Exercises and Answers to Chapter 1

1 The continuous type of random variable X has the following density function:

$$f(x) = \begin{cases} a - x, & \text{if } 0 < x < a, \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following questions.

(1) Find a.
(2) Obtain mean and variance of X.
(3) When $Y = X^2$, derive the density function of Y.

[Answer]

(1) From the property of the density function, i.e., $\int f(x)\mathrm{d}x = 1$, we need to have:

$$\int f(x)\mathrm{d}x = \int_0^a (a-x)\mathrm{d}x = \Big[ax - \frac{1}{2}x^2\Big]_0^a = \frac{1}{2}a^2 = 1.$$

Therefore, $a = \sqrt{2}$ is obtained, taking into account $a > 0$.

(2) The definitions of mean and variance are given by: $\mathrm{E}(X) = \int x f(x)\mathrm{d}x$ and $\mathrm{V}(X) = \int (x-\mu)^2 f(x)\mathrm{d}x$, where $\mu = \mathrm{E}(X)$. Therefore, mean of X is:

$$\begin{aligned}\mathrm{E}(X) &= \int x f(x)\mathrm{d}x = \int_0^a x(a-x)\mathrm{d}x = \Big[\frac{1}{2}ax^2 - \frac{1}{3}x^3\Big]_0^a = \frac{1}{6}a^3 \\ &= \frac{\sqrt{2}}{3} \qquad \longleftarrow \qquad a = \sqrt{2} \text{ is substituted.}\end{aligned}$$

Variance of X is:

$$\begin{aligned}\mathrm{V}(X) &= \int (x-\mu)^2 f(x)\mathrm{d}x = \int x^2 f(x)\mathrm{d}x - \mu^2 = \int_0^a x^2(a-x)\mathrm{d}x - \mu^2 \\ &= \Big[\frac{1}{3}ax^3 - \frac{1}{4}x^4\Big]_0^a - \mu^2 = \frac{1}{12}a^4 - \mu^2 = \frac{1}{3} - \Big(\frac{\sqrt{2}}{3}\Big)^2 = \frac{1}{9}.\end{aligned}$$

(3) Let $f(x)$ be the density function of X and $F(x)$ be the distribution function of X. And let $g(y)$ be the density function of Y and $G(y)$ be the distribution function of Y. Using $Y = X^2$, we obtain:

$$\begin{aligned} G(y) &= P(Y < y) = P(X^2 < y) = P(-\sqrt{y} < X < \sqrt{y}) = F(\sqrt{y}) - F(-\sqrt{y}) \\ &= F(\sqrt{y}) \quad \longleftarrow \quad F(-\sqrt{y}) = 0. \end{aligned}$$

Moreover, from the relationship between the density and the distribution functions, we obtain the following:

$$\begin{aligned} g(y) &= \frac{\mathrm{d}G(y)}{\mathrm{d}y} = \frac{\mathrm{d}F(\sqrt{y})}{\mathrm{d}y} = \frac{\mathrm{d}F(x)}{\mathrm{d}x}\frac{\mathrm{d}\sqrt{y}}{\mathrm{d}y} \quad \longleftarrow \quad x = \sqrt{y} \\ &= F'(x)\frac{1}{2\sqrt{y}} = f(x)\frac{1}{2\sqrt{y}} = f(\sqrt{y})\frac{1}{2\sqrt{y}} \\ &= (\sqrt{2} - \sqrt{y})\frac{1}{2\sqrt{y}}, \qquad \text{for } 0 < y < 2. \end{aligned}$$

The range of y is obtained as: $0 < x < \sqrt{2} \Longrightarrow 0 < x^2 < 2 \Longrightarrow 0 < y < 2$.

2 The continuous type of random variable X has the following density function:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Answer the following questions.

(1) Compute mean and variance of X.

(2) When $Y = X^2$, compute mean and variance of Y.

(3) When $Z = e^X$, obtain mean and variance of Z.

[Answer]

(1) The definitions of mean and variance are: $\mathrm{E}(X) = \int x f(x)\mathrm{d}x$ and $\mathrm{V}(X) = \int (x-\mu)^2 f(x)\mathrm{d}x$, where $\mu = \mathrm{E}(X)$. Therefore, mean of X is:

$$\mathrm{E}(X) = \int x f(x)\mathrm{d}x = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\mathrm{d}x = -\frac{1}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}x^2}\right]_{-\infty}^{\infty} = 0.$$

In the third equality, we utilize: $\dfrac{\mathrm{d}e^{-\frac{1}{2}x^2}}{\mathrm{d}x} = -xe^{-\frac{1}{2}x^2}$.

Variance of X is:

$$\mathrm{V}(X) = \int (x-\mu)^2 f(x)\mathrm{d}x = \int x^2 f(x)\mathrm{d}x - \mu^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\mathrm{d}x - \mu^2$$
$$= \left[-x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\mathrm{d}x - \mu^2 = 1.$$

In the fourth equality, the following formula is used.

$$\int_a^b h'(x)g(x)\mathrm{d}x = \Big[h(x)g(x)\Big]_a^b - \int_a^b h(x)g'(x)\mathrm{d}x,$$

where $g(x) = x$ and $h'(x) = x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ are set.

And in the first term of the fourth equality, we use:

$$\lim_{x\to\pm\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} = 0.$$

In the second term of the fourth equality, we utilize the property that the integration of the density function is equal to one.

(2) When $Y = X^2$, mean of Y is:

$$\mathrm{E}(Y) = \mathrm{E}(X^2) = \mathrm{V}(X) - \mu_x^2 = 1$$

From (1), note that $\mathrm{V}(X) = 1$ and $\mu_x = \mathrm{E}(X) = 0$.

Variance of Y is:

$$\begin{aligned}
\mathrm{V}(Y) &= \mathrm{E}(Y-\mu_y)^2 \quad \longleftarrow \quad \mu_y = \mathrm{E}(Y) = 1 \\
&= \mathrm{E}(Y^2) - \mu_y^2 = \mathrm{E}(X^4) - \mu_y^2 = \int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\mathrm{d}x - \mu_y^2 \\
&= \int_{-\infty}^{\infty} x^3 \cdot x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\mathrm{d}x - \mu_y^2 \\
&= \left[-x^3\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right]_{-\infty}^{\infty} + 3\int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\mathrm{d}x - \mu_y^2 \\
&= 3\mathrm{E}(X^2) - \mu_y^2 \quad \longleftarrow \quad \mathrm{E}(X^2) = 1,\ \mu_y = 1 \\
&= 2
\end{aligned}$$

In the sixth equality, the following formula on integration is utilized.

$$\int_a^b h'(x)g(x)\mathrm{d}x = \Big[h(x)g(x)\Big]_a^b - \int_a^b h(x)g'(x)\mathrm{d}x,$$

where $g(x) = x^3$ and $h'(x) = x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ are set.

In the first term of the sixth equality, we use:

$$\lim_{x\to\pm\infty} x^3\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} = 0.$$

(3) For $Z = e^X$, mean of Z is:

$$\begin{aligned}\mathrm{E}(Z) &= \mathrm{E}(e^X) = \int_{-\infty}^{\infty} e^x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\mathrm{d}x = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x^2-2x)}\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-1)^2+\frac{1}{2}}\mathrm{d}x = e^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-1)^2}\mathrm{d}x = e^{\frac{1}{2}}.\end{aligned}$$

In the sixth equality, $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-1)^2}$ is a normal distribution with mean one and variance one, and accordingly its integration is equal to one.

