{\delta}{1+\delta}\left[(1-\tau)w_t h_t - (1+r_t)e_{t-1}\right].$$

By substituting (1) into the above expressions, we obtain:

$$e_{t-1} = \gamma k_t, \tag{3}$$

$$s_t = \eta A k_t^{\alpha} h_t^{1-\alpha}, \tag{4}$$

where $\gamma \equiv \frac{(1-\tau)\beta_1(1-\alpha)}{\alpha}$ denotes the propensity of private education and $\eta \equiv \frac{\delta}{1+\delta}(1-\tau)(1-\beta_1)(1-\alpha) \in (0,1)$ denotes the savings rate.

Since the government can issue public debt, by defining b_t as the stock of public debt in period t, the government's budget constraint is given by:

$$\Delta b_{t+1} \equiv b_{t+1} - b_t = r_t b_t + E_t + G_t - \tau w_t h_t, \tag{5}$$

where G_t denotes unproductive government expenditure. Equation (5) states that the sum of repayment of an interest of public debt $r_t d_t$ and primary deficit $E_t + G_t - \tau w_t h_t$ is financed by issuing new public debt Δb_{t+1}. The government sets the level of public education E_t as a fraction $\theta \in (0,1)$ of production Y_t: [7][8]

$$E_t = \theta Y_t. \tag{6}$$

Similarly, the unproductive government expenditure G_t is set as a fraction $d \in (0,1)$ of production Y_t, as assumed in Chalk (2000) and Bräuninger (2005), that is, $G_t = dY_t$.

Finally, savings are used as physical capital, public debt, and lending to the young generation; thereby, the capital-market clearing condition becomes:

$$s_t = e_t + b_{t+1} + k_{t+1}. \tag{7}$$

[7] In reality, θ in 2013 was 4.8% (OECD average) according to the OECD (2016). Further, that in 2005 was 4.7%, and the ratio was relatively stable.

[8] Instead of (6), even if the government sets the primary deficit as a fraction θ of production (i.e., $E_t + G_t - \tau w_t h_t = \theta Y_t$), the dynamics are almost the same as this model, since public education eventually becomes a constant fraction of production under the constant tax rate and $G_t = dY_t$.

2.2 Model solution

Substituting $k_t = \frac{1}{\gamma}e_{t-1}$ from (3) into (7) yields $e_{t-1} = \frac{\gamma}{1+\gamma}(s_{t-1} - b_t)$; again, by substituting this and (4) into (7), we obtain the dynamics of k_t, such as:

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+\gamma}(s_t - b_{t+1}) = \frac{1}{1+\gamma}\left(\eta A k_t^{\alpha} h_t^{1-\alpha} - b_{t+1}\right), \\
\Leftrightarrow \quad \frac{k_{t+1}}{k_t} &= \frac{1}{1+\gamma}\left(\eta A x_t^{\alpha-1} - y_t \frac{b_{t+1}}{b_t}\right),
\end{aligned}
\tag{8}
$$

where $x_t \equiv \frac{k_t}{h_t}$ denotes the physical/human capital ratio and $y_t \equiv \frac{b_t}{k_t}$ denotes the public debt/physical capital ratio. By combining (2) with (6) and (3), the dynamics of h_t becomes:

$$
\begin{aligned}
h_{t+1} &= B(\gamma k_{t+1})^{\beta_1}(\theta Y_t)^{\beta_2} h_t^{1-\beta_1-\beta_2} = \psi \left(\frac{k_{t+1}}{k_t}\right)^{\beta_1} k_t^{\beta_1+\alpha\beta_2} h_t^{1-\beta_1-\alpha\beta_2}, \\
\Leftrightarrow \quad \frac{h_{t+1}}{h_t} &= \psi \left(\frac{k_{t+1}}{k_t}\right)^{\beta_1} x_t^{\beta_1+\alpha\beta_2},
\end{aligned}
\tag{9}
$$

where $\psi \equiv B\gamma^{\beta_1}\theta^{\beta_2}A^{\beta_2}$ denotes a scale parameter of the long-run growth rate. Note that ψ is increasing in θ because the growth rate of human capital directly increases when a higher level of public education is introduced, and that it is decreasing in τ, since a higher tax rate decreases the return of private education; in turn, this decreases the propensity of private education γ. Finally, the dynamics of b_t are derived by substituting (1) and (6) into (5) as follows:

$$
\begin{aligned}
b_{t+1} &= (1 + r_t)b_t + (\theta + d)Y_t - \tau w_t h_t = [\alpha b_t + (\theta + d - \tau(1-\alpha))k_t]A x_t^{\alpha-1}, \\
\Leftrightarrow \quad \frac{b_{t+1}}{b_t} &= \left(\alpha + \frac{\theta + d - \tau(1-\alpha)}{y_t}\right) A x_t^{\alpha-1} \equiv \left(\alpha + \frac{D}{y_t}\right) A x_t^{\alpha-1},
\end{aligned}
\tag{10}
$$

where $D \equiv \theta + d - \tau(1-\alpha)$ is a primary deficit/production ratio (i.e., $D_t \equiv \frac{E_t + G_t - \tau w_t h_t}{Y_t} = D$), and we assume $D > 0$, which means that there is a positive primary deficit. Then, by combining (8), (9), and (10), this economic system is characterized by the following two difference equations, with respect to (x_t, y_t):

$$
x_{t+1} = \phi(\eta - \alpha y_t - D)^{1-\beta_1} x_t^{\alpha\beta_3}, \tag{11}
$$

$$
y_{t+1} = (1+\gamma)\frac{\alpha y_t + D}{\eta - \alpha y_t - D} \equiv \Omega(y_t), \tag{12}
$$

where $\phi \equiv \frac{1}{\psi}\left(\frac{A}{1+\gamma}\right)^{1-\beta_1}$ and $\beta_3 \equiv 1-\beta_1-\beta_2$. Importantly, note that y_{t+1} depends only on y_t.[9] From (11), since x_t must be a real number, y_t must satisfy $\eta > \alpha y_t + D$ in order for the economy to be fiscally sustainable. In other words, if $y_t > \overline{y}$, where $\eta = \alpha\overline{y} + D$, the economy is fiscally unsustainable.

The debt-existing condition of Tirole is that the long-run growth rate without public debt is higher than the interest rate. In our model, such a parameter region is given by:[10]

$$\eta - D > (1+\gamma)\alpha, \tag{13}$$

which is assumed hereafter. Note that the condition (13) implies $\eta - D > 0$; in turn, it also implies $\overline{y} > 0$.

