# A GENERALIZATION OF CONTINUITY AND CONVEXITY CONDITIONS FOR CORRESPONDENCES IN THE ECONOMIC EQUILIBRIUM THEORY *

**Ken Urai**†
**Graduate School of Economics, Osaka University**

**and**

**Takashi Hayashi**
**Graduate School of Economics, Osaka University**

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**Abstract**

Fixed point theorems for multi-valued mappings and economic equilibrium existence theorems are generalized from the viewpoint that the continuity and/or the convexity assumptions on a mapping may be replaced for weaker conditions on the local direction indicated by the value of the mapping. The generalization gives us natural conditions on individual (possibly non-ordered) preferences or aggregated demand behaviors so that we may obtain several extensions of recent results in social and game theoretic equilibrium theories such as the results of Mehta and Tarafdar (1987) on the market equilibrium existence theorem of the Gale-Nikaido-Debreu type, the results of Nishimura and Friedman (1981) on the Nash equilibrium existence problem, and the results of Yannelis and Prabhakar (1983), Tan and Yuan (1994), and Bagh (1998) on the social equilibrium existence theorem.

**Keywords** : Gale-Nikaido-Debreu theorem, Excess demand correspondence, Non-ordered preference, Nash equilibrium, Abstract economy, N-person game, Fixed point theorem, Infinite dimensional commodity space.

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†Corresponding author. Graduate School of Economics, Osaka University, Toyonaka, Osaka 560-0043, JAPAN. Tel: +81-6-6850-5228. Fax: +81-6-6850-5274. Internet E-mail: urai@econ.osaka-u.ac.jp.

# 1 INTRODUCTION

In this paper, we generalize fixed point theorems for multi-valued mappings and the existence of economic equilibrium theorems from the viewpoint that both the continuity and the convexity conditions of a correspondence may be replaced for rather weaker conditions on the direction indicated by the value of the correspondence.

Suppose that a function $f : X \to X$ is continuous and $x \in X$ is not a fixed point of $f$. Then, "$x^n$ converges to $x$" implies that "$f(x^n) - x^n$ converges to the vector $f(x) - x \neq 0$." In other words, the variation under $f$ at each point near $x$ is also close to the vector $f(x) - x \neq 0$. More generally, it can be said that for a continuous function $f$, the directions of the variation under $f$ near a non-fixed point are almost equal. Our generalization of the fixed point theorem (Theorem 2, Theorem 4) shows that such a condition on the local non-zero direction of a mapping near each non-fixed point is not merely a *necessary* condition for a mapping to be continuous but also a *sufficient* condition for the existence of a fixed point.

Such a condition for the local direction of mappings may also be seen in the paper of Nishimura and Friedman (1981). Nishimura and Friedman proved that there exists a fixed point (a Nash equilibrium point for a best response correspondence) if the direction indicated by the mapping near a non-fixed point has a definite sign on at least one coordinate. By using the Eaves' theorem (Eaves (1974)), they have essentially dropped the convexity and/or continuity condition for a correspondence to have a fixed point. Unfortunately, however, their result does not cover the most popular (Kakutani's fixed point) case in which mappings are closed convex valued and upper semi-continuous. In this paper, the local definiteness condition on the *sign* of one coordinate of the mapping is generalized to the local definiteness condition on the *direction* of the mapping indicated by a linear form, so that we may obtain our fixed point theorem as an extension of Kakutani's fixed point theorem.

It is easy to interpret the local definiteness condition for the direction of mappings in this paper as a condition in economic terms such as "directions of excess demands near a non-equilibrium state are almost equal," "directions of better points near an individually non-optimal point are almost similar," and so on.

Conditions for the direction of excess demands give us a remarkably general version of the market equilibrium existence (so called, Gale-Nikaido-Debreu) theorem. Our results do not depend on the upper semi-continuity and/or closed convex valuedness of aggregate demand correspondences for the existence of market equilibrium (Theorem 1, Theorem 3); this gives one of the most general market equilibrium existence theorems in the literature (see, e.g., Mehta and Tarafdar (1987)).

Conditions for the local direction of individual preferences (better points) enables us to construct (generalized) N-person games without referring to any global continuity conditions for individual preferences. By applying our main theorem, we may obtain several extensions of well-known results in social and game theoretic equilibrium theories such as the non-convex generalization of the Nash equilibrium existence theorem in Nishimura and Friedman (1981) (see our Corollary 1, and Corollary 5), Nash equilibrium existence results for non-ordered preferences (see Corollary 2, Corollary 3, and Corollary 6), and the non-ordered social equilibrium existence results in Shafer and H.F.Sonnenschein (1975), Yannelis and Prabhakar (1983), Tarafdar (1991), Tan and Yuan (1994), and Bagh (1998) (see Corollary 4, and Corollary 7).

In section 2, we present all of our results in the framework of finite dimensional vector spaces. Though there seems to be some loss in generality, the arguments in finite dimensional cases still have most of the significant features of our theory together with new and stronger results which have not been covered in the literature. Most of the results in section 2 may also be generalized to the case with infinite dimensional vector spaces, as we see in section 3.

## 2 BASIC RESULTS

We begin with the market equilibrium existence problem as in Debreu (1956). Let $\Delta$ be the $\ell - 1$ dimensional unit simplex in $R^\ell$, and let $\zeta : \Delta \to R^\ell$ be a non-empty valued correspondence (an excess demand correspondence) satisfying the Walras' law,[1]

(W) $\forall p \in \Delta, \forall z \in \zeta(p), p \cdot z \leq 0,$

and the following local definiteness condition for directions of excess demands.

(LDD) For each $p \in \Delta$, if $\zeta(p) \cap -R^\ell_+ = \emptyset$, there exist a vector $y(p) \in \Delta$ and an open neighbourhood $U^p$ of $p$ in $\Delta$ such that $y(p) \cdot z > 0$ for all $z \in \zeta(q)$ for all $q \in U^p$.

Note that if $\zeta(p) \cap -R^\ell_+ = \emptyset$, and if $\zeta(p)$ is convex and closed, then $\zeta(p)$ and $-R^\ell_{++}$ may be strictly separated by a hyper plane normal to a vector $y(p)$ in $\Delta$. If $\zeta$ is also upper semi-continuous, then we have an open neighbourhood $U^p$ of $p$ such that $\zeta(q)$ is a subset of the open half space defined by the hyper plane for all $q \in U^p$. Hence, one of the sufficient conditions for (LDD) is the upper semi-continuity together with the closed convex valuedness for $\zeta$. It should also be noted, however, that in (LDD) the excess demand correspondence $\zeta$ is not assumed to be closed convex valued or upper semi-continuous. The condition merely says that excess demands under non-equilibrium prices have *a locally common direction*. That is, if $p$ is not an equilibrium price vector, then there exists at least one direction $y(p)$ such that all excess demands under prices near $p$ belong to the same side of the hyperplane defined by $y(p)$.