Variance of Z is:

$$\begin{aligned}\mathrm{V}(Z) &= \mathrm{E}(Z-\mu_z)^2 \quad\longleftarrow\quad \mu_z = \mathrm{E}(Z) = e^{\frac{1}{2}} \\ &= \mathrm{E}(Z^2) - \mu_z^2 = \mathrm{E}(e^{2X}) - \mu_z^2 = \int_{-\infty}^{\infty} e^{2x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\mathrm{d}x - \mu_z^2 \\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x^2-4x)}\mathrm{d}x - \mu_z^2 = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-2)^2+2}\mathrm{d}x - \mu_z^2 \\ &= e^2\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-2)^2}\mathrm{d}x - \mu_z^2 = e^2 - e.\end{aligned}$$

The eighth equality comes from the facts that $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-2)^2}$ is a normal distribution with mean two and variance one and that its integration is equal to one.

3 The continuous type of random variable X has the following density function:

$$f(x) = \begin{cases}\dfrac{1}{\lambda}e^{-\frac{x}{\lambda}}, & \text{if } 0 < x, \\ 0, & \text{otherwise.}\end{cases}$$

Answer the following questions.

(1) Compute mean and variance of X.

(2) Derive the moment generating function of X.

(3) Let $X_1, X_2, \cdots, X_n$ be the random variables, which are mutually independently distributed and have the density function shown above. Prove that the density function of $Y = X_1 + X_2 + \cdots + X_n$ is given by the chi-square distribution with $2n$ degrees of freedom when $\lambda = 2$. Note that the chi-square distribution with m degrees of freedom is given by:

$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

[Answer]

(1) Mean of X is:

$$\begin{aligned} \mathrm{E}(X) &= \int x f(x)\mathrm{d}x = \int_0^\infty x\frac{1}{\lambda}e^{-\frac{x}{\lambda}}\mathrm{d}x \\ &= \Big[-xe^{-\frac{x}{\lambda}}\Big]_0^\infty + \int_0^\infty e^{-\frac{x}{\lambda}}\mathrm{d}x = \Big[-\lambda e^{-\frac{x}{\lambda}}\Big]_0^\infty = \lambda. \end{aligned}$$

In the third equality, the following formula is used:

$$\int_a^b h'(x)g(x)\mathrm{d}x = \Big[h(x)g(x)\Big]_a^b - \int_a^b h(x)g'(x)\mathrm{d}x.$$

where $g(x) = x$ and $h'(x) = \dfrac{1}{\lambda}e^{-\frac{x}{\lambda}}$ are set.

And we utilize:

$$\lim_{x\to\infty} xe^{-\frac{x}{\lambda}} = 0, \qquad \lim_{x\to\infty} e^{-\frac{x}{\lambda}} = 0.$$

Variance of X is:

$$\begin{aligned} \mathrm{V}(X) &= \int (x-\mu)^2 f(x)\mathrm{d}x = \int x^2 f(x)\mathrm{d}x - \mu^2 \qquad \longleftarrow \qquad \mu = \mathrm{E}(X) = \lambda \\ &= \int_0^\infty x^2\frac{1}{\lambda}e^{-\frac{x}{\lambda}}\mathrm{d}x - \mu^2 = \Big[-x^2e^{-\frac{x}{\lambda}}\Big]_0^\infty + 2\int_0^\infty xe^{-\frac{x}{\lambda}}\mathrm{d}x - \mu^2 \\ &= \Big[-x^2e^{-\frac{x}{\lambda}}\Big]_0^\infty + 2\lambda\int_0^\infty x\frac{1}{\lambda}e^{-\frac{x}{\lambda}}\mathrm{d}x - \mu^2 \\ &= 2\lambda\mathrm{E}(X) - \mu^2 \qquad \longleftarrow \qquad \mu = \mathrm{E}(X) = \lambda \\ &= 2\lambda^2 - \lambda^2 = \lambda^2. \end{aligned}$$

In the third equality, we utilize:

$$\int_a^b h'(x)g(x)\mathrm{d}x = \Big[h(x)g(x)\Big]_a^b - \int_a^b h(x)g'(x)\mathrm{d}x,$$

where $g(x) = x^2$ and $h'(x) = \dfrac{1}{\lambda}e^{-\frac{x}{\lambda}}$.

In the sixth equality, the following formulas are used:

$$\lim_{x\to\infty} x^2 e^{-\frac{x}{\lambda}} = 0, \qquad \mu = \mathrm{E}(X) = \int_0^\infty x e^{-\frac{x}{\lambda}}\mathrm{d}x.$$

(2) The moment generating function of X is:

$$\begin{aligned}\phi(\theta) = \mathrm{E}(e^{\theta X}) &= \int e^{\theta x} f(x)\mathrm{d}x = \int_0^\infty e^{\theta x}\frac{1}{\lambda}e^{-\frac{x}{\lambda}}\mathrm{d}x = \int_0^\infty \frac{1}{\lambda}e^{-(\frac{1}{\lambda}-\theta)x}\mathrm{d}x \\ &= \frac{1/\lambda}{1/\lambda-\theta}\int_0^\infty (\frac{1}{\lambda}-\theta)e^{-(\frac{1}{\lambda}-\theta)x}\mathrm{d}x = \frac{1}{1-\lambda\theta}.\end{aligned}$$

In the last equality, since $(\dfrac{1}{\lambda}-\theta)e^{-(\frac{1}{\lambda}-\theta)x}$ is a density function, its integration is one. λ in $f(x)$ is replaced by $\dfrac{1}{\lambda}-\theta$.

(3) We want to show that the moment generating function of Y is equivalent to that of a chi-square distribution with $2n$ degrees of freedom.

Because $X_1, X_2, \cdots, X_n$ are mutually independently distributed, the moment generating function of X_i, $\phi_i(\theta)$, is:

$$\phi_i(\theta) = \frac{1}{1-2\theta} = \phi(\theta),$$

which corresponds to the case $\lambda = 2$ of (2).

For $\lambda = 2$, the moment generating function of $Y = X_1 + X_2 + \cdots + X_n$, $\phi_y(\theta)$, is:

$$\begin{aligned}\phi_y(\theta) &= \mathrm{E}(e^{\theta Y}) = \mathrm{E}(e^{\theta(X_1+X_2+\cdots+X_n)}) = \mathrm{E}(e^{\theta X_1})\mathrm{E}(e^{\theta X_2})\cdots\mathrm{E}(e^{\theta X_n}) \\ &= \phi_1(\theta)\phi_2(\theta)\cdots\phi_n(\theta) = \Big(\phi(\theta)\Big)^n = \Big(\frac{1}{1-2\theta}\Big)^n = \Big(\frac{1}{1-2\theta}\Big)^{\frac{2n}{2}}.\end{aligned}$$

Therefore, the moment generating function of Y is:

$$\phi_y(\theta) = \Big(\frac{1}{1-2\theta}\Big)^{\frac{2n}{2}}.$$

A chi-square distribution with m degrees of freedom is given by:

$$f(x) = \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}}, \quad \text{for } x > 0.$$

The moment generation function of the above density function, $\phi_{\chi^2}(\theta)$, is:

$$\begin{aligned}
\phi_{\chi^2}(\theta) = \mathrm{E}(e^{\theta X}) &= \int_0^\infty e^{\theta x} \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x \\
&= \int_0^\infty \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{1}{2}(1-2\theta)x} \mathrm{d}x \\
&= \int_0^\infty \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} \left(\frac{y}{1-2\theta}\right)^{\frac{m}{2}-1} e^{-\frac{1}{2}y} \frac{1}{1-2\theta} \mathrm{d}x \\
&= \left(\frac{1}{1-2\theta}\right)^{\frac{m}{2}-1} \frac{1}{1-2\theta} \int_0^\infty \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} e^{-\frac{1}{2}y} \mathrm{d}x = \left(\frac{1}{1-2\theta}\right)^{\frac{m}{2}}.
\end{aligned}$$

In the fourth equality, use $y = (1-2\theta)x$. In the sixth equality, since the function in the integration corresponds to the chi-square distribution with m degrees of freedom, the integration is one. Thus, $\phi_y(\theta)$ is equivalent to $\phi_{\chi^2}(\theta)$ for $m = 2n$. That is, $\phi_y(\theta)$ is the moment generating function of a chi square distribution with $2n$ degrees of freedom. Therefore, $Y \sim \chi^2(2n)$.