In what follows, we consider the phase diagram of (x_t, y_t). From (11), the $\Delta x_{t+1} \geq 0$ region is given by:

$$x_t \leq \phi^{\frac{1}{1-\alpha\beta_3}}(\eta - \alpha y_t - D)^{\frac{1-\beta_1}{1-\alpha\beta_3}}. \tag{14}$$

Moreover, from (12), by focusing on the region $y_t < \overline{y}$, the $\Delta y_{t+1} \geq 0$ region is given by:

$$\Omega(y_t) \geq y_t, \;\; \Leftrightarrow \;\; \alpha y_t^2 - [\eta - D - (1+\gamma)\alpha]y_t + (1+\gamma)D \equiv F(y_t) \geq 0. \tag{15}$$

Further the quadratic equation, $F(y) = 0$, has two distinct real roots if the following holds:

$$[\eta - D - (1+\gamma)\alpha]^2 \equiv A_1^2 > 4\alpha(1+\gamma)D \equiv A_2. \tag{16}$$

Note that from (13), A_1 is positive; thus, both roots are positive if both $D > 0$ and (16) are satisfied. Then, the roots of y are obtained as $y = \frac{1}{2\alpha}\left[A_1 \pm \sqrt{A_1^2 - A_2}\right]$. By defining y_H and y_L as the roots of $F(y) = 0$, satisfying $y_H > y_L$, the $\Delta y_{t+1} \geq 0$ region is given by:

$$y_t \leq y_L, \;\; \text{and} \;\; y_t \geq y_H. \tag{17}$$

Note that y_H is smaller than $\overline{y}$, and the economy is fiscally sustainable under both steady states, since the difference equation of y_t, $\Omega(y)$, goes to infinity as $y_t \to \overline{y}$ (See Figure 3 in Appendix A2).

Now that we can draw the phase diagram in (x_t, y_t) space, as in Figure 1, by using (14) and (17). According to Figure 1, the steady state (x_H, y_L) is locally stable, while the steady state (x_L, y_H) is a saddle point. In summary, we can obtain the following proposition:

[9] In Section 3.1, we consider the reason in detail.

[10] See Appendix A1 for deriving (13).

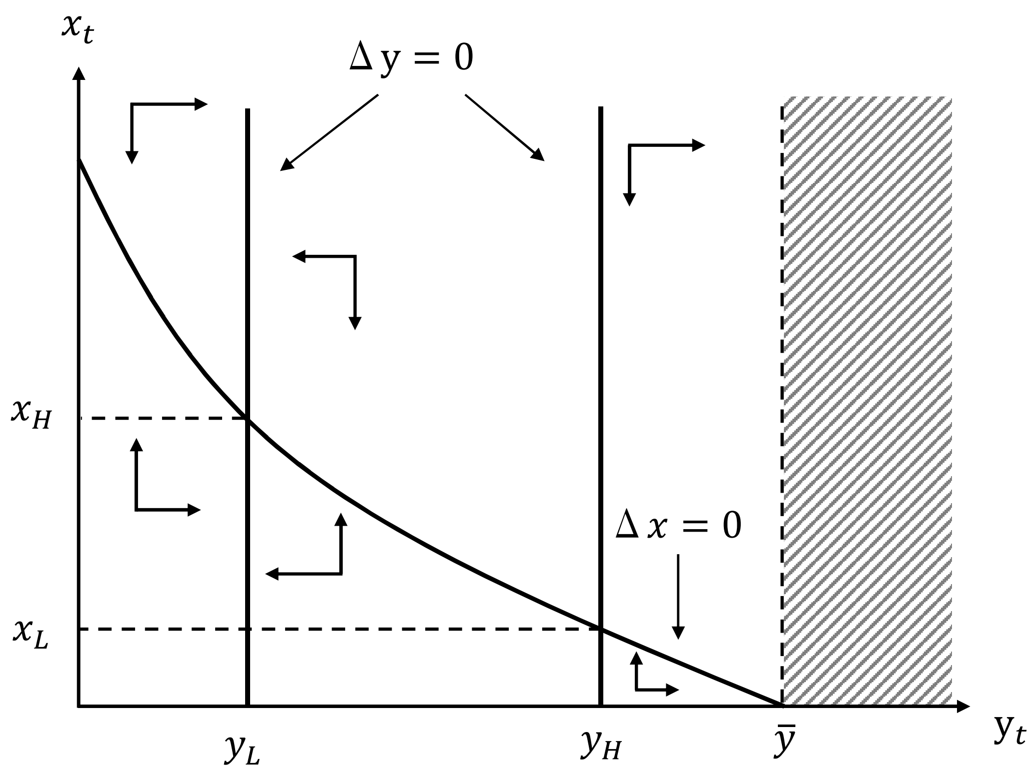

Figure 1: Phase diagram under (16)

Notes: The $\Delta x = 0$ ($\Delta y = 0$) locus is Equation (14) ((17)) with equality. In the shaded area, $y_t > \bar{y}$, and the economy is fiscally unsustainable. The steady state (x_H, y_L) is locally stable and the economy with $y_0 < y_H$ converges there. The steady state (x_H, y_L) is a saddle point and the saddle path is given by $y_t = y_H$.

Proposition 1 *Under (13) and the positive primary deficit $D > 0$:*

- *if (16) holds, the dynamic system (11) and (12) has two steady states, (x_H, y_L) and (x_L, y_H), where $x_H > x_L$ and $\bar{y} > y_H \geq y_L > 0$, respectively; the former is locally stable, while the latter is a saddle point, and*
- *if (16) is violated, there are no steady states.*

Proof. See Appendix A2 for the stability analysis. ■

Finally, the long-run growth rate, which is equivalent to that of human capital, $\lim_{t\to\infty} \frac{h_{t+1}}{h_t}$, is obtained by substituting (8) into (9), and evaluating the resulting (9) at $y_{t+1} = y_t = y_L$ and $x_{t+1} = x_t = x_H$[11] as follows:

$$g^* = \psi \left(\frac{A}{1+\gamma} \right)^{\beta_1} (\eta - D - \alpha y_L)^{\beta_1} x_H^{\alpha(1-\beta_3)} = \psi^{\frac{1}{1-\beta_1}} x_H^{\frac{\beta_1+\alpha\beta_2}{1-\beta_1}}. \tag{18}$$

[11]Since both x_t and y_t are state variables, only the stable steady state (x_H, y_L) is economically meaningful and, thus, we focus on this steady state.

The long-run growth rate g^* is increasing in both the scale parameter ψ and physical/human capital ratio x_H.

3 Policy implications

3.1 Fiscal sustainability

If (16) is satisfied, according to the phase diagram, the economy with the initial public debt/physical capital ratio $y_0 < y_H$ converges to the stable steady state (x_H, y_L), while that with $y_0 > y_H$ eventually crosses the $\overline{y}$ line (the vertical dotted line in Figure 1); thereafter the economy becomes fiscally unsustainable. Hence, if there are two steady states, given the initial physical capital, the economy with low initial public debt is fiscally sustainable, while that with high initial public debt becomes fiscally unsustainable in some time. There are two policy implications as follows.

