Lecture Note #04 of
Econometrics I & Advanced Econometrics I (2013SY)

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§ 3. Random Variables

§ 3.1 What are random variables.

As mentioned briefly before, suppose a real valued function $X = X(\omega)$, for $\omega \in \Omega$.

$$X \ : \ \omega \quad \to \quad \boldsymbol{R}$$

This X is a random variable.

Now, let $\mathcal{D}$ be a smallest σ-field of $\boldsymbol{R}$ (a Borel field). And let μ be a measure defined on $\mathcal{D}$. Then, from correspondence between X and ω, X maps the probability space to a measurable space, such as,

$$(\Omega,\ \mathcal{B},\ P) \quad \longrightarrow \quad (\boldsymbol{R},\ \mathcal{D},\ \mu).$$

Definition 3.1.1

For a arbitrary $D \in \mathcal{D}$, if

$$X^{-1}(D) \equiv \{\omega \in \Omega\ ;\ X(\omega) \in D\} \in \mathcal{B}$$

be satisfied, then we call X as a random variable. (X is said as a measurable function.)

$<$ Note $>$

It is easy to understand if we set $D = (-\infty, x)$, $\quad x \in \boldsymbol{R}$.

<u>Remark</u> When $X(\omega) = x$, an individual possible value of X is said a realized value.

If number of possible values of $X(\omega)$ is at most countable, then X is a **discrete random variable**.

If number of possible values of $X(\omega)$ is not at most countable (i.e. uncountably infinite), then X is a **continuous random variable**.

Random Vector

A random vector is an useful representation to describe multivariate random variables.

$$\boldsymbol{X} : \Omega \longrightarrow \boldsymbol{R}^k$$

$$\boldsymbol{X}(\boldsymbol{\omega}) = (X_1(\omega_1),\ X_2(\omega_2),\ \cdots,\ X_k(\omega_k))'$$

§ 3.2 Distribution Function

Let show you the definition of distribution function and its basic properties.

Definition 3.2.1

$$F : (-\infty,\ \infty) \longrightarrow [0,\ 1] \qquad \text{such that}$$

$$F(x) \equiv P(\omega : X(\omega) \le x) = \mu\left((-\infty,\ x]\right) \qquad x \in \boldsymbol{R}$$

Theorem 3.2.2

μ is existed uniquely.

Theorem 3.2.3

(1) $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$

(2) $F(+\infty) = \lim_{x\to\infty} F(x) = 1$

(3) For $a < b \quad (a,\ b \in \boldsymbol{R})$, $F(a) \le F(b)$ (i.e. monotone non-decreasing)

Theorem 3.2.4

(1) $P(a < X(\omega) \le b) = F(b) - F(a)$

(2) $\lim_{x\downarrow a} F(x) \equiv F(a+0) = F(a)$ (right continuous)

(3) $\lim_{x\uparrow a} F(x) \equiv F(a) - P(X(\omega) = a)$

Discrete random variables

Let $x_1,\ x_2,\ \cdots\ (x_1 < x_2 < \cdots)$ be realized values of a random variable X. And we call $p(x) \equiv P(X = x)$ as a **probability function** such that

$$P(X = x) = \begin{cases} p_\ell & x = x_\ell, \quad (\ell = 1, 2, \ldots) \\ 0 & \text{otherwise} \end{cases} .$$

Theorem 3.2.5

(1) $F(x) = \sum_{X_\ell \le x} P(X_\ell)$

(2) $p(x_\ell) = F(x_\ell) - \lim_{x \uparrow x_\ell} F(x)$

(3) $p(x) \ge 0 \quad {}^\forall x$, where $\sum_\ell p(x_\ell) = 1$

Continuous random variables

Definition 3.2.6

If there exists a non-negative integrable function $f(\cdot)$ which satisfies

$$F(x) \equiv P(X \le x) = \int_{-\infty}^{x} f(u)du,$$

f is called the probability density function of a random variable X. The distribution of a random variable X is absolutely continuous.

Theorem 3.2.7

Let $F(x)$, $f(x)$ be the distribution function and the probability density function of X, respectively. Then,

(1) $P(X = x) = 0$

(2) $f(x) = \frac{dF(x)}{dx}$

(3) $\int_{-\infty}^{\infty} f(x)dx = 1 \qquad f(x) \ge 0 \qquad {}^\forall x.$

(ex.) Uniform distribution on $[0,\ 2]$ (same manner if it is defined on $(0,\ 2)$)

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x}{2} & x \in (0,\ 2] \\ 1 & x > 2 \end{cases}$$

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} \frac{1}{2} & 0 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{2} f(x)dx = \left[\frac{x}{2}\right]_0^2 = 1$$

mixture of discrete and continuous random variable

Omit here.