4 The continuous type of random variable X has the following density function:

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following questions.

(1) Compute mean and variance of X.

(2) When $Y = -2\log X$, derive the moment generating function of Y. Note that the log represents the natural logarithm (i.e., $y = -2\log x$ is equivalent to $x = e^{-\frac{1}{2}y}$).

(3) Let Y_1 and Y_2 be the random variables which have the density function obtained in (2). Suppose that Y_1 is independent of Y_2. When $Z = Y_1 + Y_2$, compute the density function of Z.

[Answer]

(1) Mean of X is:

$$\mathrm{E}(X) = \int x f(x)\mathrm{d}x = \int_0^1 x\mathrm{d}x = \Big[\frac{1}{2}x^2\Big]_0^1 = \frac{1}{2}.$$

Variance of X is:

$$\begin{aligned} \mathrm{V}(X) &= \int (x-\mu)^2 f(x)\mathrm{d}x = \int x^2 f(x)\mathrm{d}x - \mu^2 \quad \longleftarrow \quad \mu = \mathrm{E}(X) = \frac{1}{2} \\ &= \int_0^1 x^2\mathrm{d}x - \mu^2 = \Big[\frac{1}{3}x^3\Big]_0^1 - \mu^2 = \frac{1}{3} - \Big(\frac{1}{2}\Big)^2 = \frac{1}{12}. \end{aligned}$$

(2) For $Y = -2\log X$, we obtain the moment generating function of Y, $\phi_y(\theta)$.

$$\begin{aligned} \phi_y(\theta) &= \mathrm{E}(e^{\theta Y}) = \mathrm{E}(e^{-2\theta\log X}) = \mathrm{E}(X^{-2\theta}) = \int x^{-2\theta} f(x)\mathrm{d}x \\ &= \int_0^1 x^{-2\theta}\mathrm{d}x = \Big[\frac{1}{1-2\theta}x^{1-2\theta}\Big]_0^1 = \frac{1}{1-2\theta}. \end{aligned}$$

(3) Let Y_1 and Y_2 be the random variables which have the density function obtained from (2). And, assume that Y_1 is independent of Y_2. For $Z = Y_1 + Y_2$, we want to have the density function of Z.

The moment generating function of Z, $\phi_z(\theta)$, is:

$$\begin{aligned} \phi_z(\theta) &= \mathrm{E}(e^{\theta Z}) = \mathrm{E}(e^{\theta(Y_1+Y_2)}) = \mathrm{E}(e^{\theta Y_1})\mathrm{E}(e^{\theta Y_2}) = \big(\phi_y(\theta)\big)^2 \\ &= \Big(\frac{1}{1-2\theta}\Big)^2 = \Big(\frac{1}{1-2\theta}\Big)^{\frac{4}{2}}, \end{aligned}$$

which is equivalent to the moment generating function of the chi square distribution with 4 degrees of freedom. Therefore, $Z \sim \chi^2(4)$. Note that the chi-square density function with n degrees of freedom is given by:

$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The moment generating function $\phi(\theta)$ is:

$$\phi(\theta) = \Big(\frac{1}{1-2\theta}\Big)^{\frac{n}{2}}.$$

5 The continuous type of random variable X has the following density function:

$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following questions. $\Gamma(a)$ is called the gamma function, defined as:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \mathrm{d}x.$$

(1) What are mean and variance of X?

(2) Compute the moment generating function of X.

[Answer]

(1) For mean:

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^\infty x f(x) \mathrm{d}x = \int_0^\infty x \frac{1}{\Gamma(\frac{n}{2})} 2^{-\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x \\ &= \frac{2^{-\frac{n}{2}}}{2^{-\frac{n+2}{2}}} \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n}{2})} \int_0^\infty \frac{1}{\Gamma(\frac{n+2}{2})} 2^{-\frac{n+2}{2}} x^{\frac{n+2}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x \\ &= 2\frac{n}{2} \int_0^\infty \frac{1}{\Gamma(\frac{n'}{2})} 2^{-\frac{n'}{2}} x^{\frac{n'}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x = n. \end{aligned}$$

Note that $\Gamma(s+1) = s\Gamma(s)$, $\Gamma(1) = 1$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Using $n' = n + 2$, from the property of the density function, we have:

$$\int_{-\infty}^\infty f(x) \mathrm{d}x = \int_0^\infty \frac{1}{\Gamma(\frac{n'}{2})} 2^{-\frac{n'}{2}} x^{\frac{n'}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x = 1,$$

which is utilized in the fifth equality.

For variance, from $\mathrm{V}(X) = \mathrm{E}(X^2) - \mu^2$ we compute $\mathrm{E}(X^2)$ as follows:

$$\begin{aligned} \mathrm{E}(X^2) &= \int_{-\infty}^\infty x^2 f(x) \mathrm{d}x = \int_0^\infty x^2 \frac{1}{\Gamma(\frac{n}{2})} 2^{-\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x \\ &= \int_0^\infty \frac{1}{\Gamma(\frac{n}{2})} 2^{-\frac{n}{2}} x^{\frac{n+4}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x \end{aligned}$$

$$= \frac{2^{-\frac{n}{2}}}{2^{-\frac{n+4}{2}}} \frac{\Gamma(\frac{n+4}{2})}{\Gamma(\frac{n}{2})} \int_0^\infty \frac{1}{\Gamma(\frac{n+4}{2})} 2^{-\frac{n+4}{2}} x^{\frac{n+4}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x$$
$$= 4(\frac{n+2}{2}\frac{n}{2}) \int_0^\infty \frac{1}{\Gamma(\frac{n'}{2})} 2^{-\frac{n'}{2}} x^{\frac{n'}{2}-1} e^{-\frac{x}{2}} \mathrm{d}x = n(n+2),$$

where $n' = n + 4$ is set. Therefore, $\mathrm{V}(X) = n(n+2) - n^2 = 2n$ is obtained.

(2) The moment generating function of X is:

$$\phi(\theta) = \mathrm{E}(e^{\theta X}) = \int_{-\infty}^\infty e^{\theta x} f(x)\mathrm{d}x = \int_0^\infty e^{\theta x} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp(-\frac{x}{2})\mathrm{d}x$$
$$= \int_0^\infty \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp\left(-\frac{1}{2}(1-2\theta)x\right) \mathrm{d}x$$
$$= \int_0^\infty \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \left(\frac{y}{1-2\theta}\right)^{\frac{n}{2}-1} \exp(-\frac{1}{2}y) \frac{1}{1-2\theta}\mathrm{d}y$$
$$= \left(\frac{1}{1-2\theta}\right)^{\frac{n}{2}} \int_0^\infty \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} \exp(-\frac{1}{2}y)\mathrm{d}y = \left(\frac{1}{1-2\theta}\right)^{\frac{n}{2}}.$$

Use $y = (1-2\theta)x$ in the fifth equality. Note that $\frac{\mathrm{d}x}{\mathrm{d}y} = (1-2\theta)^{-1}$. In the seventh equality, the integration corresponds to the chi-square distribution with n degrees of freedom.