1. In addition to (13), (16) is required for the economy with low initial public debt to be fiscally sustainable. That is, the debt-existing condition of Tirole, (13), is necessary, since otherwise, y_L becomes negative, but it is not sufficient for the debt-existing condition of this model. Since (16) is likely to be satisfied when D is small, the primary deficit/production ratio must be small for the economy to be fiscally sustainable.[12]

2. The threshold line for whether the economy is eventually fiscally sustainable, that is, the saddle path given by $y_t = y_H$, does not depend on x_0. This means that the initial human capital is neutral to the threshold level. Hence, whether the economy has a high initial human capital level is irrelevant to whether it is fiscally sustainable. To explain the reason for this, we consider why y_{t+1} depends only on y_t, and for explanatory simplicity, let us assume that the productivity of private education β_1 is zero, which also means that the propensity of private education γ is zero. Then, by dividing (7) by b_{t+1}, we obtain:

$$y_{t+1} = \frac{k_{t+1}}{b_{t+1}} = \frac{s_t}{b_{t+1}} - 1 = \frac{\eta Y_t}{b_{t+1}} - 1, \tag{19}$$

[12] If θ is small, D is also small, and (16) is likely to be satisfied, since there are no terms that include θ, except for D. If τ is high, it decreases D, but at the same time, it decreases η and γ. Since an increase in γ increases (decreases) the right-hand (left-hand) side of (16), and we show that $\eta - D$ is increasing in τ in Appendix A3, (16) tends to be satisfied when τ is high. Therefore, (16) is likely to be satisfied when θ is small and/or τ is high.

where the third equality holds from (4). The next level of public debt b_{t+1} is the sum of current public debt including the interest payment and primary deficit. Then, when the production Y_t increases, savings increase by η, while debt obligations $(1 + r_t)b_t$ increase by αy_t because of an increase in the interest, and the primary deficit increases by D. Thus, from (19), an an increase of Y_t does not affect y_{t+1} since it proportionally increases s_t and b_{t+1}, and the effect is completely offset. Since an increase of x_t is equivalent to that of Y_t, x_t does not affect y_{t+1}.

If (16) is violated, there are no steady states. In this case, $F(y_t)$ is always positive since there are no real roots of $F(y_t) = 0$; hence, Δy_{t+1} is positive for any y_t. Therefore, public debt continuously increases and the economy eventually crosses the $\bar{y}$ line and becomes fiscally unsustainable thereafter for any endowment. As mentioned above, the economy with high D tends to violate (16) and lapse into fiscal unsustainability. In summary, we obtain the following corollary regarding fiscal sustainability.

Corollary 1 *Under (13),*

- *if (16) holds, the economy with $y_0 < y_H$ is fiscally sustainability and that with $y_0 > y_H$ eventually becomes fiscally unsustainable, where y_H is given by*

$$y_H = \frac{1}{2\alpha}\left[A_1 + \sqrt{A_1^2 - A_2}\right],$$

and

- *if (16) is violated, the economy eventually becomes fiscally unsustainable for any initial endowment.*

3.2 Comparative statics of steady-state value

In this model, there are two policy variables—the tax rate τ and public education/production ratio θ—and we consider the effect of a marginal increase in each policy variable on the stable steady-state value (x_H, y_L). At the same time, we also consider the comparative statics of y_H, which is a threshold level of the initial public debt/physical capital ratio, to consider whether the marginal increase of each policy variable widens or narrows the fiscally sustainable region. The results of the comparative statics are given as the following proposition.

Proposition 2 *The signs of the derivatives of* (x_H, y_L, y_H)*, with respect to* (τ, θ)*, become:*

$$\begin{aligned} &\frac{dx_H}{d\tau} > 0, \quad \frac{dy_L}{d\tau} < 0, \quad \frac{dy_H}{d\tau} > 0, \\ &\frac{dx_H}{d\theta} < 0, \quad \frac{dy_L}{d\theta} > 0, \quad \frac{dy_H}{d\theta} < 0. \end{aligned} \tag{20}$$

Proof. See Appendix A3. ■

Because of its simplicity, we first consider the effect of an increase in θ on y_L. When θ increases, the primary deficit increases since public education increases. Then, given the same level of tax, public debt must increase to finance it; thus, y_L increases. Considering this, x_H is affected by: (i) the scale parameter of the long-run growth rate ψ increasing, which leads to a decrease of parameter ϕ, and (ii) y_L increasing. Since (i) stems from an increase in the return of private education given by $(1-\tau)w_t \frac{dh_t}{de_t}$, investment in education becomes attractive and human capital increases, and (ii) physical capital is crowded out as public debt increases according to the capital market clearing condition, both the first and second effects decrease x_H. The sixth inequality, $dy_H/d\theta < 0$, means that the higher θ is, the more the economy eventually becomes fiscally unsustainable. This is because a higher public education/production ratio under the same tax rate worsens the primary deficit.

On the other hand, the effect of an increase in τ on y_L is threefold: (i) savings rate η decreases since total income decreases, (ii) primary deficit DY_t decreases since tax revenue increases, and (iii) propensity of private education γ decreases since the return of private education decreases. The first effect decreases k_t, while the second effect increases k_t, since pubic debt decreases; however, we can show that the effect of a decrease in D dominates that of η;[13] Hence, the combination of first and second effects increases k_t. The third effect also increases k_t since the household substitutes investment into private education with savings, and the sum of three effects increases k_t, which, in turn, decreases y_L, since the denominator increases. The effect of an increase in τ on x_H can be categorized into three channels: (i) $\eta - D$ increases, (ii) γ decreases, and (iii) y_L decreases. All three effects reallocate the investment in human capital with that in physical capital; therefore, x_H increases. Finally, y_H increases as τ increases since the primary deficit improves, which means that more of the

[13] In Appendix A3, we show that

$$\frac{d}{d\tau}(\eta - D) = (1-\alpha)\left(1 - \frac{\delta}{1+\delta}(1-\beta_1)\right) > 0.$$

economy can be fiscally sustainable.

3.3 Growth-maximizing policy

In this subsection, we show the effect of marginal increase in each policy variable on the long-run growth rate g^* given by (18), and consider whether there is a growth-maximizing level of policy variables. Regarding the growth-maximizing policy, we obtain the following proposition:

Proposition 3

1. *The growth-maximizing tax rate satisfies the following equation:*

$$\frac{\beta_1+\alpha\beta_2}{\eta-\alpha y_L-D}\left(\frac{d}{d\tau}(\eta-D)-\alpha\frac{dy_L}{d\tau}\right)=\frac{(1-\alpha)\beta_1}{\gamma(1+\gamma)}\left((1-\tau)(\beta_1+\beta_2)-1\right)\frac{d\gamma}{d\tau}, \tag{21}$$

   where $\frac{d}{d\tau}(\eta-D)>0$, $(1-\tau)(\beta_1+\beta_2)<1$, *and* $\frac{d\gamma}{d\tau}<0$. *There exists a unique tax rate that satisfies (21),* $\tau^*\in(\hat{\tau},1)$, *where* $\hat{\tau}$ *is defined as* $A_1^2|_{\tau=\hat{\tau}}=A_2|_{\tau=\hat{\tau}}$.

2. *The growth-maximizing public education/production ratio satisfies the following equation:*

$$\frac{\beta_1+\alpha\beta_2}{\eta-\alpha y_L-D}\left(1+\alpha\frac{dy_L}{d\theta}\right)=\frac{\beta_2(1-\alpha)}{\theta}. \tag{22}$$

   There exists a unique public education/production ratio that satisfies (22), $\theta^*\in(0,\hat{\theta})$, *where* $\hat{\theta}$ *is defined as* $A_1^2|_{\theta=\hat{\theta}}=A_2|_{\theta=\hat{\theta}}$.