§ 3.3 Joint Distribution Function

Joint distribution function

Let $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be random variables (random vector).

Then, suppose $P(\boldsymbol{X} \leq \boldsymbol{x})$ for $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. As

$$P(\boldsymbol{X} \leq \boldsymbol{x}) = P\{(X_1 \leq x_1) \cap (X_2 \leq x_2)\}$$

is a function of x_1 and x_2, then we can represent it as

$$F(x_1,\ x_2) = P\{(X_1 \leq x_1) \cap (X_2 \leq x_2)\}.$$

This $F(x_1,\ x_2)$ is called a joint distribution function.

Theorem 3.3.1

Let $F(x_1,\ x_2)$ be the joint distribution function of $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Then, it satisfies following properties.

(1) $F(-\infty,\ x_2) = F(x_1,\ -\infty) = 0$

(2) $F(\infty,\ \infty) = 1$

(3) for $a_1 < a_2, \quad b_1 < b_2 \ \in \boldsymbol{R}$
$F(a_1,\ x_2) \leq F(a_2,\ x_2) \quad {}^{\forall}x_2$
$F(x_1,\ b_1) \leq F(x_1,\ b_2) \quad {}^{\forall}x_1$

(4) for $a_1 < a_2, \quad b_1 < b_2 \quad \in \boldsymbol{R}$

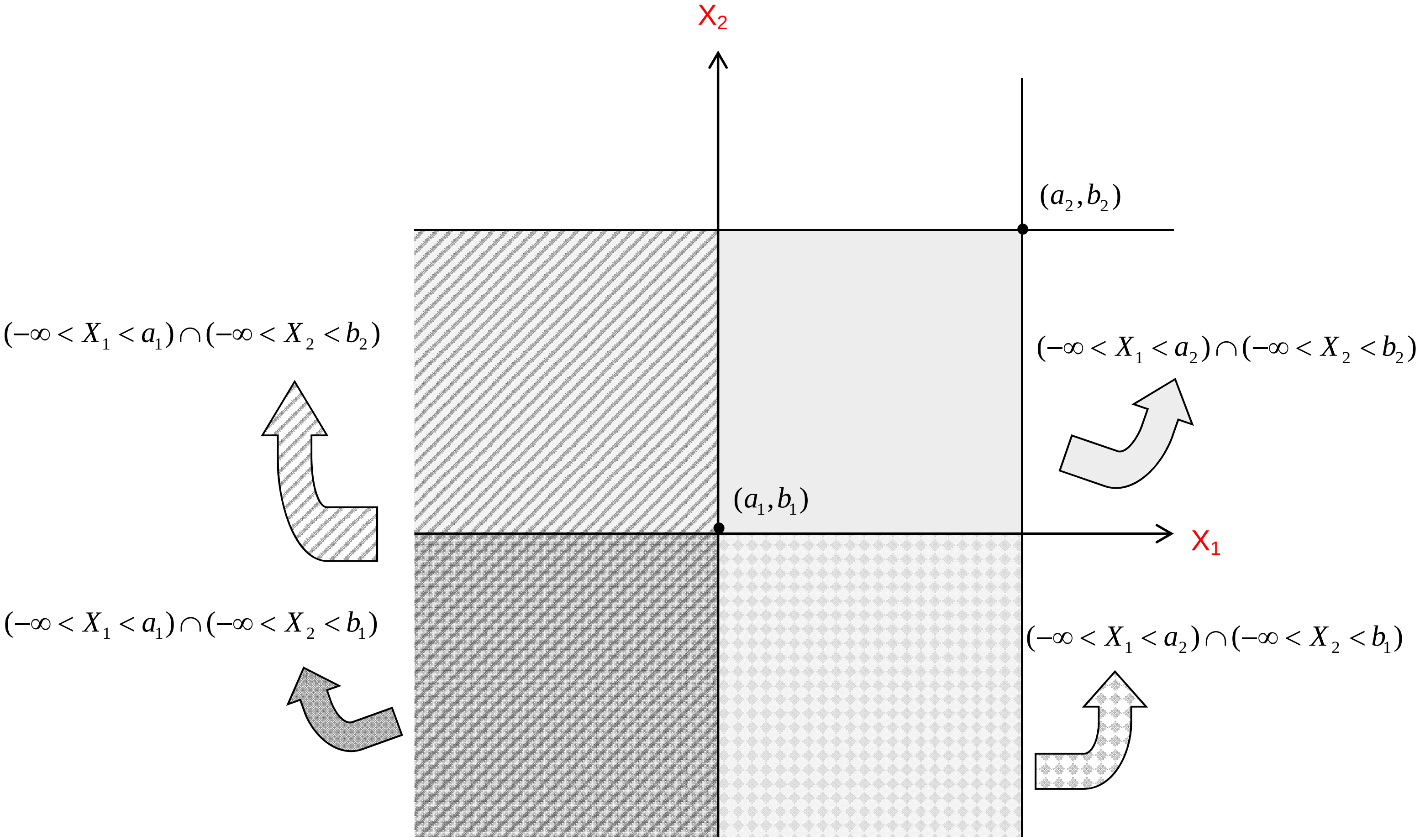
$$\begin{aligned} &P\{(a_1 < X_1 \leq a_2) \cap (b_1 < X_2 \leq b_2)\} \\ &= F(a_2,\ b_2) - F(a_1,\ b_2) - F(a_2,\ b_1) + F(a_1,\ b_1) \end{aligned}$$ See Figure 3.3.1