The first result in this paper is the existence of market equilibrium price, $p^* \in \Delta$, $\zeta(p^*) \cap -R^\ell_+ \neq \emptyset$, under conditions (W) and (LDD). An equilibrium existence theorem of this type is called a Gale-Nikaido-Debreu Theorem, and the framework of the theorem is used in various contexts in the economic equilibrium theory. Though the result may be generalized to the case with infinite dimensional commodity spaces (Theorem 3) in section 3, the following version of the Gale-Nikaido-Debreu theorem may not be covered in the existing literature, even as a result for finite dimensional commodity spaces (see, e.g., Mehta and Tarafdar (1987)).

THEOREM 1 : *For $\zeta : \Delta \to R^\ell$ satisfying conditions* (W) *and* (LDD), *there is an equilibrium price $p^*$, $\zeta(p^*) \cap -R^\ell_+ \neq \emptyset$.*

PROOF : Suppose that there is no equilibrium price vector. Then under (LDD), we have, for all $p \in \Delta$, a vector $y(p) \in \Delta$, an $\epsilon(p) > 0$, and an open neighbourhood $V^p = \{q \in \Delta \,|\, \|q - p\| < \epsilon(p)\}$ of $p$ such that for all $q \in V^q$ and $z \in \zeta(q)$, $y(p) \cdot z > 0$. Since $\Delta$ is compact, there is a finite set $\{p^1, \cdots, p^m\} \subset \Delta$ such that $\Delta \subset \bigcup_{t=1}^m V^{p^t}$. For each $p \in \Delta$ and for each $t = 1, \cdots, m$, let $\theta_t(p) = \max\{\epsilon(p^t) - \|p - p^t\|, 0\}$. where $\|\cdot\|$ denotes the Euclidean norm. Define for each $t = 1, \cdots, m$, a function $\beta_t : \Delta \to [0,1]$ as $\beta_t(p) = \theta_t(p)/\sum_{s=1}^m \theta_s(p)$, (the partition of unity subordinated to the covering $\{V^{p^t}\}_{t=1}^m$ of $\Delta$). Then, the mapping $f : \Delta \to \Delta$,

$f : \Delta \ni p \mapsto \sum_{t=1}^m \beta_t(p) y(p^t) \in \Delta,$

[1] Throughout this section, the inner product of two vectors, $x$, $y$, is denoted by $x \cdot y$.

is easily seen to be continuous. Since $\forall p \in \Delta$, $\beta_t(p) > 0$ iff $p \in V^{p^t}$ and since for all $t$, $p \in V^{p^t}$, and $z \in \zeta(p)$, $y(p^t) \cdot z > 0$, we have $f(p) \cdot z > 0$ for all $p \in \Delta$ and $z \in \zeta(p)$. Then, since we have for all $p \in \Delta$ and $z \in \zeta(p)$, $p \cdot z \leq 0$ and $f(p) \cdot z > 0$, it follows that $p \neq f(p)$ for all $p \in \Delta$. That is, $f$ has no fixed point, which contradicts the Brouwer's fixed point theorem. □

Besides the closed convex valuedness together with the upper semi-continuity, one of the sufficient condition for (LDD) is the following local definiteness condition on the *sign* of excess demands.

> (LDS) For all $p \in \Delta$, if $\zeta(p) \cap -R^\ell_+ = \emptyset$, then there exist a coordinate $i^p \in \{1, 2, \cdots, \ell\}$ and an open neighbourhood $U^p$ of $p$ in $\Delta$ such that the $i^p$-th coordinate correspondence $\zeta_{i^p}$ of $\zeta$ satisfies $\zeta_{i^p}(q) \subset R_{++}$ for all $q \in U^p$.

That is, if $p$ is not an equilibrium price vector, there exists at least one commodity such that for all prices near $p$ the sign of excess demands for the commodity is always positive. When $\zeta$ is single valued, condition (LDS) is weaker than the continuity of $\zeta$ though it is still stronger than condition (LDD).[2] Unfortunately, however, when $\zeta$ is multi-valued, condition (LDS) may not be satisfied even if $\zeta$ is upper semicontinuous and closed convex valued.

The concept underlying the condition (LDD) as well as the method for proving Theorem 1 may be directly applicable to fixed point arguments for multi-valued mappings. Let $X$ be a non-empty compact convex subset of $R^\ell$ and let $\varphi$ be a non-empty closed convex valued upper semi-continuous correspondence on $X$ to itself. (Hence, the situation is that for which we may apply the fixed point theorem of Kakutani (1941).) If $x \in X$ is not a fixed point of $\varphi$, then $x$ does not belong to the non-empty closed convex set $\varphi(x)$, so that there is a hyperplane $H_x$ which strictly separates $\{x\}$ and $\varphi(x)$. Since $\varphi$ is upper semi-continuous, there is an open neighbourhood $U(x)$ of $x$ such that $H_x$ also strictly separates $z$ and $\varphi(z)$ for all $z \in U(x)$, i.e., all elements $v \in \varphi(z) - z$ for all $z$ near $x$ may be evaluated positively via the inner product with a vector $p(x)$ which is normal to $H_x$. In other words, we can say that the set of variation $\varphi(z) - z$ from $z$ near $x$ has *a locally common direction* represented by the vector $p(x)$.

The next theorem shows that the above condition on variations of the correspondence is indeed a *sufficient* condition for the existence of fixed points.

THEOREM 2 : (A generalization of Kakutani's fixed point theorem) *Let $\varphi$ be a non-empty valued correspondence on a compact convex subset $X$ of $R^\ell$ to $X$ satisfying one of the following two conditions:*

> (LDV1) *For each $x \in X$ such that $x \notin \varphi(x)$, there exist a vector $p(x) \in R^\ell$ and an open neighbourhood $U^x$ of $x$ such that $p(x) \cdot (w - z) > 0$ for all $z \in U^x$ and $w \in \varphi(z)$.*
>
> (LDV2) *For each $x \in X$ such that $x \notin \varphi(x)$, there exist a vector $y(x) \in X$ and an open neighbourhood $U^x$ of $x$ such that $(y(x) - z) \cdot (w - z) > 0$ for all $z \in U^x$ and $w \in \varphi(z)$.*

*Then, $\varphi$ has a fixed point $p^* \in \varphi(p^*)$.*

PROOF : The theorem under (LDV1) will be generalized in the next section.