6 The continuous type of random variables X and Y are mutually independent and assumed to be $X \sim N(0,1)$ and $Y \sim N(0,1)$. Define $U = X/Y$. Answer the following questions. When $X \sim N(0,1)$, the density function of X is represented as:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

(1) Derive the density function of U.

(2) Prove that the first moment of U does not exist.

[Answer]

(1) The density of U is obtained as follows. The densities of X and Y are:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2), \quad -\infty < x < \infty,$$
$$g(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2), \quad -\infty < y < \infty.$$

Since X is independent of Y, the joint density of X and Y is:

$$\begin{aligned} h(x,y) = f(x)g(y) &= \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2) \\ &= \frac{1}{2\pi} \exp(-\frac{1}{2}(x^2+y^2)). \end{aligned}$$

Using $u = \frac{x}{y}$ and $v = y$, the transformation of the variables is performed. For $x = uv$ and $y = v$, we have the Jacobian:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}.$$

Using transformation of variables, the joint density of U and V, $s(u,v)$ is given by:

$$s(u,v) = h(uv,v)|J| = \frac{1}{2\pi} \exp(-\frac{1}{2}v^2(1+u^2))|v|.$$

The marginal density of U is:

$$\begin{aligned} p(u) = \int s(u,v)\mathrm{d}v &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |v| \exp(-\frac{1}{2}v^2(1+u^2))\mathrm{d}v \\ &= \frac{1}{\pi}\int_0^{\infty} v \exp(-\frac{1}{2}v^2(1+u^2))\mathrm{d}v \\ &= \frac{1}{\pi}\left[-\frac{1}{1+u^2}\exp(-\frac{1}{2}v^2(1+u^2))\right]_{v=0}^{\infty} = \frac{1}{\pi(1+u^2)}, \end{aligned}$$

which corresponds to Cauchy distribution.

(2) We prove that the first moment of U is infinity, i.e.,

$$\begin{aligned} \mathrm{E}(U) = \int u f(u)\mathrm{d}u &= \int_{-\infty}^{\infty} u\frac{1}{\pi(1+u^2)}\mathrm{d}u \\ &= \int_1^{\infty} \frac{1}{2\pi}\frac{1}{x}\mathrm{d}x \quad \longleftarrow \quad x = 1+u^2 \text{ is used.} \\ &= \left[\frac{1}{2\pi}\log x\right]_1^{\infty} \quad \longleftarrow \quad \frac{\mathrm{d}\log x}{\mathrm{d}x} = \frac{1}{x} \\ &= \infty. \end{aligned}$$

For $-\infty < u < \infty$, the range of $x = 1 + u^2$ is give by $1 < x < \infty$.

7 The continuous type of random variables has the following joint density function:

$$f(x,y) = \begin{cases} x+y, & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following questions.

(1) Compute the expectation of XY.

(2) Obtain the correlation coefficient between X and Y.

(3) What is the marginal density function of X?

[Answer]

(1) The expectation of XY is:

$$\begin{aligned} \mathrm{E}(XY) &= \int_0^1 \int_0^1 xyf(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 xy(x+y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \left[\frac{1}{3}yx^3 + \frac{1}{2}y^2x^2\right]_0^1 \mathrm{d}y = \int_0^1 (\frac{1}{3}y + \frac{1}{2}y^2)\mathrm{d}y \\ &= \left[\frac{1}{6}y^2 + \frac{1}{6}y^3\right]_0^1 = \frac{1}{3}. \end{aligned}$$

(2) We want to obtain the correlation coefficient between X and Y, which is represented as: $\rho = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{V}(X)\mathrm{V}(Y)}$. Therefore, $\mathrm{E}(X)$, $\mathrm{E}(Y)$, $\mathrm{V}(X)$, $\mathrm{V}(Y)$ and $\mathrm{Cov}(X,Y)$ have to be computed.

$\mathrm{E}(X)$ is:

$$\begin{aligned} \mathrm{E}(X) &= \int_0^1 \int_0^1 xf(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 x(x+y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \left[\frac{1}{3}x^3 + \frac{1}{2}yx^2\right]_0^1 \mathrm{d}y = \int_0^1 (\frac{1}{3} + \frac{1}{2}y)\mathrm{d}y \\ &= \left[\frac{1}{3}y + \frac{1}{4}y^2\right]_0^1 = \frac{7}{12}. \end{aligned}$$

In the case where x and y are exchangeable, the functional form of $f(x,y)$ is unchanged. Therefore, $\mathrm{E}(Y)$ is:

$$\mathrm{E}(Y) = \mathrm{E}(X) = \frac{7}{12}.$$

For V(X),

$$\begin{aligned}
\mathrm{V}(X) &= \mathrm{E}\Big((X-\mu)^2\Big) \quad \longleftarrow \quad \mu = \mathrm{E}(X) = \frac{7}{12} \\
&= \mathrm{E}(X^2) - \mu^2 = \int_0^1 \int_0^1 x^2 f(x,y)\mathrm{d}x\mathrm{d}y - \mu^2 \\
&= \int_0^1 \int_0^1 x^2(x+y)\mathrm{d}x\mathrm{d}y - \mu^2 = \int_0^1 \Big[\frac{1}{4}x^4 + \frac{1}{3}yx^3\Big]_0^1 \mathrm{d}y - \mu^2 \\
&= \int_0^1 (\frac{1}{4} + \frac{1}{3}y)\mathrm{d}y - \mu^2 = \Big[\frac{1}{4}y + \frac{1}{6}y^2\Big]_0^1 - \mu^2 \\
&= \frac{5}{12} - \Big(\frac{7}{12}\Big)^2 = \frac{11}{144}.
\end{aligned}$$

For V(Y),

$$\mathrm{V}(Y) = \mathrm{V}(X) = \frac{11}{144}.$$

For Cov(X, Y),

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathrm{E}\Big((X-\mu_x)(Y-\mu_y)\Big) = \mathrm{E}(XY) - \mu_x\mu_y \\
&= \frac{1}{3} - \frac{7}{12}\frac{7}{12} = -\frac{1}{144},
\end{aligned}$$

where

$$\mu_x = \mathrm{E}(X) = \frac{7}{12}, \quad \mu_y = \mathrm{E}(Y) = \frac{7}{12}.$$

Therefore, ρ is:

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{V}(X)\mathrm{V}(Y)}} = \frac{-1/144}{\sqrt{(11/144)(11/144)}} = -\frac{1}{11}.$$

(3) The marginal density function of X, $f_x(x)$, is:

$$f_x(x) = \int f(x,y)\mathrm{d}y = \int_0^1 (x+y)\mathrm{d}y = \Big[xy + \frac{1}{2}y^2\Big]_{y=0}^1 = x + \frac{1}{2},$$

for $0 < x < 1$.

8 The discrete type of random variable X has the following density function:

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \cdots.$$

Answer the following questions.

(1) Prove $\sum_{x=0}^{\infty} f(x) = 1$.

(2) Compute the moment generating function of X.

(3) From the moment generating function, obtain mean and variance of X.

[Answer]

(1) We can show $\sum_{x=0}^{\infty} f(x) = 1$ as:

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1.$$

Note that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, because we have $f^{(k)}(x) = e^x$ for $f(x) = e^x$. As shown in Appendix 1.3, the formula of Taylor series expansion is:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k.$$

The Taylor series expansion around $x = 0$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) x^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Here, replace x by λ and k by x.

