Lecture Note #02 of
Econometrics I & Advanced Econometrics I (2013SY)

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April 17, 2013

§ 1. The Foundation of Set Theory

Probability represents a scale (or measure) on likeliness of a certain event (ω) to be realized. In this sense, the probability is a function such that $P : \omega(= \text{event}) \longrightarrow [0, 1]$, i.e. probability measure.

Events are introduced in the set theory. Their features concerning on probability are as follows;

Feature 1 Introducing the collection of sets; family class / field of sets.

Feature 2 Introducing the sequence of sets and its limit.

§ 1.1 Basic Concepts of Sets

We define a **set** as a collection of **elements**. We denote

$\omega \in A$, when an element ω is included in the set A

and

$\omega \notin A$, when an element ω is not included in the set A,

respectively. We call set A as 'finite set' when the number of elements in A is finite and call it as 'infinite set' when it is infinite. In addition, we call A as 'countable set' if we could label each element in it with unique number.

When sets A and B exist, if

$$\omega \in A \quad \Rightarrow \quad \omega \in B$$

then we call A as a **subset** of B and represent as $A \subset B$.

When $A \subset B$ and $A \supset B$ are satisfied, A equals to B, that is to say $A = B$.

Next, we define special sets.

empty set The set with no element. It is denoted by $\emptyset$. (Remind that it is different from Greek letter ϕ.)

universal set The set of all elements under consideration. It is denoted by Ω. (In the context of probability theory, it is called the **sample space** or the **state space**.)

Union, intersection, and difference

We will define union, intersection, and difference of sets A and B.

Intersection

$$A \cap B \equiv \{\omega : \omega \in A \text{ and } \omega \in B\}$$

When $A \cap B = \emptyset$, we note that A and B are disjoint.

Union

$$A \cup B \equiv \{\omega : \omega \in A \text{ or } \omega \in B\}$$

Difference

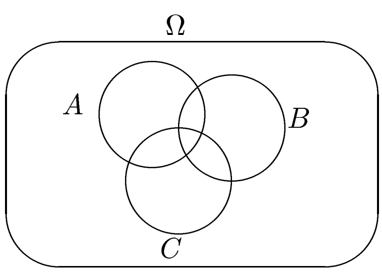
$$\begin{aligned} A \setminus B &\equiv \{\omega : \omega \in A \text{ and } \omega \notin B\} \\ &= A \cap B^C \end{aligned}$$

Compliment

$$\begin{aligned} A^C &\equiv \{\omega : \omega \notin A \text{ and } \omega \in \Omega\} \\ &= \Omega \setminus A \end{aligned}$$

Theorem 1.1.1 Basic Theorems for Sets

$$\begin{array}{lrcl} \text{Associativity} & A \cap (B \cap C) &=& (A \cap B) \cap C \\ & A \cup (B \cup C) &=& (A \cup B) \cup C \\ \text{Distributivity} & A \cup (B \cap C) &=& (A \cup B) \cap (A \cup C) \\ & A \cap (B \cup C) &=& (A \cap B) \cup (A \cap C) \end{array}$$



Theorem1.1.2 De Morgan's Laws (HMC p.6 Example 1.2.17)

$$\begin{aligned}(A \cup B)^C &= A^C \cap B^C \\ (A \cap B)^C &= A^C \cup B^C\end{aligned}$$

§ 1.2 Sequences of sets

Let $A_1, A_2, A_3, \cdots, A_n$ be sequences of sets (including the case of $n = \infty$). We denote unions and intersections of these sequences by

$$\begin{aligned}\bigcap_{i=1}^{n} A_i &= A_1 \cap A_2 \cap \cdots \cap A_n, \\ \bigcup_{i=1}^{n} A_i &= A_1 \cup A_2 \cup \cdots \cup A_n.\end{aligned}$$

In particular, if $A_i \cap A_j = \emptyset$ for $^\forall i \neq j$, we have

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i.$$

Theorem 1.2.1

$$\begin{aligned}\left(\bigcup_{i=1}^{n} A_i\right)^c &= \bigcap_{i=1}^{n} A_i^c \\ \left(\bigcap_{i=1}^{n} A_i\right)^c &= \bigcup_{i=1}^{n} A_i^c\end{aligned}$$

The Limit of a Sequence of Sets

Next, we will define the limit of a sequence of sets. The limit supremum and the limit infimum of a sequence of sets are defined as follows;

$$\begin{aligned}\limsup_n A_n &\equiv \bigcap_{n=1}^{\infty} (\bigcup_{i=n}^{\infty} A_i), \\ \liminf_n A_n &\equiv \bigcup_{n=1}^{\infty} (\bigcap_{i=n}^{\infty} A_i).\end{aligned}$$

The limit of a sequence of sets exists if the limit supremum of it coincides with the limit infimum of it. That is to say,

$$\lim_n A_n = \limsup_n A_n = \liminf_n A_n.$$

$$\left.\begin{array}{ll}\text{increasing sequence} & A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \\ \text{decreasing sequence} & A_1 \supset A_2 \supset \cdots\end{array}\right\} \text{monotone sequence}$$

Theorem 1.2.2

We have following properties for the limit of a monotone sequence.

$$\begin{array}{llcll} (1) & \lim A_n & = & \bigcup_{n=1}^{\infty} A_n & \text{if increase sequence} \\ (2) & \lim A_n & = & \bigcap_{n=1}^{\infty} A_n & \text{if decreasing sequence} \end{array}$$

§ 1.3 Family class (Field of sets, Algebra)

Next, suppose the collection of sets, i.e. the set having sets as its elements. We denote all of subsets of Ω including $\emptyset$ by 2^{Ω}. We will introduce the concepts of a **family class** (or a **field**) for $\mathcal{F}$, a subset of 2^{Ω}, i.e. $\mathcal{F} \subset 2^{\Omega}$.