(Case: LDV1) Suppose $x \notin \varphi(x)$ for all $x \in X$. Then, by (LDV1), for each $x \in X$, there exist $p(x) \in X$ and an open neighborhood $U^x$ of $x$ in $X$ such that $p(x) \cdot (w - z) > 0$ for all $z \in U^x$ and $w \in \varphi(z)$. Since

[2] Hayashi (1997) shows the existence of an equilibrium price $p^*$, $\zeta(p^*) \cap R^\ell_- \neq \emptyset$, for a single valued $\zeta$ under (W), (LDS), and a certain kind of boundary condition by using Eaves' theorem (Eaves, 1974).

$X \subset \bigcup_{x\in X} U^x$ and since $X$ is compact, the covering $\{U^x | x \in X\}$ has a finite subcovering $\{U^{x^t}\}_{t=1}^m$. Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, m$, be the partition of unity subordinated to $\{U^{x^t}\}_{t=1}^m$. Let us consider a function $f : X \to R^\ell$ such that $f(x) = \sum_{t=1}^m \beta_t(x)p(x^t)$. Moreover, let $\Psi : R^\ell \to X$ be the correspondence defined as $\Psi(v) = \{x \in X | v \cdot x = \max_{z\in X} v \cdot z\}$. Clearly, $f$ is continuous and $\Psi$ is a non-empty closed convex valued correspondence having a closed graph, so that we have a fixed point $\hat{x}$ for $\Psi \circ f$ by Kakutani's fixed point theorem. By definitions of $f$ and $\Psi$, we have $\sum_{t=1}^m \beta_t(\hat{x})p(x^t) \cdot \hat{x} \geq \sum_{t=1}^m \beta_t(\hat{x})p(x^t) \cdot z$ for all $z \in X$. On the other hand, since $\hat{x} \in U^{x^t}$ for at least one $t \in \{1, \cdots, m\}$, we have also $\sum_{t=1}^m \beta_t(\hat{x})p(x^t) \cdot (w - \hat{x}) > 0$ for all $w \in \varphi(\hat{x})$, i.e., $\sum_{t=1}^m \beta_t(\hat{x})p(x^t) \cdot w > \sum_{t=1}^m \beta_t(\hat{x})p(x^t) \cdot \hat{x}$ for all $w \in \varphi(\hat{x})$, a contradiction.

(Case: LDV2) Suppose that $\varphi$ does not have a fixed point. Then, under (LDV2), we have, for each $x \in X$, a vector $y(x) \in X$ and an open neighbourhood $U^x$ of $x$ such that for all $z \in U^x$ and $v \in \varphi(z) - z$, $(y(x) - z) \cdot v > 0$. Since $X$ is compact, we have a finite set $\{x^1, \cdots, x^m\} \subset X$ such that $X \subset \bigcup_{t=1}^m U^{x^t}$. Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, m$ be the partition of unity subordinated to $\{U^{x^t}\}_{t=1}^m$. Then, the mapping $f : X \to X$ defined as $f : X \ni x \mapsto x + \sum_{t=1}^m \beta_t(x)(y(x^t) - x) \in X$ is continuous. Since $\forall x \in X$, $\beta_t(x) > 0$ iff $x \in V^{x^t}$, and since for each $t$, $x \in V^{x^t}$ and $v \in \varphi(x) - x$ implies $(y(x^t) - x) \cdot v > 0$, we have for all $x \in X$, $(f(x) - x) \cdot v > 0$ for all $v \in \varphi(x) - x$. It follows that $f(x) \neq x$ for all $x \in X$, which contradicts the fact that $f$ has a fixed point by the Brouwer's fixed point theorem. □

As an immediate corollary to Theorem 2, we obtain a generalization of the result in Nishimura and Friedman (1981). Nishimura-Friedman's theorem is well-known in the economic equilibrium theory as the only instance of an equilibrium theorem for N-person games which does not essentially depend on the convexity for preferences. Indeed, Nishimura-Friedman's theorem is a Nash equilibrium existence theorem based on the best reply correspondences satisfying a definiteness condition on the local *sign* of at least one coordinate of the mapping. As is the case with excess demand correspondences under condition (LDS), however, the condition for Nishimura-Friedman's theorem may not be satisfied for the usual case in which the best reply correspondences are closed convex valued and upper semi-continuous. Our work offers a generalization of Nishimura-Friedman's theorem based on the best reply correspondences satisfying a definiteness condition on the local *direction* indicated by the value of the mapping, so that we may apply it for correspondences which are upper semi-continuous and closed convex valued.

Corollary 1 : (A generalization of Nishimura-Friedman's theorem) *Consider a strategic form game such that for each player $i \in I = \{1, 2, \cdots, n\}$,*[3] *the strategy set $S_i \subset R^\ell$ is compact and convex. Suppose that there is a family of non-empty valued best reply correspondences, $r_i : S = \prod_{j\in I} S_j \to S_i$, $i \in I$, satisfying one of the following two conditions.*

(i) *For each $s = (s_j)_{j\in I} \in S$ such that $s \notin \prod_{j\in I} r_j(s)$, there exist a player $i$, a vector $p_i(s) \in R^\ell$, and an open neighbourhood $U(s)$ of $s$ satisfying that for all $t = (t_j)_{j\in I} \in U(s)$, $p_i(s) \cdot (r_i(t) - t_i) > 0$.*

(ii) *For each $s = (s_j)_{j\in I} \in S$ such that $s \notin \prod_{j\in I} r_j(s)$, there exist a player $i$, a point $y_i(s) \in S_i$, and an open neighbourhood $U(s)$ of $s$ satisfying that for all $t = (t_j)_{j\in I} \in U(s)$, $(y_i(s) - t_i) \cdot (r_i(t) - t_i) > 0$.*

*Then, there is an equilibrium point $s^* = (s_i^*)_{i\in I} \in S$, $s_i^* \in r_i(s^*)$ for all $i \in I$.*

[3] As in this corollary, we take the index set of players, $I$, as a finite set throughout this section, though there is no difficulty to extend all of our results to allow for an arbitrary cardinality of $I$ as we shall see in section 3.