Proof. See Appendix A4. ■

The effect of an increase in tax rate τ on g^* can be decomposed into: (i) γ decreases because of a decrease in the return of private education, which leads to a decrease in a scale parameter of the long-run growth rate ψ, and (ii) x_H increases as shown in (20). The left-hand side (LHS) of (21) captures the former marginal cost, while the right-hand side (RHS) captures the latter marginal benefit. In Appendix A4, we show that the LHS of (21) is decreasing in τ, goes to infinity as $\tau\to\hat{\tau}$, where $\hat{\tau}$ satisfies $A_1^2|_{\tau=\hat{\tau}}=A_2|_{\tau=\hat{\tau}}$, and is positive when $\tau=1$, while the RHS of (21) is increasing in τ, is positive when $\tau=\hat{\tau}$, and goes to infinity as $\tau\to1$. Taking these features into account, we obtain the left panel of Figure 2 and show the existence and uniqueness of the growth-maximizing tax rate $\tau^*\in(\hat{\tau},1)$.

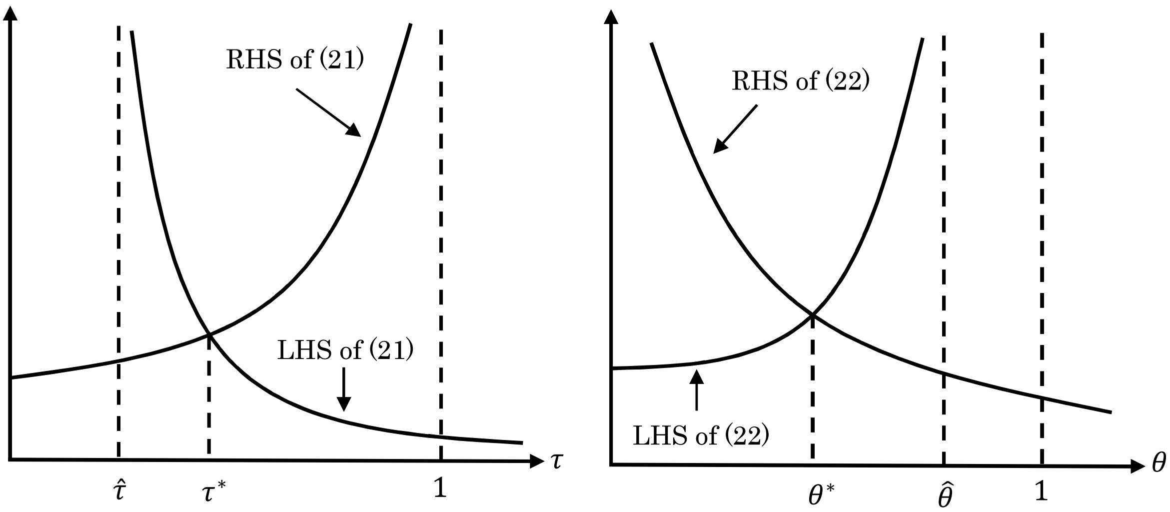

Figure 2: Growth-maximizing tax rate τ^* (left panel) and public education/production ratio θ^* (right panel)

On the other hand, an increase in public education/production ratio θ affects g^* in the following two ways: (i) ψ increases since higher θ directly increases $\frac{h_{t+1}}{h_t}$ through higher public education, and (ii) x_H decreases from a crowding-out effect as shown in (20). The RHS of (22) captures the first marginal benefit, while the LHS captures the second marginal cost. Similar to the case of τ from Appendix A4, we show that the LHS of (22) is increasing in τ, goes to infinity as $\theta \to \hat{\theta}$, where $\hat{\theta}$ satisfies $A_1^2|_{\theta=\hat{\theta}} = A_2|_{\theta=\hat{\theta}}$, and is positive when $\theta = 0$, while the RHS of (21) is decreasing in τ, is positive when $\theta = \hat{\theta}$, and goes to infinity as $\theta \to 0$. Thus, we obtain the right panel of Figure 2 and show that there uniquely exists $\theta^* \in (0, \hat{\theta})$, which maximizes the long-run growth rate. Note that if the government can set $\tau = \tau^*$ and/or $\theta = \theta^*$, $A_1^2 > A_2$ holds; therefore, (16) is satisfied and the economy with $y_0 < y_H$ is fiscally sustainable.[14]

4 Conclusion

In this study, we consider fiscal sustainability by using the OLG model with private and public education as well as public debt. Based on our model, if any, we obtain multiple steady states with respect to physical/human capital ratio x_t and public debt/physical capital ratio y_t; one of them characterized

[14]However, when the government implements the growth-maximizing policy, the primary surplus may accrue (i.e., $D < 0$). In that case, the stable steady-state value of the public debt/physical capital ratio, y_L, becomes negative and the government becomes a saver in the long run.

by low public debt is locally stable, while the other characterized by high public debt is a saddle point. The threshold level of public debt above which the economy eventually becomes fiscally unsustainable is given by the saddle path toward the saddle steady state, and we show that it is neutral to the initial human capital level. This means that whether a country is educationally developed has nothing to do with fiscal sustainability. Moreover, the debt-existing condition of Tirole, that is, that the long-run growth rate without public debt is higher than the interest rate, is a necessary, but sufficient condition in our model, and we can analytically show the additional debt-existing condition given by (16), which tends to be satisfied when the primary deficit is low.

Further, we consider the effect of a marginal increase in each policy variable on the long-run growth rate in Section 3.3. There are two opposite effects of raising each policy variable on the rate: the improvement of primary deficit versus a decrease in private education in terms of the tax rate, and an increase in public education versus the crowding-out effect of physical capital in terms of the public education/production ratio. Taking these effects into account, we show the existence and uniqueness of the growth-maximizing tax rate and public education/production ratio, and that the economy satisfies (16) under this level. This means that if the government can set the tax rate and/or public education/production ratio as the growth-maximizing level, the economy with a moderate, initial public debt/physical capital ratio, which is given by $y_0 < y_H$, is fiscally sustainable.

Finally, in this paper, we consider the constant public education/production policy. Instead, if the government implements a similar policy for budget deficit (i.e., $\Delta b_{t+1} = \theta Y_t$), as assumed in Bräuninger (2005), Yakita (2008), and Arai (2011), the dynamics will change since the public education/production ratio does not become constant under the constant tax rate and unproductive government expenditure/production. It would be interesting to analyze how it changes the threshold of fiscal sustainability and how the long-run growth rate is affected by the budget deficit/production ratio. The analysis is left to future studies.

Appendices

A1. Derivation of (13)

In this model, the debt-existing condition of Tirole is given by:[15]

$$g^*|_{y^*=0} - 1 > \lim_{t\to\infty} \frac{\partial F(k_t,h_t)}{\partial k_t}\Big|_{y^*=0} - 1.$$

$g^*|_{y^*=0}$ is calculated from (18) as $\psi^{\frac{1}{1-\beta_1}} x|_{y^*=0}^{*\frac{\beta_1+\alpha\beta_2}{1-\beta_1}}$, while $\lim_{t\to\infty} \frac{\partial F(k_t,h_t)}{\partial k_t}\Big|_{y^*=0}$ becomes $A\alpha x|_{y^*=0}^{*\alpha-1}$. Substituting these expressions into the above expression yields:

$$x|_{y^*=0}^{*\frac{1-\alpha\beta_3}{1-\beta_1}} > A\alpha\left(\frac{1}{\psi}\right)^{\frac{1}{1-\beta_1}}.$$

From (14), $x|_{y^*=0}^{*\frac{1-\alpha\beta_3}{1-\beta_1}}$ becomes $\phi^{1-\beta_1}(\eta - D)$, and by substituting this into the above expression and after some manipulation by using the definitions of ϕ and ψ, we obtain:

$$\eta - D > (1+\gamma)\alpha,$$

which is equivalent to (13).