(2) The moment generating function of X is:

$$\begin{aligned}
\phi(\theta) = \mathrm{E}(e^{\theta X}) &= \sum_{x=0}^{\infty} e^{\theta x} f(x) = \sum_{x=0}^{\infty} e^{\theta x} e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda} \frac{(e^{\theta}\lambda)^x}{x!} \\
&= e^{-\lambda} \exp(e^{\theta}\lambda) \sum_{x=0}^{\infty} \exp(-e^{\theta}\lambda) \frac{(e^{\theta}\lambda)^x}{x!} = e^{-\lambda} \exp(e^{\theta}\lambda) \sum_{x=0}^{\infty} e^{-\lambda'} \frac{\lambda'^x}{x!} \\
&= \exp(-\lambda) \exp(e^{\theta}\lambda) = \exp\Big(\lambda(e^{\theta} - 1)\Big).
\end{aligned}$$

Note that $\lambda' = \exp(e^{\theta}\lambda)$.

(3) Based on the moment generating function, we obtain mean and variance of X.

For mean, because of $\phi(\theta) = \exp\big(\lambda(e^\theta - 1)\big)$, $\phi'(\theta) = \lambda e^\theta \exp\big(\lambda(e^\theta - 1)\big)$ and $\mathrm{E}(X) = \phi'(0)$, we obtain:

$$\mathrm{E}(X) = \phi'(0) = \lambda.$$

For variance, from $\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$, we obtain $\mathrm{E}(X^2)$. Note that $\mathrm{E}(X^2) = \phi''(0)$ and $\phi''(\theta) = (1 + \lambda e^\theta)\lambda e^\theta \exp\big(\lambda(e^\theta - 1)\big)$. Therefore,

$$\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \phi''(0) - (\phi'(0))^2 = (1 + \lambda)\lambda - \lambda^2 = \lambda.$$

9 $X_1, X_2, \cdots, X_n$ are mutually independently and normally distributed with mean μ and variance σ^2, where the density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}.$$

Then, answer the following questions.

(1) Obtain the maximum likelihood estimators of mean μ and variance σ^2.

(2) Check whether the maximum likelihood estimator of σ^2 is unbiased. If it is not unbiased, obtain an unbiased estimator of σ^2. (Hint: use the maximum likelihood estimator.)

(3) We want to test the null hypothesis $H_0 : \mu = \mu_0$ by the likelihood ratio test. Obtain the test statistic and explain the testing procedure.

[Answer]

(1) The joint density is:

$$\begin{aligned}
f(x_1, x_2, \cdots, x_n; \mu, \sigma^2) &= \prod_{i=1}^{n} f(x_i; \mu, \sigma^2) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right) \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right) = l(\mu, \sigma^2).
\end{aligned}$$

Taking the logarithm, we have:

$$\log l(\mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

The derivatives of the log-likelihood function $\log l(\mu, \sigma^2)$ with respect to μ and σ^2 are set to be zero.

$$\begin{aligned}
\frac{\partial \log l(\mu, \sigma^2)}{\partial \mu} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0, \\
\frac{\partial \log l(\mu, \sigma^2)}{\partial \sigma^2} &= -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i - \mu)^2 = 0.
\end{aligned}$$

Solving the two equations, we have the solution of (μ, σ^2), denoted by $(\hat{\mu}, \hat{\sigma}^2)$:

$$\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}, \\
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2.
\end{aligned}$$

Therefore, the maximum likelihood estimators of μ and σ^2, $(\hat{\mu}, \hat{\sigma}^2)$, are as follows:

$$\overline{X}, \quad S^{**2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

(2) Take the expectation to check whether S^{**2} is unbiased.

$$\begin{aligned}
\mathrm{E}(S^{**2}) &= \mathrm{E}\Big(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\Big) = \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n}(X_i - \overline{X})^2\Big) \\
&= \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n}((X_i - \mu) - (\overline{X} - \mu))^2\Big) \\
&= \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n}((X_i - \mu)^2 - 2(X_i - \mu)(\overline{X} - \mu) + (\overline{X} - \mu)^2)\Big) \\
&= \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n}(X_i - \mu)^2 - 2(\overline{X} - \mu)\sum_{i=1}^{n}(X_i - \mu) + n(\overline{X} - \mu)^2\Big)
\end{aligned}$$

$$= \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n}(X_i-\mu)^2 - 2n(\overline{X}-\mu)^2 + n(\overline{X}-\mu)^2\Big)$$

$$= \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n}(X_i-\mu)^2 - n(\overline{X}-\mu)^2\Big)$$

$$= \frac{1}{n}\mathrm{E}\Big(\sum_{i=1}^{n}(X_i-\mu)^2\Big) - \frac{1}{n}\mathrm{E}\Big(n(\overline{X}-\mu)^2\Big)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\Big((X_i-\mu)^2\Big) - \mathrm{E}\Big((\overline{X}-\mu)^2\Big)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathrm{V}(X_i) - \mathrm{V}(\overline{X}) = \frac{1}{n}\sum_{i=1}^{n}\sigma^2 - \frac{\sigma^2}{n}$$

$$= \sigma^2 - \frac{1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2 \neq \sigma^2.$$

Therefore, S^{**2} is not unbiased. Based on S^{**2}, we obtain the unbiased estimator of σ^2. Multiplying $n/(n-1)$ on both sides of $\mathrm{E}(S^{**2}) = \sigma^2(n-1)/n$, we obtain:

$$\frac{n}{n-1}\mathrm{E}(S^{**2}) = \sigma^2.$$

Therefore, the unbiased estimator of σ^2 is:

$$\frac{n}{n-1}S^{**2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2 = S^2.$$

(3) The likelihood ratio is defined as:

$$\lambda = \frac{\max_{\sigma^2} l(\mu_0, \sigma^2)}{\max_{\mu,\sigma^2} l(\mu, \sigma^2)} = \frac{l(\mu_0, \widetilde{\sigma}^2)}{l(\hat{\mu}, \hat{\sigma}^2)}.$$

Since the number of restriction is one, we have:

$$-2\log\lambda \longrightarrow \chi^2(1).$$

$l(\mu, \sigma^2)$ is given by:

$$l(\mu, \sigma^2) = (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right).$$

Taking the logarithm, $\log l(\mu, \sigma^2)$ is:

$$\log l(\mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

On the numerator, under the restriction $\mu = \mu_0$, $\log l(\mu_0, \sigma^2)$ is maximized with respect to σ^2 as follows:

$$\frac{\partial \log l(\mu_0, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i - \mu_0)^2 = 0.$$

This solution of σ^2 is $\widetilde{\sigma}^2$, which is represented as:

$$\widetilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2.$$

Then, $l(\mu_0, \widetilde{\sigma}^2)$ is:

$$l(\mu_0, \widetilde{\sigma}^2) = (2\pi\widetilde{\sigma}^2)^{-n/2}\exp\left(-\frac{1}{2\widetilde{\sigma}^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right) = (2\pi\widetilde{\sigma}^2)^{-n/2}\exp\left(-\frac{n}{2}\right).$$

On the denominator, from the question (1), we have:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}x_i, \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2.$$

Therefore, $l(\hat{\mu}, \hat{\sigma}^2)$ is:

$$l(\hat{\mu}, \hat{\sigma}^2) = (2\pi\hat{\sigma}^2)^{-n/2}\exp\left(-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n}(x_i - \hat{\mu})^2\right) = (2\pi\hat{\sigma}^2)^{-n/2}\exp\left(-\frac{n}{2}\right).$$

The likelihood ratio is:

$$\lambda = \frac{\max_{\sigma^2} l(\mu_0, \sigma^2)}{\max_{\mu, \sigma^2} l(\mu, \sigma^2)} = \frac{l(\mu_0, \widetilde{\sigma}^2)}{l(\hat{\mu}, \hat{\sigma}^2)} = \frac{(2\pi\widetilde{\sigma}^2)^{-n/2}\exp(-n/2)}{(2\pi\hat{\sigma}^2)^{-n/2}\exp(-n/2)} = \left(\frac{\widetilde{\sigma}^2}{\hat{\sigma}^2}\right)^{-n/2}.$$

As n goes to infinity, we obtain:

$$-2\log\lambda = n(\log\widetilde{\sigma}^2 - \log\hat{\sigma}^2) \sim \chi^2(1).$$

When $-2\log\lambda > \chi^2_\alpha(1)$, the null hypothesis $H_0: \ \mu = \mu_0$ is rejected by the significance level α, where $\chi^2_\alpha(1)$ denotes the $100 \times \alpha$ percent point of the Chi-square distribution with one degree of freedom.