PROOF : Suppose that condition (i) is satisfied. Then, the mapping $\varphi = \prod_{i\in I} r_i : S \to S$ satisfies (LDV1) in Theorem 2 on $S$ if we set up $p(s)$ for each $s \in S$ such that $s \notin r(s)$ as $p(s) = (p_j(s))_{j\in I}$, where $p_j(s) = 0$ for all $j$ different from $i$ stated in condition (i) for $s$. Hence, we have a fixed point $s^*$ for $r$, a Nash equilibrium point. Next, suppose that the condition (ii) is satisfied and that there is no equilibrium point. For each $j \in I$, denote by $U_j$ the set of all $s \in S$ such that condition (ii) is satisfied if we chose $j$ as the player $i$ in (ii). Clearly, each $U_j$ is open and $\{U_j\}_{j\in I}$ covers $S$. Note that if $(s_i)_{i\in I} \in U_j$ then $s_j \notin r_j((s_i)_{i\in I})$. Let us define for each $j \in I$, a correspondence $\varphi_j : S \to S_j$ as $\varphi_j((s_i)_{i\in I}) = \{s_j\}$ if $(s_i)_{i\in I} \notin U_j$, and $\varphi_j(s) = r_j(s)$ if $s \in U_j$. Then, condition (ii) implies (LDV2) in Theorem 2 for $\varphi = \prod_{j\in I} \varphi_j : S \to S$. Hence, $\varphi$ has a fixed point, which is impossible since $\{U_j\}_{j\in I}$ covers $S$. □

Though it is possible to apply Theorem 2, directly, to the Nash equilibrium existence problem, we should note that for certain kinds of correspondences such as preference correspondences, the set $\varphi(x) - x$ may typically fail to satisfy conditions stated in (LDV1) or (LDV2) in Theorem 2. (For example, preference correspondences represented by a strictly concave and differentiable utility function necessarily fail to satisfy those conditions.) In corollary 1, each correspondence $r_i$ is not the preference correspondence but a selection of the preference correspondence, and by assuming the condition on local directions of such a selection, we have obtained a rather weak condition for the Nash equilibrium existence. More generally, it can be said that if we could suitably modify each preference correspondence into another correspondence satisfying the condition on local direction of the mapping, then the Nash equilibrium exists. As one of the simplest and the most powerful ways for such a deformation, we introduce the following notion: let us consider the set $Y(x) = \{v | v \cdot w > 0 \text{ for all } w \in \varphi(x) - x\}$, say, the strict polar (cone) of the set $\varphi(x) - x$, and assume that such a polar to have a locally common element. In Corollary 2, we present two conditions, (N1) and (N2), on the polar of local variations, together with condition (K) of the direct application of (LDV1), to assure the existence of Nash equilibria. If a preference correspondence is convex and irreflexive, (N1) is automatically satisfied if it has an open graph, and (N2) is satisfied if it has open lower sections.[4] Each of (N1) and (N2), however, is far more general than to assume those global continuity conditions, e.g., we may apply (N1) or (N2) even for preferences having lexicographic orderings.[5]

COROLLARY 2 : (Existence of Nash equilibria I) *Consider an n-person game such that for each player $i = 1, \cdots, n$, the strategy set $S_i \subset R^\ell$ is non-empty, compact, and convex. For each $i = 1, \cdots, n$, let $\mathscr{P}_i : S = \prod_{i=1}^n S_i \to S_i$ be a (possibly empty valued) correspondence. Moreover, for each $i = 1, \cdots, n$, and $t = (t_1, \cdots, t_n) \in S$, let $Y_i(t) = \{v \in R^\ell | v \cdot w > 0$ for all $w \in \mathscr{P}_i(t) - t_i\}$. Suppose that each player i satisfies one of the following conditions.*

(K) *For each $s \in S$ such that $\mathscr{P}_i(s) \neq \emptyset$, there are an open neighbourhood $U(s)$ of $s$ and a vector $p_i(s) \in R^\ell$ satisfying that for all $t = (t_1, \cdots, t_n) \in U(s)$, $\mathscr{P}_i(t) \neq \emptyset$, and $\forall w \in \mathscr{P}_i(t) - t_i$, $p_i(s) \cdot w > 0$.*

(N1) *For each $s \in S$ such that $\mathscr{P}_i(s) \neq \emptyset$, there are an open neighbourhood $U(s)$ of $s$ and a vector $p_i(s) \in R^\ell$ satisfying that for all $t = (t_1, \cdots, t_n) \in U(s)$, we have $Y_i(t) \neq \emptyset$ and $\forall v \in Y_i(t)$, $p_i(s) \cdot v > 0$.*

[4] If a preference $\mathscr{P}$ on $X \subset R^\ell$ has an open graph and $\mathscr{P}(x) \neq \emptyset$, there are a neighbourhood $W(x)$ of $x$ and a point $y \in \mathscr{P}(x)$ such that $y + (W(x) - x) \subset X$ and $\forall w \in y + W(x)$, $\forall z \in W(x)$, $w \in \mathscr{P}(z)$. Hence, on $U(x)$ we have a locally common superior direction, $y - x$, so that (N1) is satisfied if we set the direction as $p(x)$. If $\mathscr{P}$ has open lower sections, $y \in \mathscr{P}(x)$ means that for a neighbourhood $U(x)$ of $x$, $y \in \mathscr{P}(z)$ for all $z \in U(x)$. Hence, on $U(x)$ we have a locally common superior $y$, so that (N2) is satisfied if we set the $y$ as $y(x)$.

[5] Though we cannot expect lexicographic preference orderings to have open lower sections, it is still possible to assume that they have at each point locally common superior directions, (e.g., $(1, 1, \cdots, 1)$ for monotonic preferences,) and/or a locally common superior point. The former is a sufficient condition for (N1) and the latter is a sufficient condition for (N2).

(N2) *For each* $s \in S$ *such that* $\mathscr{P}_i(s) \neq \emptyset$, *there are an open neighbourhood* $U(s)$ *of* $s$ *and a point* $y_i(s) \in S_i$ *satisfying that for all* $t = (t_1, \cdots, t_n) \in U(s)$, *we have* $Y_i(t) \neq \emptyset$ *and* $\forall v \in Y_i(t)$, $v \cdot (y_i(s) - t_i) > 0$.

*Then, there is an equilibrium* $s^* \in S$, $\mathscr{P}_i(s^*) = \emptyset$ *for all* $i$.

PROOF : Omitted since the next corollary is more general. □

As conditions for preference correspondences, (N1) or (N2) may be sufficiently general for assuring the existence of Nash equilibrium. In many applications such as in the social equilibrium existence problem (Corollary 4), however, it is more convenient to show the existence of Nash equilibria under the generality we shall see in the next corollary. The condition (NK1) (resp., (NK2)) may be considered, intuitively, as a mixed condition of (LDV1) and (N1) (resp., (LDV1) and (N2)). These conditions are so useful that we may apply them to a correspondence which is upper semi-continuous in a certain area and has open lower sections in another area, and so on, (see Corollary 4).