A2. Stability analysis of (11) and (12)

Under (16), there exist two steady states, (x_H, y_L) and (x_L, y_H), where $x_H > x_L$ and $y_H > y_L$, respectively, and we express these steady states by (x^*, y^*). Then, by linearly approximating (11) and (12) around the steady state (x^*, y^*), we obtain the following matrix form:

$$\begin{pmatrix} dx_{t+1} \\ dy_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha\beta_3 & \frac{\alpha(1-\beta_1)x^*}{\eta-\alpha y^*-D} \\ 0 & \Omega'(y^*) \end{pmatrix} \begin{pmatrix} dx_t \\ dy_t \end{pmatrix},$$

where $dx_t \equiv x_t - x^*$ and $dy_t \equiv y_t - y^*$. Since the coefficient matrix is a triangular matrix, characteristic roots are: $\lambda_1 = \alpha\beta_3 \in (0,1)$ and $\lambda_2 = \Omega'(y^*)$. To consider the value of $\Omega'(y^*)$, note that $\Omega(y_t)$ is increasing and convex for y_t, since both $\Omega'(y_t)$ and $\Omega''(y_t)$ are positive; hence, $\Omega'(y_L) \in (0,1)$ and $\Omega'(y_H) > 1$ must hold (see Figure 3). Therefore, the steady state (x_H, y_L) is locally stable since both λ_1 and λ_2 are in $(0,1)$, while the steady state (x_L, y_H) is a saddle point since $\lambda_1 \in (0,1)$ and $\lambda_2 > 1$. As x_t and y_t are state variables, only the stable steady state (x_H, y_L) is economically meaningful.

[15]Recall that we assume that the depreciation rate is one and define g^* such as $\lim_{t\to\infty} \frac{h_{t+1}}{h_t}$, not as $\lim_{t\to\infty} \frac{h_{t+1}-h_t}{h_t}$.

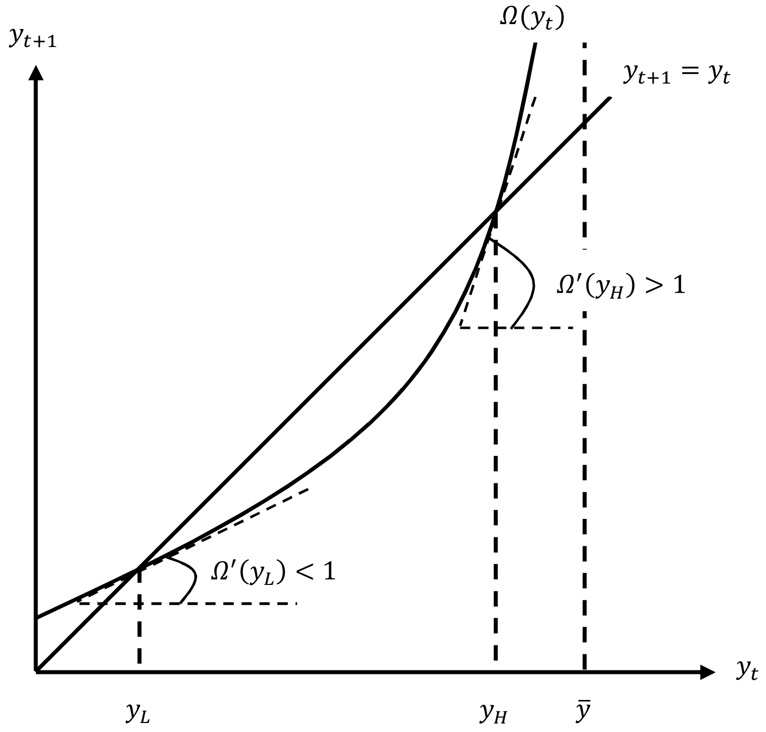

Figure 3: Phase diagram of y_t

A3. Proof of Proposition 2

First, we consider the effect of a marginal increase in τ. To show this, we calculate the derivative of $A_1 \equiv \eta - D - (1+\gamma)\alpha$, with respect to τ, as follows:

$$\frac{dA_1}{d\tau} = \frac{d}{d\tau}(\eta - D) - \alpha\frac{d\gamma}{d\tau}.$$

Since $\frac{d}{d\tau}(\eta - D) = \left(1 - \frac{\delta}{1+\delta}(1-\beta_1)\right)(1-\alpha) > 0$ and $\alpha\frac{d\gamma}{d\tau} = -\beta_1(1-\alpha) < 0$, $\frac{dA_1}{d\tau} > 0$. Similarly, the derivative of $A_2 \equiv 4\alpha(1+\gamma)D$, with respect to τ, is:

$$\frac{dA_2}{d\tau} = 4\alpha\frac{d}{d\tau}(1+\gamma)D = 4\alpha\left[(1+\gamma)\frac{dD}{d\tau} + D\frac{d\gamma}{d\tau}\right].$$

Since $\frac{dD}{d\tau} = -(1-\alpha) < 0$ and $\frac{d\gamma}{d\tau} < 0$, $\frac{dA_2}{d\tau} < 0$. Taking these into account, the sign of the derivative of y_L, with respect to τ, is given by:

$$
\begin{aligned}
\frac{dy_L}{d\tau} &= \frac{1}{2\alpha}\left[\frac{dA_1}{d\tau} - \frac{1}{2}\left(A_1^2 - A_2\right)^{-\frac{1}{2}}\left(2A_1\frac{dA_1}{d\tau} - \frac{dA_2}{d\tau}\right)\right] \\
&= \frac{\left(A_1^2 - A_2\right)^{-\frac{1}{2}}}{2\alpha}\frac{dA_1}{d\tau}\left[(A_1^2 - A_2)^{\frac{1}{2}} - \left(A_1 - \frac{1}{2}\frac{dA_2/d\tau}{dA_1/d\tau}\right)\right] \\
&= \frac{\left(A_1^2 - A_2\right)^{-\frac{1}{2}}}{2\alpha}\underbrace{\frac{dA_1}{d\tau}}_{>0}\left[\underbrace{\left((A_1^2 - A_2)^{\frac{1}{2}} - A_1\right)}_{<0} + \underbrace{\frac{1}{2}\frac{dA_2/d\tau}{dA_1/d\tau}}_{<0}\right] < 0. \qquad (23)
\end{aligned}
$$