(5) for $a,\ b \in \boldsymbol{R}$

$$\begin{cases} F(a+0,\ x_2) &= F(a,\ x_2) \quad {}^{\forall}x_2 \\ F(x_1,\ b+0) &= F(x_1,\ b) \quad {}^{\forall}x_1 \\ F(a-0,\ x_2) &= F(a,\ x_2) - P\Big((X_1 = a) \cap (X_2 = x_2)\Big) \quad {}^{\forall}x_2 \\ F(x_1,\ b-0) &= F(x_1,\ b) - P\Big((X_1 = x) \cap (X_2 = b)\Big) \quad {}^{\forall}x_1 \end{cases}$$

(Remark: The second term of left hand side become zero if $\boldsymbol{X}$ is continuous.)

Figure 3.3.1



Marginal distribution function

We can define a marginal distribution function by use of a joint distribution function. Suppose $F(x_1,\ x_2)$ when $x_2=\infty$. Then, we get the marginal distribution function of X_1 as,

$$\begin{aligned} F(x_1,\ \infty) &= P((X_1 \le x_1) \cap (X_2 < \infty)) \\ &= P(X_1 \le x_1) \\ &= F_1(x_1). \end{aligned}$$

In the same way,

$$F(\infty,\ x_2) = F_2(x_2)$$

is called the marginal distribution function of X_2.

Joint Density Function and Marginal Density Function (for discrete random variables)

Let X_1, X_2 be discrete random variables, and

$$\begin{aligned} X_1 &= \{a_1,\ a_2,\ \cdots\} \\ X_2 &= \{b_1,\ b_2,\ \cdots\} \end{aligned}$$

Then, their joint probability function is defined as

$$p(a_j,\ b_k) = P(X_1 = a_j \cap X_2 = b_k) \qquad j, k, = 1, 2, \cdots.$$

Theorem 3.3.2

Let a joint probability function of X_1 and X_2 be $p(x_1, x_2)$. If $F(x_1, x_2)$ be their joint distribution function, then the following properties are satisfied.

(1) $p(x_1, x_2) \ge 0 \qquad {}^{\forall}x_1, {}^{\forall}x_2$

(2) $\sum_j \sum_k p(a_j, b_k) = 1$

(3) $F(x_1, x_2) = \sum_{a_j \le x_1} \sum_{b_k \le x_2} p(a_j, b_k)$

Theorem 3.3.3

Let the marginal probability function of X_1 be $p_1(a_j)$, and that of X_2 be $p_2(b_k)$. Then,

$$\begin{aligned} p_1(a_j) &= \sum_k p(a_j, b_k) \\ p_2(b_k) &= \sum_j p(a_j, b_k) \end{aligned}$$

(ex.) Let a_j and b_k be random variables with binary values. Then,

$$p_1(X_1(\omega_1)) = p_1(X_1(\omega_1) \cap X_2(\omega_2)) + p_1(X_1(\omega_1) \cap X_2(\omega_2^c)).$$

Joint Density Function and Marginal Density Function (for continuous random variables)

Definition 3.3.4

(An extension of Definition 3.2.6) If

$$\begin{aligned} F(x_1, x_2) &= P(X_1 \le x_1 \cap X_2 \le x_2) \\ &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u_1, u_2) du_2 du_1 \end{aligned}$$

exists, we call $f(x_1, x_2)$ as a **joint density function**.

Theorem 3.3.5

(1) $f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$

(2) $f(x_1, x_2) \ge 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2) dx_1, dx_2 = 1 \qquad {}^{\forall} x_1, {}^{\forall} x_2$$

(3) Let be $S \in \mathcal{B}(\boldsymbol{R} \times \boldsymbol{R})$, then

$$P((X_1, X_2) \in S) = \int \int_S f(x_1, x_2) dx_1 dx_2.$$