COROLLARY 3 : (Existence of Nash equilibria II) *Consider an* $n$*-person game such that for each player* $i \in I = \{1, \cdots, n\}$, *the strategy set* $S_i \subset R^\ell$ *is non-empty, compact, and convex. For each* $i \in I$, *let* $\mathscr{P}_i : S = \prod_{j \in I} S_j \to S_i$ *be a (possibly empty valued) correspondence. Moreover, for each* $i \in I$, *and* $t = (t_1, \cdots, t_n) \in S$, *let* $Y_i(t) = \{v \in R^\ell | \, v \cdot w > 0$ *for all* $w \in \mathscr{P}_i(t) - t_i\}$. *Suppose that for each player* $i$ *one of the following two condition is satisfied.*

(NK1) *For each* $s \in S$ *such that* $\mathscr{P}_i(s) \neq \emptyset$, *there are an open neighbourhood* $U(s)$ *of* $s$ *and a vector* $p_i(s) \in R^\ell$ *satisfying:*

(i) $\forall t = (t_1, \cdots, t_n) \in U(s)$, $\mathscr{P}_i(t) \neq \emptyset$ *and* $\forall w \in \mathscr{P}_i(t) - t_i$, $p_i(s) \cdot w > 0$, *or*

(ii) $\forall t = (t_1, \cdots, t_n) \in U(s)$, $Y_i(t) \neq \emptyset$ *and* $\forall v \in Y_i(t)$, $v \cdot p_i(s) > 0$.

(NK2) *For each* $s \in S$ *such that* $\mathscr{P}_i(s) \neq \emptyset$, *there are an open neighbourhood* $U(s)$ *of* $s$ *and a vector* $p_i(s) \in R^\ell$ *or a point* $y_i(s) \in S_i$ *satisfying:*

(i) $\forall t = (t_1, \cdots, t_n) \in U(s)$, $\mathscr{P}_i(t) \neq \emptyset$ *and* $\forall w \in \mathscr{P}_i(t) - t_i$, $p_i(s) \cdot w > 0$, *or*

(ii) $\forall t = (t_1, \cdots, t_n) \in U(s)$, $Y_i(t) \neq \emptyset$ *and* $\forall v \in Y_i(t)$, $v \cdot (y_i(s) - t_i) > 0$.

*Then, there is an equilibrium* $s^* \in S$, $\mathscr{P}_i(s^*) = \emptyset$ *for all* $i \in I$.

PROOF : (Case: NK1) Suppose that (NK1) is satisfied and there is no equilibrium point. Then for each $s \in S$ there is at least one player $i$ such that $\mathscr{P}_i(s) \neq \emptyset$. Since $S$ is compact, there are a finite set $\{s^1, \cdots, s^k, s^{k+1}, \cdots, s^{n'}\} \subset S$, a finite family of open subsets of $S$, $\{U(s^1), \cdots, U(s^k), U(s^{k+1}), \cdots, U(s^{n'})\}$, and a sequence of vectors (together with indices of players) $p_{i^1}(s^1), \cdots, p_{i^k}(s^k), p_{i^{k+1}}(s^{k+1}), \cdots, p_{i^{n'}}(s^{n'})$, such that $S \subset \bigcup_{m=1}^{n'} U(s^m)$, each of $(U(s^m), p_{i^m}(s^m)), m = 1, \cdots, k$, satisfies (i) of (NK1), and each of $(U(s^m), p_{i^m}(s^m)), m = k+1, \cdots, n'$, satisfies (ii) of (NK1). Note that, in finite dimensional cases, for each $m = k+1, \cdots, n'$, we may suppose (by the well known bipolar theorem) that $p_{i^m}(s^m) \in \{\alpha v | \, v \in \mathrm{co}\, \mathscr{P}_{i^m}(t) - t_{i^m}, \alpha > 0\}$ for all $t = (t_1, \cdots, t_n) \in U(s^m)$, so that there exists an $\alpha(t, s^m) > 0$, $t + \alpha(t, s^m) p_{i^m}(s^m) \in \mathrm{co}\, \mathscr{P}_{i^m}(t) \subset S_{i^m}$ for all $t = (t_1, \cdots, t_n) \in U(s^m)$. Let $\beta_m : U(s^m) \to [0,1], m = 1, \cdots, n'$ be a partition of unity subordinated to $\{U(s^1), \cdots, U(s^{n'})\}$, and let $f$ be a function on $U = \bigcup_{m=k+1}^{n'} U(s^m)$ into $S$ such that

$$f(t) = t + \sum_{m=k+1}^{n'} \frac{\beta_m(t)}{\sum_{m'=k+1}^{n'} \beta_{m'}(t)} (0, \cdots, 0, \alpha(t, s^m) p_{i^m}(s^m), 0, \cdots, 0).$$

Moreover, for each $i = 1, \cdots, n$, and $t = (t_1, \cdots, t_n) \in S$, let $r_i(t) = \mathscr{P}_i(t)$ if $\mathscr{P}_i(t) \neq \emptyset$ and $r_i(t) = \{t_i\}$ if otherwise. Define a correspondence $\varphi : S \to S$ as $\varphi(t) = \{f(t)\}$ for each $t \in U$ and $\varphi(t) = \prod_{i=1}^{n} r_i(t)$ for each $t \notin U$. Then, $\varphi$ satisfies (LDV1) in Theorem 2, (here, we use the condition $\mathscr{P}_i(t) \neq \emptyset$ in (i) of (NK1)) so that $\varphi$ has a fixed point $\hat{s} \in S$. If $\hat{s} \in U$, then for each $i^m$ such that $\hat{s} \in U(s^m)$, $Y_{i^m}(\hat{s}) \neq \emptyset$ and $i^m$-th entry of $f(t) - t$ should positively appreciate an arbitrary member of $Y_{i^m}(\hat{s})$, so that $i^m$-th entry of $\hat{s}$ cannot be equal to the $i^m$-th entry of $f(\hat{s})$, i.e., $\hat{s} \notin \varphi(\hat{s})$. On the other hand, if $\hat{s} \notin U$, then for each $i^m$, $m \geq k+1$, $\hat{s} \in U(s^m)$, $\mathscr{P}_{i^m}(\hat{s})$ cannot contain the $i^m$-th entry of $s$ since the condition (i) should be satisfied for the $i^m$. Hence, we have $\hat{s} \notin \varphi(\hat{s})$, a contradiction.

(Case: NK2) Assume that (NK2) is satisfied and there is no equilibrium point. Then, for each $s \in S$ there is a player $i$ such that $\mathscr{P}_i(s) \neq \emptyset$. Since $S$ is compact, we have a finite set $\{s^1, \cdots, s^k, s^{k+1}, \cdots, s^{n'}\}$ of points of $S$, a family of open sets $\{U(s^1), \cdots, U(s^k), U(s^{k+1}), \cdots, U(s^{n'})\}$ which covers $S$, and vectors $p_{i^1}(s^1), \cdots, p_{i^k}(s^k), y_{i^{k+1}}(s^{k+1}), \cdots, y_{i^{n'}}(s^{n'})$ (together with the indices of players, $i^1, \cdots, i^{n'}$), such that each of $(U(s^m), p_{i^m}(s^m)), m = 1, \cdots, k$, satisfies (i) of (NK2), and each of $(U(s^m), y_{i^m}(s^m)), m = k+1, \cdots, n'$, satisfies (ii) of (NK2). Let $\beta_m : U(s^m) \to [0,1], m = 1, \cdots, n'$ be a partition of unity subordinated to $\{U(s^1), \cdots, U(s^{n'})\}$. For each $i \in I$, let $f_i$ be a function on $U_i = \bigcup_{i^m = i, m \geq k+1} U(s^m)$ into $S_i$ such that for each $t = (t_j)_{j \in I} \in U_i$,

$$f_i(t) = t_i + \sum_{i^m = i, m \geq k+1} \frac{\beta_m(t)}{\sum_{i^{m'} = i, m' \geq k+1} \beta_{m'}(t)} (0, \cdots, 0, y_{i^m}(s^m) - t_{i^m}, 0, \cdots).$$