We use the sign of $\frac{d^2y_L}{d\tau^2}$ in the proof of Proposition 3 in Appendix A4, and we show it as follows:

$$
\begin{aligned}
2\alpha\frac{d^2y_L}{d\tau^2} &= \frac{d}{d\tau}\left[\frac{dA_1}{d\tau} - (A_1^2 - A_2)^{-\frac{1}{2}}\left(A_1\frac{dA_1}{d\tau} - \frac{1}{2}\frac{dA_2}{d\tau}\right)\right] \\
&= \frac{1}{2}(A_1^2 - A_2)^{-\frac{1}{2}-1}\left(2A_1\frac{dA_1}{d\tau} - \frac{dA_2}{d\tau}\right)\left(A_1\frac{dA_1}{d\tau} - \frac{1}{2}\frac{dA_2}{d\tau}\right) - (A_1^2 - A_2)^{-\frac{1}{2}}\left(\left(\frac{dA_1}{d\tau}\right)^2 - \frac{1}{2}\frac{d^2A_2}{d\tau^2}\right) \\
&= (A_1^2 - A_2)^{-\frac{1}{2}-1}\left[\left(A_1\frac{dA_1}{d\tau} - \frac{1}{2}\frac{dA_2}{d\tau}\right)^2 - (A_1^2 - A_2)\left(\left(\frac{dA_1}{d\tau}\right)^2 - \frac{1}{2}\frac{d^2A_2}{d\tau^2}\right)\right] \\
&= (A_1^2 - A_2)^{-\frac{1}{2}-1}\left[-A_1\underbrace{\frac{dA_1}{d\tau}}_{>0}\underbrace{\frac{dA_2}{d\tau}}_{<0} + \frac{1}{4}\left(\frac{dA_2}{d\tau}\right)^2 + A_2\left(\frac{dA_1}{d\tau}\right)^2 + \underbrace{\frac{(A_1^2 - A_2)}{2}}_{>0}\underbrace{\frac{d^2A_2}{d\tau^2}}_{>0}\right] > 0, \qquad (24)
\end{aligned}
$$

where, note that:

$$
\frac{d^2A_2}{d\tau^2} = 8\alpha\underbrace{\frac{dD}{d\tau}}_{<0}\underbrace{\frac{d\gamma}{d\tau}}_{<0} > 0.
$$

From (14), x_H is determined as $x_H = \phi^{\frac{1}{1-\alpha\beta_3}}(\eta - \alpha y_L - D)^{\frac{1-\beta_1}{1-\alpha\beta_3}}$, and taking logarithms in both sides yields:

$$
\ln x_H = \frac{1}{1-\alpha\beta_3}\ln\phi + \frac{1-\beta_1}{1-\alpha\beta_3}\ln(\eta - \alpha y_L - D). \qquad (25)
$$

Hence, the sign of the derivative of x_H, with respect to τ, is:

$$
\frac{1}{x_H}\frac{dx_H}{d\tau} = \frac{1}{\phi(1-\alpha\beta_3)}\underbrace{\frac{d\phi}{d\tau}}_{>0} + \frac{1-\beta_1}{1-\alpha\beta_3}\frac{1}{\eta - \alpha y_L - D}\left[\underbrace{-\alpha\frac{dy_L}{d\tau}}_{>0} + \underbrace{\frac{d}{d\tau}(\eta - D)}_{>0}\right] > 0, \qquad (26)
$$

where $\frac{d\phi}{d\tau} > 0$ holds because $\frac{d\gamma}{d\tau} < 0 \Rightarrow \frac{d\psi}{d\tau} < 0$ and $\frac{d\phi}{d\psi} < 0$. The sign of the derivative of y_H, with respect to τ, is directly given by:

$$\frac{dy_H}{d\tau} = \frac{1}{2\alpha}\left[\underbrace{\frac{dA_1}{d\tau}}_{>0} + \frac{1}{2}\left(A_1^2 - A_2\right)^{-\frac{1}{2}}\left(2A_1 \underbrace{\frac{dA_1}{d\tau}}_{>0} - \underbrace{\frac{dA_2}{d\tau}}_{<0}\right)\right] > 0. \tag{27}$$

In what follows, similar to the case of τ, we derive the derivative of the stable steady state (x_H, y_L), with respect to θ. By using $\frac{dA_1}{d\theta} = -1$ and $\frac{dA_2}{d\theta} = 4\alpha(1+\gamma)$, we obtain the derivative of y_L, with respect to θ, as:

$$\begin{aligned}
\frac{dy_L}{d\theta} &= \frac{1}{2\alpha}\left[\frac{dA_1}{d\theta} - \frac{1}{2}\left(A_1^2 - A_2\right)^{-\frac{1}{2}}\left(2A_1\frac{dA_1}{d\theta} - \frac{dA_2}{d\theta}\right)\right] \\
&= \frac{1}{2\alpha}\left[-1 - \left(A_1^2 - A_2\right)^{-\frac{1}{2}}\left(-A_1 - 2\alpha(1+\gamma)\right)\right] \\
&= \frac{(A_1^2 - A_2)^{-\frac{1}{2}}}{2\alpha}\left[\underbrace{A_1 - \left(A_1^2 - A_2\right)^{\frac{1}{2}}}_{>0} + 2\alpha(1+\gamma)\right] > 0.
\end{aligned} \tag{28}$$

Note that the sign of $\frac{d^2 y_L}{d\theta^2}$ is positive according to the following calculation:

$$\begin{aligned}
\frac{d^2 y_L}{d\theta^2} &= \frac{d}{d\theta}\frac{1}{2\alpha}\left[-1 - \left(A_1^2 - A_2\right)^{-\frac{1}{2}}\left(-A_1 - 2\alpha(1+\gamma)\right)\right] \\
&= \frac{1}{2\alpha}\frac{d}{d\theta}\left(A_1^2 - A_2\right)^{-\frac{1}{2}}\left(A_1 + 2\alpha(1+\gamma)\right) \\
&= \frac{(A_1^2 - A_2)^{-\frac{1}{2}-1}}{2\alpha}\left[-\frac{1}{2}\left(2A_1\frac{dA_1}{d\theta} - \frac{dA_2}{d\theta}\right)(A_1 + 2\alpha(1+\gamma)) + (A_1^2 - A_2)\frac{dA_1}{d\theta}\right] \\
&= \frac{(A_1^2 - A_2)^{-\frac{1}{2}-1}}{2\alpha}\left[(A_1 + 2\alpha(1+\gamma))^2 - (A_1^2 - A_2)\right] > 0.
\end{aligned} \tag{29}$$

Similar to (25), the sign of the derivative of x_H, with respect to θ, becomes:

$$\frac{1}{x_H}\frac{dx_H}{d\theta} = \frac{1}{\phi(1-\alpha\beta_3)}\underbrace{\frac{d\phi}{d\theta}}_{<0} - \frac{1-\beta_1}{1-\alpha\beta_3}\frac{\alpha}{\eta - \alpha y_L - D}\underbrace{\frac{dy_L}{d\theta}}_{>0} < 0. \tag{30}$$

Finally, the sign of the derivative of y_H, with respect to θ, is simply given by:

$$\frac{dy_H}{d\theta} = \frac{1}{2\alpha}\left[\underbrace{\frac{dA_1}{d\theta}}_{<0} + \frac{1}{2}\left(A_1^2 - A_2\right)^{-\frac{1}{2}}\left(2A_1 \underbrace{\frac{dA_1}{d\theta}}_{<0} - \underbrace{\frac{dA_2}{d\theta}}_{>0}\right)\right] < 0. \quad \blacksquare \tag{31}$$