10 Answer the following questions.

(1) The discrete type of random variable X is assumed to be Bernoulli. The Bernoulli distribution is given by:

$$f(x) = p^x (1-p)^{1-x}, \qquad x = 0, 1.$$

Let $X_1, X_2, \cdots, X_n$ be random variables drawn from the Bernoulli trials. Compute the maximum likelihood estimator of p.

(2) Let Y be a random variable from a binomial distribution, denoted by $f(y)$, which is represented as:

$$f(y) = {}_nC_y p^y (1-p)^{n-y}, \qquad y = 0, 1, 2, \cdots, n.$$

Then, prove that Y/n goes to p as n is large.

(3) For the random variable Y in the question (2), Let us define:

$$Z_n \equiv \frac{Y - np}{\sqrt{np(1-p)}}.$$

Then, Z_n goes to a standard normal distribution as n is large.

(4) The continuous type of random variable X has the following density function:

$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

where $\Gamma(a)$ denotes the Gamma function, i.e.,

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \mathrm{d}x.$$

Then, show that X/n approaches one when $n \longrightarrow \infty$.

[Answer]

(1) When X is a Bernoulli random variable, the probability function of X is given by:

$$f(x; p) = p^x (1-p)^{1-x}, \qquad x = 0, 1.$$

The joint probability function of X_1, X_2, $\cdots$,X_n is:

$$f(x_1, x_2, \cdots, x_n; p) = \prod_{i=1}^{n} f(x_i; p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$
$$= p^{\sum_i x_i}(1-p)^{n-\sum_i x_i} = l(p).$$

Take the logarithm of $l(p)$.

$$\log l(p) = (\sum_i x_i)\log(p) + (n - \sum_i x_i)\log(1-p).$$

The derivative of the log-likelihood function $\log l(p)$ with respect to p is set to be zero.

$$\frac{\mathrm{d}\log l(p)}{\mathrm{d}p} = \frac{\sum_i x_i}{p} - \frac{n-\sum_i x_i}{1-p} = \frac{\sum_i x_i - np}{p(1-p)} = 0.$$

Solving the above equation, we have:

$$p = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}.$$

Therefore, the maximum likelihood estimator of p is:

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}.$$

(2) Mean and variance of Y are:

$$\mathrm{E}(Y) = np, \qquad \mathrm{V}(Y) = np(1-p).$$

Therefore, we have:

$$\mathrm{E}(\frac{Y}{n}) = \frac{1}{n}\mathrm{E}(Y) = p, \qquad \mathrm{V}(\frac{Y}{n}) = \frac{1}{n^2}\mathrm{V}(Y) = \frac{p(1-p)}{n}.$$

Chebyshev's inequality indicates that for a random variable X and $g(x) \geq 0$ we have:

$$P(g(X) \geq k) \leq \frac{\mathrm{E}(g(X))}{k},$$

where $k > 0$.

Here, when $g(X) = (X - \mathrm{E}(X))^2$ and $k = \epsilon^2$ are set, we can rewrite as:

$$P(|X - \mathrm{E}(X)| \geq \epsilon) \leq \frac{\mathrm{V}(X)}{\epsilon^2},$$

where $\epsilon > 0$.

Replacing X by $\dfrac{Y}{n}$, we apply Chebyshev's inequality.

$$P(|\frac{Y}{n} - \mathrm{E}(\frac{Y}{n})| \geq \epsilon) \leq \frac{\mathrm{V}(\frac{Y}{n})}{\epsilon^2}.$$

That is, as $n \longrightarrow \infty$,

$$P(|\frac{Y}{n} - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2} \longrightarrow 0.$$

Therefore, we obtain:

$$\frac{Y}{n} \longrightarrow p.$$

(3) Let $X_1, X_2, \cdots, X_n$ be Bernoulli random variables, where $P(X_i = x) = p^x(1-p)^{1-x}$ for $x = 0, 1$. Define $Y = X_1 + X_2 + \cdots + X_n$. Because Y has a binomial distribution, Y/n is taken as the sample mean from X_1, X_2, $\cdots$, X_n, i.e., $Y/n = (1/n)\sum_{i=1}^n X_i$. Therefore, using $\mathrm{E}(Y/n) = p$ and $\mathrm{V}(Y/n) = p(1-p)/n$, by the central limit theorem, as $n \longrightarrow \infty$, we have:

$$\frac{Y/n - p}{\sqrt{p(1-p)/n}} \longrightarrow N(0,1).$$

Moreover,

$$Z_n \equiv \frac{Y - np}{\sqrt{np(1-p)}} = \frac{Y/n - p}{\sqrt{p(1-p)/n}}.$$

Therefore,

$$Z_n \longrightarrow N(0,1).$$

(4) When $X \sim \chi^2(n)$, we have $\mathrm{E}(X) = n$ and $\mathrm{V}(X) = 2n$. Therefore, $\mathrm{E}(X/n) = 1$ and $\mathrm{V}(X/n) = 2/n$.

Apply Chebyshev's inequality. Then, we have:

$$P(|\frac{X}{n} - \mathrm{E}(\frac{X}{n})| \ge \epsilon) \le \frac{\mathrm{V}(\frac{X}{n})}{\epsilon^2},$$

where $\epsilon > 0$. That is, as $n \longrightarrow \infty$, we have:

$$P(|\frac{X}{n} - 1| \ge \epsilon) \le \frac{2}{n\epsilon^2} \longrightarrow 0.$$

Therefore,

$$\frac{X}{n} \longrightarrow 1.$$

11 Consider n random variables X_1, X_2, $\cdots$, X_n, which are mutually independently and exponentially distributed. Note that the exponential distribution is given by:

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0.$$

Then, answer the following questions.

(1) Let $\hat{\lambda}$ be the maximum likelihood estimator of λ. Obtain $\hat{\lambda}$.

(2) When n is large enough, obtain mean and variance of $\hat{\lambda}$.