Definition 1.3.1

$\mathcal{F}$, a collection of a subset of non-empty set Ω satisfies the following properties, we call $\mathcal{F}$ as a family class (or field, algebra).

$$\begin{array}{lrcl} (1) & \Omega \in \mathcal{F} & & \\ (2) & A \in \mathcal{F} & \Rightarrow & A^C = \Omega \setminus A \in \mathcal{F} \\ (3) & A, B \in \mathcal{F} & \Rightarrow & A \cup B \in \mathcal{F} \end{array}$$

Note:

We can extend (3) to

$$(3)' \quad A_i \in \mathcal{F} \quad (i = 1, 2, \ldots, n) \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F} \qquad \text{finite additivity}$$

for $n < \infty$.

Theorem 1.3.2

In addtion (1) and (2), if

$$(4) \quad A_i \in \mathcal{F} \quad (i = 1, 2, \ldots) \quad \Rightarrow \quad \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \qquad \text{complete additivity}$$

are satisfied, we call $\mathcal{F}$ as σ-field (or σ-algebra).

Theorem 1.3.3

If it satisfies complete additivity, then it satisfies finite additivity.

Note: Borel set

Ex : When $\Omega = \boldsymbol{R}$, we call σ-algebra generated by $\{(a,b]; a < b,\ a, b \in \boldsymbol{R}\}$ as (Euclidean)Borel σ-algebra.

§ 2. Probability and Probability Space

§ 2.1 Family Class and Measure

Definition 2.1.1 Measure

Let $\mathcal{A}$ be an algebra (which satisfies finite additivity) generated by a subset of Ω.
For $A \cap B = \emptyset$ 、 $A, B \in \mathcal{A}$, if a function that $\mu : A \in \mathcal{A} \to [0, \infty]$ is satisfied with

$$\mu(A + B) = \mu(A) + \mu(B),$$

then we denote μ as **a finite additive measure**. We define

$$\mu \text{ is finite} \quad \text{if } \mu(\Omega) < \infty$$
$$\mu \text{ is a probability measure and denoted by } P(\cdot). \quad \text{if } \mu(\Omega) = 1$$

In fact, to be a probability measure, an algebra $\mathcal{A}$ must be σ-algebra.

Theorem 2.1.2 Probability Measure

Let $\mathcal{B}$ be a σ-algebra generated a subset of Ω.
If a function $P : A \in \mathcal{B} \to [0, 1]$ satisfies the following properties, we call P as a **probability measure** (or simply **probability**).

(1) $P(A) \geq 0$

(2) $P(\Omega) = 1$

(3) For A_i , $A_j \in \mathcal{B}$ and $A_i \cap A_j = \emptyset \quad (i \neq j)$,

$$P\{\sum_{i=1}^{\infty} A_i\} = \sum_{i=1}^{\infty} P(A_i) \qquad \text{(complete additivity)}$$

Definition 2.1.3 Probability Space ("Triple")

A combination of Ω and (smallest) σ-algebra generated by a subset of Ω, and the probability measure defined on it is called as a **probability space** and is denoted by $(\Omega, \mathcal{B}, P)$.

Theorem 2.1.4

Let $(\Omega, \mathcal{B}, P)$ be a probability space $(i = 1, \cdots, \infty)$.
For $A,\ B,\ C_i,\ D_i \in \mathcal{B}$, the following properties are satisfied.

(a) $P(\emptyset) = 0$

(b) $P(\sum_{i=1}^{n} C_i) = \sum_{i=1}^{n} P(C_i), \qquad C_i \cap C_j = \emptyset\ (i \neq j)$ (finite addtivity)

(c) $P(A^C) = 1 - P(A)$

(d) $A \subset B \Rightarrow P(A) \leq P(B)$

(e) $P(A) \leq 1$

(f) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (additive theorem)

(g) $D_n \subset D_{n+1} \quad n = 1, 2, \cdots \ \Rightarrow\ P(D_n) \uparrow P(\bigcup_{n=1}^{\infty} D_n)$

(h) $D_n \supset D_{n+1} \quad n = 1, 2, \cdots \ \Rightarrow\ P(D_n) \downarrow P(\bigcap_{n=1}^{\infty} D_n)$

(i) $P(\bigcup_{i=1}^{n} D_n) \leq \sum_{i=1}^{n} P(D_n) \qquad n = 1, 2, \cdots, \infty$

(j) If D_n is monotone, $\lim P(D_n) = P \lim(D_n)$