A4. Proof of Proposition 3

From (18), the derivative of g^*, with respect to τ, is given by:

$$\ln g^* = \frac{1}{1-\beta_1}\ln\psi + \frac{\beta_1+\alpha\beta_3}{1-\beta_1}\ln x_H,$$
$$\Leftrightarrow \quad \frac{1-\beta_1}{g^*}\frac{dg^*}{d\tau} = \frac{1}{\psi}\frac{d\psi}{d\tau} + \frac{\beta_1+\alpha\beta_2}{x_H}\frac{dx_H}{d\tau}. \tag{32}$$

Moreover, from the definition of x_H, we obtain:

$$\ln x_H = \frac{1}{1-\alpha\beta_3}\ln\phi + \frac{1-\beta_1}{1-\alpha\beta_3}\ln(\eta-\alpha y_L - D),$$
$$\Leftrightarrow \quad (1-\alpha\beta_3)\ln x_H = -\ln\psi + (1-\beta_1)\ln A - (1-\beta_1)\ln(1+\gamma) + (1-\beta_1)\ln(\eta-\alpha y_L-D),$$
$$\Leftrightarrow \quad \frac{1-\alpha\beta_3}{x_H}\frac{dx_H}{d\tau} = -\frac{1}{\psi}\frac{d\psi}{d\tau} - \frac{1-\beta_1}{1+\gamma}\frac{d\gamma}{d\tau} + \frac{1-\beta_1}{\eta-\alpha y_L-D}\left(\frac{d}{d\tau}(\eta-D) - \alpha\frac{dy_L}{d\tau}\right). \tag{33}$$

Substituting (33) into (32) yields:

$$\frac{1-\beta_1}{g^*}\frac{dg^*}{d\tau} = \frac{1}{\psi}\frac{d\psi}{d\tau} - \frac{\beta_1+\alpha\beta_2}{1-\alpha\beta_3}\left(\frac{1}{\psi}\frac{d\psi}{d\tau} + \frac{1-\beta_1}{1+\gamma}\frac{d\gamma}{d\tau}\right) + \frac{\beta_1+\alpha\beta_2}{1-\alpha\beta_3}\frac{1-\beta_1}{\eta-\alpha y_L-D}\left(\frac{d}{d\tau}(\eta-D) - \alpha\frac{dy_L}{d\tau}\right). \tag{34}$$

The sum of the first and second terms of the RHS of (34) becomes:

$$\left(1-\frac{\beta_1+\alpha\beta_2}{1-\alpha\beta_3}\right)\frac{1}{\psi}\frac{d\psi}{d\tau} - \frac{\beta_1+\alpha\beta_2}{1-\alpha\beta_3}\frac{1-\beta_1}{1+\gamma}\frac{d\gamma}{d\tau} = \frac{1-\beta_1}{1-\alpha\beta_3}\left(\frac{1-\alpha}{\psi}\frac{d\psi}{d\tau} - \frac{\beta_1+\alpha\beta_2}{1+\gamma}\frac{d\gamma}{d\tau}\right), \tag{35}$$

where the second equality holds, since $\left(1-\frac{\beta_1+\alpha\beta_2}{1-\alpha\beta_3}\right) = \frac{(1-\alpha)(1-\beta_1)}{1-\alpha\beta_3}$. From the definition of ψ, $\frac{d\psi}{d\tau} = \frac{\beta_1\psi}{\gamma}\frac{d\gamma}{d\tau}$, and by substituting this into (35), it becomes:

$$\frac{1-\beta_1}{1-\alpha\beta_3}\frac{1}{\gamma(1+\gamma)}[\beta_1(1-\alpha)(1+\gamma) - \gamma(\beta_1+\alpha\beta_2)]\frac{d\gamma}{d\tau}. \tag{36}$$

Since:

$$\beta_1(1-\alpha)(1+\gamma) - \gamma(\beta_1+\alpha\beta_2) = (1-\alpha)\beta_1 - \alpha\gamma(\beta_1+\beta_2)$$
$$= (1-\alpha)\beta_1\left(1-\frac{\alpha\gamma}{(1-\alpha)\beta_1}(\beta_1+\beta_2)\right) = (1-\alpha)\beta_1(1-(1-\tau)(\beta_1+\beta_2)) > 0,$$

substituting this into (36) yields:

$$\frac{1-\beta_1}{1-\alpha\beta_3}\frac{(1-\alpha)\beta_1}{\gamma(1+\gamma)}(1-(1-\tau)(\beta_1+\beta_2))\frac{d\gamma}{d\tau},$$

and by substituting this into (34), we obtain:

$$\frac{1-\beta_1}{g^*}\frac{dg^*}{d\tau} = \frac{1-\beta_1}{1-\alpha\beta_3}\frac{(1-\alpha)\beta_1}{\gamma(1+\gamma)}(1-(1-\tau)(\beta_1+\beta_2))\frac{d\gamma}{d\tau} + \frac{\beta_1+\alpha\beta_2}{1-\alpha\beta_3}\frac{1-\beta_1}{\eta-\alpha y_L-D}\left(\frac{d}{d\tau}(\eta-D)-\alpha\frac{dy_L}{d\tau}\right),$$

$$\Leftrightarrow \quad \frac{1-\alpha\beta_3}{g^*}\frac{dg^*}{d\tau} = \frac{\beta_1+\alpha\beta_2}{\eta-\alpha y_L-D}\left(\frac{d}{d\tau}(\eta-D)-\alpha\frac{dy_L}{d\tau}\right) - \frac{(1-\alpha)\beta_1}{\gamma(1+\gamma)}((1-\tau)(\beta_1+\beta_2)-1)\frac{d\gamma}{d\tau}.$$

The first-order condition of g^*, with respect to τ, is given by $\frac{dg^*}{d\tau}=0$ as:

$$\frac{\beta_1+\alpha\beta_2}{\eta-\alpha y_L-D}\left(\frac{d}{d\tau}(\eta-D)-\alpha\frac{dy_L}{d\tau}\right) = \frac{(1-\alpha)\beta_1}{\gamma(1+\gamma)}((1-\tau)(\beta_1+\beta_2)-1)\frac{d\gamma}{d\tau}, \tag{37}$$

which is equivalent to (21). The LHS of (37) is decreasing in τ[16] and goes to infinity as $\tau \to \hat{\tau}$, where $A_1^2|_{\tau=\hat{\tau}} = A_2|_{\tau=\hat{\tau}}$, since $(A_1^2 - A_2)^{-\frac{1}{2}} \to \infty$ and $\frac{dy_L}{d\tau} < 0$. Hence, $\frac{dy_L}{d\tau} \to -\infty$, and it is positive when $\tau = 1$. On the other hand, the RHS of (37) is increasing in τ, and goes to infinity as $\tau \to 1$, since $\gamma \to 0$. Therefore, $\gamma^{-1} \to \infty$, while it is positive when $\tau = \hat{\tau}$. This means that there is the growth-maximizing tax rate $\tau^* \in (\hat{\tau}, 1)$ (see the left panel of Figure 2).