Furthermore, for each $i \in I$, and $t = (t_j)_{j \in I} \in S \setminus U_i$, let $g_i(t) = \mathscr{P}_i(t)$ if $\mathscr{P}_i(t) \neq \emptyset$ and $g_i(t) = \{t_i\}$ if otherwise. Then, let us define for each $i \in I$ a correspondence $r_i : S \to S$ as $r_i(t) = \{f_i(t)\}$ for $t \in U_i$ and $r_i(t) = g_i(t)$ for $t \notin U_i$, and define a correspondence $\varphi : S \to S$ as $\varphi(t) = \prod_{i \in I} r_i$. Since $f_i$ is continuous on $U_i$, $\varphi$ satisfies (LDV1) in Theorem 2 on $U = \bigcup_{i \in I} U_i$. $\varphi$ also satisfies (LDV1) on the interior of $S \setminus U$ since for each $s \in \int (S \setminus U)$ there is at least one $i^m$, $1 \leq m \leq k$, such that $s \in U(s^m) \cap \operatorname{int}(S \setminus U)$. Suppose that $s = (s_j)_{j \in I}$ is an element of the boundary of $U$. Then, there is at least one $U(s^m)$, $1 \leq m \leq k$ such that $s \in U(s^m)$ since $U(s^1), \cdots, U(s^{n'})$ is a cover of $S$. If $s$ is not an element of the boundary of $U_{i^m}$, then $\varphi$ satisfies (LDV1) since $r_{i^m} = g_{i^m} = \mathscr{P}_{i^m}$ satisfies (i) of (NK1). If $s$ is an element of the boundary of $U_{i^m}$, then for each $t \in U(s^m) \cap U(s^{m'}) \subset U_{i^m}$, $i^m = i^{m'}$, we have $p_{i^m}(s^m) \cdot (y_{i_{m'}}(s^{m'}) - t_i) > 0$ since $p_{i^m}(s^m) \in Y_{i^m}(t) = Y_{i_{m'}}(t)$. Therefore, we have $p_{i^m}(s^m) \cdot (f_{i^m}(t) - t_{i^m}) > 0$ for all $t \in U(s^m) \cap U_{i^m}$, so that $\varphi$ satisfies (LDV1) at $s$ because for every $t \in U(s^m) \setminus U_{i^m}, r_{i^m}(t) = \mathscr{P}_{i^m}(t) \neq \emptyset$. Hence, the mapping $\varphi$ satisfies (LDV1) in Theorem 2 on $S$. Let $\hat{s} = (\hat{s}_j)_{j \in I} \in S$ be a fixed point of $\varphi$. If $\hat{s} \in U$, then for each $i^m$ such that $s \in U(s^m)$, $k+1 \leq m \leq n'$, and for each $v_{i^m}(\hat{s}) \in Y_{i^m}(\hat{s})$, $v_{i^m}(\hat{s}) \cdot (f_{i^m}(\hat{s}) - \hat{s}_{i^m}) > 0$ implies that $f(\hat{s}) \neq \hat{s}$ and $\hat{s} \notin \varphi(\hat{s})$. On the other hand, if $\hat{s} \notin U$, then there is at least one $i^m$, $1 \leq m \leq k$, $\hat{s} \in U(s^m)$, and we have $\hat{s}_{i^m} \notin r_{i^m}(\hat{s}) = g_{i^m}(\hat{s})$ under the condition (i) of (NK1) for player $i^m$. Hence, we have $\hat{s} \notin \varphi(\hat{s})$, a contradiction. □

By Corollary 3, we immediately obtain the following social equilibrium existence theorem.

Corollary 4 : (Existence of Social Equilibria) *Consider an abstract economy such that for each agent $i \in I = \{1, \cdots, n\}$, his strategy set $S_i \subset R^\ell$ is compact and convex. For each $i \in I$, let $K_i : S = \prod_{i=1}^{n} S_i \to S_i$ be a non-empty valued (constraint) correspondence, and let $P_i : S \to S_i$ be a (better set) correspondence. Suppose that:*

(i) *For each $i$, $\{s \in S | \, P_i(s) \cap K_i(s) \neq \emptyset\}$ is open.*

(ii) *All of* $P_i$*'s, and all of* $K_i$*'s at* $s = (s_j)_{j \in I}$, $s_i \notin K_i(s)$, *satisfy the same condition as* $\mathscr{P}_i$ *in (NK1) of Corollary 3, or (NK2) of Corollary 3.*

*Then, there is a social equilibrium point* $s^* = (s_i^*)_{i \in I} \in S$, $\forall i \in I$, $s_i^* \in K_i(s^*)$, $P_i(s^*) \cap K_i(s^*) = \emptyset$.

PROOF : For each $i \in I$, and $s = (s_1, \cdots, s_n) \in S$, if $P_i(s) \cap K_i(s) \neq \emptyset$, let $\alpha_i(s) = P_i(s)$, else if $s_i \notin K_i(s)$, let $\alpha_i(s) = K_i(s)$, else let $\alpha_i(s) = \emptyset$. Then $s^* \in S$ is a social equilibrium iff $\forall i \in I$, $\alpha_i(s^*) = \emptyset$. Under (i), the condition (ii) implies that all $\alpha_i$'s satisfy the condition (NK1) in Corollary 3, or (NK2) in Corollary 3. □

Since (NK1) and (NK2) in Corollary 3 are weaker than the open graph property, the open lower section property, and/or the upper semi-continuity, it is a routine task to ascertain that conditions (i) and (ii) in Corollary 4 are automatically satisfied under standard assumptions for the existence of social equilibrium theorems with or without non-ordered preferences of agents (e.g., Debreu (1952), Shafer and H.F.Sonnenschein (1975)). As a theorem for finite dimensional action spaces, it also covers a part of more recent results such as theorems in Borglin and Keiding (1976), Yannelis and Prabhakar (1983), Tan and Yuan (1994), and Bagh (1998). (Indeed, in paracompact spaces, every $\mathscr{L}$-majorized map[6] $P : S \to S^i$ and every upper semi-continuous correspondence $K : S \to S^i$ satisfy condition (ii) of Corollary 4 (see Urai (1999)). Hence, Corollary 2 is a generalization of the maximal element existence theorem in Yannelis and Prabhakar (1983; Corollary 5.1), and Corollary 4 gives a more general result than Theorem 1.6 of Bagh (1998).