Further, the derivative of g^*, with respect to θ, is derived as follows; similar to (32), $\frac{dg^*}{d\theta}$ is calculated as:

$$\frac{1-\beta_1}{g^*}\frac{dg^*}{d\theta} = \frac{1}{\psi}\frac{d\psi}{d\theta} + \frac{\beta_1+\alpha\beta_2}{x_H}\frac{dx_H}{d\theta}. \tag{38}$$

By using a similar procedure to (33), we obtain $\frac{dx_H}{d\theta}$ as:

$$\frac{1-\alpha\beta_3}{x_H}\frac{dx_H}{d\theta} = -\frac{1}{\psi}\frac{d\psi}{d\theta} - \frac{1-\beta_1}{1+\gamma}\underbrace{\frac{d\gamma}{d\theta}}_{=0} + \frac{1-\beta_1}{\eta-\alpha y_L-D}\left(\underbrace{\frac{d}{d\theta}(\eta-D)}_{=-1}-\alpha\frac{dy_L}{d\theta}\right). \tag{39}$$

Combining (38) with (39) yields:

$$\begin{aligned}\frac{1-\beta_1}{g^*}\frac{dg^*}{d\theta} &= \frac{1}{\psi}\frac{d\psi}{d\theta} - \frac{\beta_1+\alpha\beta_2}{1-\alpha\beta_3}\left[\frac{1}{\psi}\frac{d\psi}{d\theta} + \frac{1-\beta_1}{\eta-\alpha y_L-D}\left(1+\alpha\frac{dy_L}{d\theta}\right)\right]\\ &= \frac{(1-\alpha)(1-\beta_1)}{1-\alpha\beta_3}\frac{1}{\psi}\frac{d\psi}{d\theta} - \frac{\beta_1+\alpha\beta_2}{\eta-\alpha y_L-D}\left(1+\alpha\frac{dy_L}{d\theta}\right).\end{aligned}$$

By using $\frac{d\psi}{d\theta} = \frac{\beta_2\psi}{\theta}$, we obtain:

$$\frac{1-\alpha\beta_3}{g^*}\frac{dg^*}{d\theta} = \frac{\beta_2(1-\alpha)}{\theta} - \frac{\beta_1+\alpha\beta_2}{\eta-\alpha y_L-D}\left(1+\alpha\frac{dy_L}{d\theta}\right),$$

[16] From (24), we show that $\frac{dy_L}{d\tau}$ is increasing in τ. Considering this, the derivative of the LHS of (37), with respect to τ, is positively proportional to

$$-\alpha\frac{d^2y_L}{d\tau^2}(\eta-\alpha y_L-D) - \left(\frac{d}{d\tau}(\eta-D)-\alpha\frac{dy_L}{d\tau}\right)^2,$$

which is negative; thus, the term is decreasing in τ.

and the first-order condition, $\frac{dg^*}{d\theta} = 0$, is given by:

$$\frac{\beta_1 + \alpha\beta_2}{\eta - \alpha y_L - D}\left(1 + \alpha\frac{dy_L}{d\theta}\right) = \frac{\beta_2(1-\alpha)}{\theta}, \tag{40}$$

which is equivalent to (22). The LHS of (40) is increasing in θ,[17] And goes to infinity as $\theta \to \hat{\theta}$, where $A_1^2|_{\theta=\hat{\theta}} = A_2|_{\theta=\hat{\theta}}$, since $(A_1^2 - A_2)^{-\frac{1}{2}} \to \infty$ and $\frac{dy_L}{d\theta} > 0$. Hence, $\frac{dy_L}{d\theta} \to \infty$ and it is positive when $\theta = 0$. On the other hand, the RHS of (40) is decreasing in θ, goes to infinity as $\tau \to 0$, and is positive when $\theta = \hat{\theta}$. This means that there is growth-maximizing public education/production ratio $\theta^* \in (0, \hat{\theta})$ (see the right panel of Figure 2). ■

References

[1] Agénor, P.-R., (2011) "Schooling and public capital in a model of endogenous growth", Economica, 78, 108-132.

[2] Arai, R., (2011) "Productive government expenditure and fiscal sustainability", FinanzArchiv: Public Finance Analysis, 67, Number 4, 327-351.

[3] Barro, R. J., (1990) "Government spending in a simple model of endogenous growth", Journal of Political Economics, 98, 103-125.

[4] Blankenau, R. W., and Simpson, B. N., (2004) "Public education expenditures and growth", Journal of Development Economics, 73, 583-605.

[5] Blankenau, R. W., Simpson, B. N., and Tomljanovich, M., (2007) "Public education expenditures, taxation, and growth: Linking data to theory", American Economic Review, 97, No.2, 393-397.

[6] Boldrin, M., and Montes, A., (2005) "The Intergenerational State Education and Pensions", Review of Economic Studies, 72, 651-664.

[7] Bräuninger, M., (2005) "The Budget Deficit, Public Debt, and Endogenous Growth", Journal of Public Economic Theory, 7 (5), 827-840.

[17] From (29), we show that $\frac{dy_L}{d\theta}$ is increasing in θ. Then, the denominator of the LHS of (40) decreases, while the numerator increases as θ increases; thus, the term is increasing in θ.

[8] Chalk, N, A., (2000) “The sustainability of bond-finance deficits: An overlapping generations approach”, Journal of Monetary Economics, 45, 293-328.

[9] Diamond, P., (1965) “National debt in a neoclassical growth model”, American Economic Review, 55, 1126-1150.

[10] Futagami, K., Morita, Y., and Shibata, A., (1993) “Dynamic analysis of an endogenous growth model with public capital”, Scandinavia Journal of Economics, 95, 607-625.

[11] Greiner, A., (2008) “Human capital formation, public debt and economic growth”, Journal of Macroeconomics, 30, 415-427.

[12] Glomm, G., and Ravikumar, B., (1992) “Public versus private investment in human capital: Endogenous growth and income inequality”, Journal of Political Economy, 100, No.4, 818-834.

[13] Gottardi, P., Kajii, A., and Nakajima, T., (2015) “Optimal taxation and debt with uninsurable risks to human capital accumulation”, American Economic Review, 105(11), 3443-3470.

[14] Grossman, G, M., and Yanagawa, N, (1993) “Asset bubbles and endogenous growth”, Journal of Monetary Economics, 31, 3-19.

[15] Lucas, R E., Jr., (1988) “On the mechanics of economics development”, Journal of Monetary Economics, 22, July, 3-42.

[16] OECD, (2016) “Education at a glance 2016: OECD indicators”, OECD Publishing, Paris. http://dx.doi.org/10.1787/eag-2016-en.

[17] Ono, T., and Uchida, Y., (2016) “Human Capital, Public debt, and Economic Growth: A Political Economy Analysis”, Discussion Papers In Economics and Business, Graduate School of Economics and Osaka School of International Public Policy (OSIPP), Osaka University, 16-01, August.

[18] Tirole, J., (1985) “Asset bubbles and overlapping generations”, Econometrica, 53, 1499-1528.

[19] Uzawa, H., (1965) “Optimal technical change in an aggregate model of economic growth”, International Economic Review, 6, January, 18-31.

[20] Yakita, A., (2008) “Sustainability of public debt, public capital formation, and endogenous growth in an overlapping generations setting”, Journal of Public Economics, 92, 897-914.

[21] Zhang, J., (2003) “Optimal debt, endogenous fertilities, externalities in a model with altruistic bequests”, Journal of Public Economics, 87, 1825-1835.