It should also be noted that the generalization in Corollary 4 is merely an example among other possibilities such as "to use a mixed condition of (LDV2) and (N1) instead of (NK1) or (NK2)," "to assume (NK1) or (NK2) directly on the variation of $\alpha_i$," "to assume (LDV1) or (LDV2) on the intersection of $\alpha_i$ and its polar $Y_i$," and so on. Each of these conditions gives one of the most general results for the existence of social equilibria. Even more important is the fact that at least for economic equilibrium results treated in this paper, our generalization of Kakutani's fixed point theorem enables us *to reconstruct the economic equilibrium theory without referring to any convexity and/or global continuity conditions,* without loss of generality.

## 3 INFINITE DIMENSIONAL CASES

In this section, we generalize some results in the previous section to the case with infinite dimensional topological vector spaces. Let $E$ be a locally convex topological vector space over the real field $R$, and let $E'$ be the topological dual of $E$. Let $P$ be a compact and closed convex nonempty subset of $E'$ with $\sigma(E', E)$ topology. From the economic context, we may consider $E$ as a commodity space and $P$ as a price set. Moreover, let $K$ be the cone generated by $P$, $K = \{\lambda p | p \in P, \lambda > 0\}$, and $\Gamma$ be the polar of $K$, $\Gamma = \{x \in E | \forall p \in K, p(x) \leq 0\}$. $\Gamma$ is a closed and convex subset of $E$.

Let us consider a non-empty valued correspondence $\zeta : P \to E$ satisfying the following *Walras' law.*

(W) $p(z) \leq 0$ for all $z \in \zeta(p)$ for each $p \in P$.

Of course, $\zeta$ may be considered as an excess demand correspondence from the economic context. A vector $p^* \in P$ is called an *equilibrium price* for $\zeta$ if $\zeta(p^*) \cap \Gamma \neq \emptyset$. At first, we generalize Theorem 1, so that

[6] For the definition, see Yannelis and Prabhakar (1983) and Tan and Yuan (1994).

we show the existence of equilibrium price under the following local definiteness condition for directions of excess demands.

(LDD) For any $p \in P$, whenever $\zeta(p) \cap \Gamma = \emptyset$, there exist a vector $y^p \in P$ and an open neighborhood $U^p$ of $p$ in $P$, such that $y^p(z) > 0$ for all $z \in \zeta(q)$ for all $q \in U^p$.

As in the finite dimensional case, it is easy to check that the condition is automatically satisfied when the excess demand correspondence is upper semicontinuous and closed convex valued.[7] Hence, the following result is a generalization of the market equilibrium existence theorem of Debreu (1956).

THEOREM 3 : *Under conditions* (W) *and* (LDD), *there exists a* $p^* \in P$ *such that* $\zeta(p^*) \cap \Gamma \neq \emptyset$.

PROOF : Consider a correspondence $G : P \to P$ such that

$$G(p) = \{v \in P | \exists U^p, \forall q \in U^p, \forall z \in \zeta(q), v(z) > 0\}$$

Suppose $\zeta(p) \cap \Gamma = \emptyset$ for any $p \in P$. Then from the condition (LDD), $G$ is nonempty valued. Moreover, it is easily seen that $G$ is convex valued and has open lower sections. Therefore, $G$ has a continuous selection $g : P \to P$ (see Klein and Thompson (1984; p.97, Theorem 8.1.3)). By the definition of $G$, $g(p)(z) > 0$ for all $z \in \zeta(p)$ at each $p \in P$. Under the condition (W), however, $p(z) \leq 0$. Hence, we have $g(p) \neq p$ for all $p \in P$, which contradicts Tychonoff's fixed point theorem. □

We should remark that Theorem 3 is one of the most general versions of, the so called, Gale-Nikaido-Debreu theorem in the literature. Let us compare our results with Theorem 8 in Mehta and Tarafdar (1987). With respect to the continuity, their result seems to have the same generality as ours, though they use the convex and compact valuedness for the correspondence.

We may also generalize the first part of Theorem 2 to the case with locally convex topological vector spaces. Of course, the result may also be considered as a generalization of Fan-Glicksberg's Fixed point theorem (Fan (1952), Glicksberg (1952)) in the sense that every upper semi-continuous closed convex valued correspondence satisfies the condition (LDV).

THEOREM 4 : (Further generalization of Kakutani's fixed point theorem) *Let* $X$ *be a non-empty compact convex subset of a locally convex topological vector space* $E$ *over* $R$. *Assume that a non-empty valued correspondence* $\varphi : X \to X$ *satisfies the following local definiteness condition for variations.*

(LDV) *For each* $x \in X$ *such that* $x \notin \varphi(x)$, *there exist a vector* $p^x \in E'$ *and an open neighborhood* $U^x$ *of* $x$ *in* $X$ *such that* $p^x(w - z) > 0$ *for all* $z \in U^x$ *and* $w \in \varphi(z)$.

*Then, there exists a fixed point* $x^* \in \varphi(x^*)$.

PROOF : Suppose $x \notin \varphi(x)$ for all $x \in X$. Then, by (LDV), for each $x \in X$, there exist $p^x \in E'$ and an open neighborhood $U^x$ of $x$ in $X$ such that $p^x(w - z) > 0$ for all $z \in U^x$ and $w \in \varphi(z)$. Since

[7] It is sufficient to show that any upper semi-continuous and nonempty compact convex valued correspondence $\zeta$ satisfies the condition (LDD). Suppose $\zeta(p) \cap \Gamma = \emptyset$ for $p \in P$. By the second separation theorem (see Schaefer (1971; p.65, 9.2)), there exists $y^p \in E'$ such that $\sup_{x \in \Gamma} y^p(x) < 0 < \inf_{z \in \zeta(p)} y^p(z)$. From the bipolar theorem (see Schaefer (1971; p.126, 1.5)), $y^p \in K$, and we may chose $y^p$ in $P$. The upper semicontinuity of $\zeta$ means clearly that there exists an open neighborhood $U^p$ of $p$ in $P$ such that $\zeta(U^p) \subset \{z \in E | \ y^p(z) > 0\}$.

$X \subset \bigcup_{x \in X} U^x$ and since $X$ is compact, the covering $\{U^x |\ x \in X\}$ has a finite subcovering $\{U^{x^t}\}_{t=1}^m$. Let $\beta_t : X \to [0,1]$, $t = 1, \cdots, m$, be the partition of unity subordinated to $\{U^{x^t}\}_{t=1}^m$. Let us consider a function $\Phi : X \to E'$ such that $\Phi(x) = \sum_{t=1}^n \beta_t(x) p^{x^t}$. Moreover, let $\Psi$ be a correspondence on $E'$ to $X$ such that $\Psi(p) = \{x \in X | p(x) = \max_{y \in X} p(y)\}$. Since $X$ is compact, and since each $\beta_t$, $p^{x^t}$ are continuous, $\Phi$ is continuous and $\Psi$ is non-empty compact convex valued upper semi-continuous correspondence. Hence, $\Psi \circ \Phi$ has a fixed point $\hat{x} \in \Psi(\Phi(\hat{x}))$ under Fan-Glicksberg's fixed point theorem. By definitions of $\Phi$ and $\Psi$, we have $\sum_{t=1}^m \beta_t(\hat{x}) p^{x^t}(y) \leq \sum_{t=1}^m \beta_t(\hat{x}) p^{x^t}(\hat{x})$ for all $y \in X$. On the other hand, since $\hat{x} \in U^{x^t}$ for at least one $t \in \{1, \cdots, m\}$, we have also $\sum_{t=1}^m \beta_t(\hat{x}) p^{x^t}(w - x) > 0$ for all $w \in \varphi(x)$, i.e., $\sum_{t=1}^m \beta_t(\hat{x}) p^{x^t}(\hat{x}) < \sum_{t=1}^m \beta_t(\hat{x}) p^{x^t}(w)$ for all $w \in \varphi(\hat{x})$, a contradiction. □

It is immediately apparent that Theorem 4 gives the following generalization of Corollary 1. In the following, the cardinal number of the set $I$ of players is arbitrary as noted in the previous section.

Corollary 5 : (Further generalization of Nishimura-Friedman's theorem) *Let $E$ be a locally convex topological vector space over $R$ and let $E'$ be its topological dual. Consider a strategic form game such that for each agent $i \in I$, his strategy set $S_i \subset E$ is compact and convex. Suppose that there is a family of non-empty valued best reply correspondences, $r_i : S = \prod_{i \in I} \to S_i$, $i \in I$, satisfying the following condition.*

> *For each $s = (s_i)_{i \in I} \in S$ such that $s \notin \prod_{i \in I} r_i(s)$, there are a player $i$, a vector $p_i^s \in E'$, and an open neighbourhood $U(s)$ of $s$ satisfying for all $t = (t_i)_{i \in I} \in U(s)$, $p_i^s(r_i(t) - t_i) > 0$.*

*Then, there is an equilibrium point $s^* = (s_i^*)_{i \in I} \in S$, $s_i^* \in r_i(s^*)$ for all $i \in I$.*

Proof : Use Theorem 4 instead of Theorem 2 in the proof for case (i) of Corollary 1. □

The following is a generalization of case (NK2) of Corollary 3.[8]

Corollary 6 : (Existence of Nash equilibria: Infinite Dimensional Case) *Let $E$ be a locally convex topological vector space over $R$ and let $E'$ be its topological dual. Consider a strategic form game such that for each player $i \in I$, the strategy set $S_i \subset E$ is non-empty, compact, and convex. For each $i \in I$, let $\mathscr{P}_i : S = \prod_{i \in I} S_i \to S_i$ be a (possibly empty valued) correspondence. Moreover, for each $i \in I$, and $t = (t_j)_{j \in I} \in S$, let $Y_i(t) = \{p \in E' | p(w) > 0$ for all $w \in \mathscr{P}_i(t) - t_i\}$. Suppose that for each player $i \in I$ the following condition is satisfied.*

> (NK) *For each $s \in S$ such that $\mathscr{P}_i(s) \neq \emptyset$, there are an open neighbourhood $U(s)$ of $s$ and a vector $p_i^s \in Y_i(s)$ or a point $y_i^s \in S_i$ satisfying:*
>
> (i) *$\forall t = (t_j)_{j \in I} \in U(s)$, $\mathscr{P}_i(t) \neq \emptyset$, and $\forall w \in \mathscr{P}_i(t) - t_i$, $p_i^s(w) > 0$, or*
>
> (ii) *$\forall t = (t_j)_{j \in I} \in U(s)$, $Y_i(t) \neq \emptyset$ and $\forall p \in Y_i(t)$, $p(y_i^s - t_i) > 0$.*

*Then, there is an equilibrium $s^* \in S$, $\mathscr{P}_i(s^*) = \emptyset$ for all $i \in I$.*

Proof : Use Theorem 4 instead of Theorem 2 in the proof for case (NK2) of Corollary 3. □

[8] We omit the generalization of case (NK1). An alternative approach for the existence of Nash equilibrium together with a generalization of the fixed point theorem of Browder (1968) may be seen in Urai (1998). For a complete treatment of this problem, see Urai (1999).

The social equilibrium existence theorem as a generalization of Corollary 4 is an immediate consequence of Corollary 6. As stated in the previous section, the result is one of the most general social equilibrium existence theorems which is not covered in the literature (e.g., Yannelis and Prabhakar (1983), Tan and Yuan (1994), Bagh (1998), etc.) The theorem gives in particular the most general social equilibrium existence theorem among these as long as the dimension of $E$ and the cardinality of $I$ are finite.

COROLLARY 7 : (Existence of Social Equilibria: Infinite Dimensional Case) *Let $E$ be a locally convex topological vector space over $R$ and let $E'$ be its topological dual. Consider an abstract economy such that for each agent $i \in I$, the strategy set $S_i \subset E$ is non-empty, compact, and convex. For each $i \in I$, let $K_i : S = \prod_{i \in I} S_i \to S_i$ be a non-empty valued (constraint) correspondence, and let $P_i : S \to S_i$ be a (better set) correspondence. Suppose that:*

(i) *For each $i$, $\{s \in S |\, P_i(s) \cap K_i(s) \neq \emptyset\}$ is open.*

(ii) *All of $P_i$'s, and all of $K_i$'s at $s = (s_j)_{j \in I}$, $s_i \notin K_i(s)$, satisfy the same condition as $\mathscr{P}_i$ in (NK) of Corollary 6.*

*Then, there is a social equilibrium point $s^* = (s_i^*)_{i \in I} \in S$, $\forall i \in I$, $s_i^* \in K_i(s^*)$, $P_i(s^*) \cap K_i(s^*) = \emptyset$.*

PROOF : Use Corollary 6 instead of Corollary 3 in the proof of Corollary 4. □

Of course, conditions (i) and (ii) are automatically satisfied when each $K_i$ is closed convex valued and continuous and each $P_i$ satisfies the same condition as $\mathscr{P}_i$ in Corollary 6.

*(Graduate School of Economics, Osaka University, Toyonaka, Osaka 560-0043, JAPAN.)*

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