[Answer]

(1) Since $X_1, \cdots, X_n$ are mutually independently and exponentially distributed, the likelihood function $l(\lambda)$ is written as:

$$l(\lambda) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum x_i}.$$

The log-likelihood function is:

$$\log l(\lambda) = n \log(\lambda) - \lambda \sum_{i=1}^{n} x_i.$$

We want the λ which maximizes $\log l(\lambda)$. Solving the following equation:

$$\frac{\mathrm{d} \log l(\lambda)}{\mathrm{d}\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0,$$

and replacing x_i by X_i, the maximum likelihood estimator of λ, denoted by $\hat{\lambda}$, is:

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i}.$$

(2) $X_1, X_2, \cdots, X_n$ are mutually independent. Let $f(x_i;\lambda)$ be the density function of X_i. For the maximum likelihood estimator of λ, i.e., $\hat{\lambda}_n$, as $n \longrightarrow \infty$, we have the following property:

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \longrightarrow N\Big(0, \sigma^2(\lambda)\Big),$$

where

$$\sigma^2(\lambda) = \frac{1}{\mathrm{E}\left[\left(\dfrac{\mathrm{d}\log f(X;\lambda)}{\mathrm{d}\lambda}\right)^2\right]}.$$

Therefore, we obtain $\sigma^2(\lambda)$. The expectation in $\sigma^2(\hat{\lambda}_n)$ is:

$$\begin{aligned}\mathrm{E}\left[\left(\frac{\mathrm{d}\log f(X;\lambda)}{\mathrm{d}\lambda}\right)^2\right] &= \mathrm{E}\left[\left(\frac{1}{\lambda} - X\right)^2\right] = \mathrm{E}\left(\frac{1}{\lambda^2} - \frac{2}{\lambda}X + X^2\right) \\ &= \frac{1}{\lambda^2} - \frac{2}{\lambda}\mathrm{E}(X) + \mathrm{E}(X^2) = \frac{1}{\lambda^2},\end{aligned}$$

where $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$ are:

$$\mathrm{E}(X) = \frac{1}{\lambda}, \qquad \mathrm{E}(X^2) = \frac{2}{\lambda^2}.$$

Therefore, we have:

$$\sigma^2(\lambda) = \frac{1}{\mathrm{E}\left[\left(\dfrac{\mathrm{d}\log f(X;\lambda)}{\mathrm{d}\lambda}\right)^2\right]} = \lambda^2.$$

As n is large, $\hat{\lambda}_n$ approximately has the following distribution:

$$\hat{\lambda}_n \sim N(\lambda, \frac{\lambda^2}{n}).$$

Thus, as n goes to infinity, mean and variance are given by λ and λ^2/n.

12 The n random variables X_1, X_2, $\cdots$, X_n are mutually independently distributed with mean μ and variance σ^2. Consider the following two estimators of μ:

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \widetilde{X} = \frac{1}{2}(X_1 + X_n).$$

Then, answer the following questions.

(1) Is $\overline{X}$ unbiased? How about $\widetilde{X}$?
(2) Which is more efficient, $\overline{X}$ or $\widetilde{X}$?
(3) Examine whether $\overline{X}$ and $\widetilde{X}$ are consistent.

[Answer]

(1) We check whether $\overline{X}$ and $\widetilde{X}$ are unbiased.

$$\mathrm{E}(\overline{X}) = \mathrm{E}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\mathrm{E}(\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu,$$
$$\mathrm{E}(\widetilde{X}) = \frac{1}{2}\Big(\mathrm{E}(X_1) + \mathrm{E}(X_n)\Big) = \frac{1}{2}(\mu + \mu) = \mu.$$

Thus, both are unbiased.

(2) We examine which is more efficient, $\overline{X}$ or $\widetilde{X}$.

$$\mathrm{V}(\overline{X}) = \mathrm{V}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\mathrm{V}(\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{V}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n},$$
$$\mathrm{V}(\widetilde{X}) = \frac{1}{4}\Big(\mathrm{V}(X_1) + \mathrm{V}(X_n)\Big) = \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{\sigma^2}{2}.$$

Therefore, because of $\mathrm{V}(\overline{X}) < \mathrm{V}(\widetilde{X})$, $\overline{X}$ is more efficient than $\widetilde{X}$ when $n > 2$.

(3) We check if $\overline{X}$ and $\widetilde{X}$ are consistent. Apply Chebyshev's inequality. For $\overline{X}$,

$$P(|\overline{X} - \mathrm{E}(\overline{X})| \geq \epsilon) \leq \frac{\mathrm{V}(\overline{X})}{\epsilon^2},$$

where $\epsilon > 0$. That is, when $n \longrightarrow \infty$, we have:

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \longrightarrow 0.$$

Therefore, we obtain:

$$\overline{X} \longrightarrow \mu.$$

Next, for $\widetilde{X}$, we have:

$$P(|\widetilde{X} - \mathrm{E}(\widetilde{X})| \geq \epsilon) \leq \frac{\mathrm{V}(\widetilde{X})}{\epsilon^2},$$

where $\epsilon > 0$. That is, when $n \longrightarrow \infty$, the following equation is obtained:

$$P(|\widetilde{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{2\epsilon^2} \not\longrightarrow 0.$$

$\overline{X}$ is a consistent estimator of μ, but $\widetilde{X}$ is not consistent.

13 The 9 random samples:

21 23 32 20 36 27 26 28 30

which are obtained from the normal population $N(\mu, \sigma^2)$. Then, answer the following questions.

(1) Obtain the unbiased estimates of μ and σ^2.

(2) Obtain both 90 and 95 percent confidence intervals for μ.

(3) Test the null hypothesis $H_0 : \mu = 24$ and the alternative hypothesis $H_1 : \mu > 24$ by the significance level 0.10. How about 0.05?

[Answer]

(1) The unbiased estimators of μ and σ^2, denoted by $\overline{X}$ and S^2, are given by:

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

The unbiased estimates of μ and σ^2 are:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2.$$

Therefore,

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{9}(21 + 23 + 32 + 20 + 36 + 27 + 26 + 28 + 30) = 27,$$

$$\begin{aligned} s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \\ &= \frac{1}{8}\Big((21-27)^2 + (23-27)^2 + (32-27)^2 + (20-27)^2 \\ &\quad +(36-27)^2 + (27-27)^2 + (26-27)^2 + (28-27)^2 + (30-27)^2\Big) \\ &= \frac{1}{8}(36 + 16 + 25 + 49 + 81 + 0 + 1 + 1 + 9) = 27.25. \end{aligned}$$

(2) We obtain the confidence intervals of μ. The following sample distribution is utilized:

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1).$$

Therefore,

$$P\Big(\Big|\frac{\overline{X} - \mu}{S/\sqrt{n}}\Big| < t_{\alpha/2}(n-1)\Big) = 1 - \alpha,$$

where $t_{\alpha/2}(n-1)$ denotes the $100 \times \alpha/2$ percent point of the t distribution, which is obtained given probability α and $n-1$ degrees of freedom. Therefore, we have:

$$P\Big(\overline{X} - t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}} < \mu < \overline{X} + t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}}\Big) = 1 - \alpha.$$

Replacing $\overline{X}$ and S^2 by $\overline{x}$ and s^2, the $100\times(1-\alpha)$ percent confidence interval of μ is:

$$\Big(\overline{x} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}, \overline{x} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\Big).$$

Since $\overline{x} = 27$, $s^2 = 27.25$, $n = 9$, $t_{0.05}(8) = 1.860$ and $t_{0.025}(8) = 2.306$, the 90 percent confidence interval of μ is:

$$(27 - 1.860\sqrt{\frac{27.25}{9}}, 27 + 1.860\sqrt{\frac{27.25}{9}}) = (23.76, 30.24),$$

and the 95 percent confidence interval of μ is:

$$(27 - 2.306\sqrt{\frac{27.25}{9}}, 27 + 2.306\sqrt{\frac{27.25}{9}}) = (22.99, 31.01).$$

(3) We test the null hypothesis $H_0 : \ \mu = 24$ and the alternative hypothesis $H_1 : \ \mu > 24$ by the significance levels 0.10 and 0.05. The distribution of $\overline{X}$ is:

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1).$$

Therefore, under the null hypothesis $H_0 : \ \mu = \mu_0$, we obtain

$$\frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1).$$

Note that μ is replaced by μ_0. For the alternative hypothesis $H_1: \ \mu > \mu_0$, since we have:

$$P\Big(\frac{\overline{X}-\mu_0}{S/\sqrt{n}} > t_\alpha(n-1)\Big) = \alpha,$$

we reject the null hypothesis $H_0: \ \mu = \mu_0$ by the significance level α when we have:

$$\frac{\overline{x}-\mu_0}{s/\sqrt{n}} > t_\alpha(n-1).$$

Substitute $\overline{x} = 27$, $s^2 = 27.25$, $\mu_0 = 24$, $n = 9$, $t_{0.10}(8) = 1.397$ and $t_{0.05}(8) = 1.860$ into the above formula. Then, we obtain:

$$\frac{\overline{x}-\mu_0}{s/\sqrt{n}} = \frac{27-24}{\sqrt{27.25/9}} = 1.724 > t_{0.10}(8) = 1.397.$$

Therefore, we reject the null hypothesis $H_0: \ \mu = 24$ by the significance level $\alpha = 0.10$. And we obtain:

$$\frac{\overline{x}-\mu_0}{s/\sqrt{n}} = \frac{27-24}{\sqrt{27.25/9}} = 1.724 < t_{0.05}(8) = 1.860.$$

Therefore, the null hypothesis $H_0: \ \mu = 24$ is accepted by the significance level $\alpha = 0.05$.

14 The 16 samples $X_1, X_2, \cdots, X_{16}$ are randomly drawn from the normal population with mean μ and known variance $\sigma^2 = 2^2$. The sample average is given by $\overline{x} = 36$. Then, answer the following questions.

(1) Obtain the 95 percent confidence interval for μ.

(2) Test the null hypothesis $H_0: \ \mu = 35$ and the alternative hypothesis $H_1: \ \mu = 36.5$ by the significance level 0.05.

(3) Compute the power of the test in the above question (2).

[Answer]

(1) We obtain the 95 percent confidence interval of μ. The distribution of $\overline{X}$ is:

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

Therefore,

$$P\Big(\Big|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\Big| < z_{\alpha/2}\Big) = 1-\alpha,$$

where $z_{\alpha/2}$ denotes the $100\times\dfrac{\alpha}{2}$ percent point, which is obtained given probability α. Therefore,

$$P\Big(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\Big) = 1-\alpha.$$

Replacing $\overline{X}$ by $\overline{x}$, the $100(1-\alpha)$ percent confidence interval of μ is:

$$\Big(\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\Big).$$

Substituting $\overline{x} = 36$, $\sigma^2 = 2^2$, $n = 16$ and $z_{0.025} = 1.960$, the $100\times(1-\alpha)$ percent confidence interval of μ is:

$$(36 - 1.960\frac{2}{\sqrt{16}}, 36 + 1.960\frac{2}{\sqrt{16}}) = (35.02, 36.98).$$

(2) We test the null hypothesis $H_0: \ \mu = 35$ and the alternative hypothesis $H_1: \ \mu = 36.5$ by the significance level 0.05. The distribution of $\overline{X}$ is:

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

Under the null hypothesis $H_0: \ \mu = \mu_0$,

$$\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1).$$

For the alternative hypothesis $H_1: \ \mu > \mu_0$, we obtain:

$$P\Big(\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha\Big) = \alpha.$$

If we have:

$$\frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} > z_\alpha,$$

the null hypothesis $H_0 : \mu = \mu_0$ is rejected by the significance level α. Substituting $\overline{x} = 36$, $\sigma^2 = 2^2$, $n = 16$ and $z_{0.05} = 1.645$, we obtain:

$$\frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{36 - 35}{2/\sqrt{16}} = 2 > z_\alpha = 1.645.$$

The null hypothesis $H_0 : \mu = 35$ is rejected by the significance level $\alpha = 0.05$.

(3) We compute the power of the test in the question (2). The power of the test is the probability which rejects the null hypothesis under the alternative hypothesis. That is, under the null hypothesis $H_0 : \mu = \mu_0$, the region which rejects the null hypothesis is: $\overline{X} > \mu_0 + z_\alpha \sigma/\sqrt{n}$, because

$$P\Big(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha\Big) = \alpha.$$

We compute the probability which rejects the null hypothesis under the alternative hypothesis $H_1 : \mu = \mu_1$. That is, under the alternative hypothesis $H_1 : \mu = \mu_1$, the following probability is known as the power of the test:

$$P\Big(\overline{X} > \mu_0 + z_\alpha \sigma/\sqrt{n}\Big).$$

Under the alternative hypothesis $H_1 : \mu = \mu_1$, we have:

$$\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Therefore, we want to compute the following probability

$$P\Big(\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} > \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} + z_\alpha\Big).$$

Substituting $\sigma = 2$, $n = 16$, $\mu_0 = 35$, $\mu_1 = 36.5$ and $z_\alpha = 1.645$, we obtain:

$$\begin{aligned} P\Big(\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} > \frac{35 - 36.5}{2/\sqrt{16}} + 1.645\Big) &= P\Big(\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} > -1.355\Big) \\ &= 1 - P\Big(\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} > 1.355\Big) \\ &= 1 - 0.0877 = 0.9123. \end{aligned}$$

Note that $z_{0.0885} = 1.35$ and $z_{0.0869} = 1.36$.

15 X_1, X_2, $\cdots$, X_n are assumed to be mutually independent and be distributed as a Poisson process, where the Poisson distribution is given by:

$$P(X = x) = f(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \cdots.$$

Then, answer the following questions.

(1) Obtain the maximum likelihood estimator of λ, which is denoted by $\hat{\lambda}$.

(2) Prove that $\hat{\lambda}$ is an unbiased estimator.

(3) Prove that $\hat{\lambda}$ is an efficient estimator.

(4) Prove that $\hat{\lambda}$ is an consistent estimator.

[Answer]

(1) We obtain the maximum likelihood estimator of λ, denoted by $\hat{\lambda}$. The Poisson distribution is:

$$P(X = x) = f(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \cdots.$$

The likelihood function is:

$$l(\lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}.$$

The log-likelihood function is:

$$\log l(\lambda) = \log(\lambda) \sum_{i=1}^{n} x_i - n\lambda - \log(\prod_{i=1}^{n} x_i!).$$

The derivative of the log-likelihood function with respect to λ is:

$$\frac{\partial \log l(\lambda)}{\partial \lambda} = \frac{1}{\lambda} \sum_{i=1}^{n} x_i - n = 0.$$

Solving the above equation, the maximum likelihood estimator $\hat{\lambda}$ is:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}.$$

(2) We prove that $\hat{\lambda}$ is an unbiased estimator of λ.

$$\mathrm{E}(\hat{\lambda}) = \mathrm{E}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\lambda = \lambda.$$

(3) We prove that $\hat{\lambda}$ is an efficient estimator of λ, where we show that the equality holds in the Cramer-Rao inequality. First, we obtain $\mathrm{V}(\hat{\lambda})$ as:

$$\mathrm{V}(\hat{\lambda}) = \mathrm{V}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{V}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\lambda = \frac{\lambda}{n}.$$

The Cramer-Rao lower bound is given by:

$$\begin{aligned}
\frac{1}{n\mathrm{E}\left[\left(\dfrac{\partial \log f(X;\lambda)}{\partial\lambda}\right)^2\right]} &= \frac{1}{n\mathrm{E}\left[\left(\dfrac{\partial (X\log\lambda - \lambda - \log X!)}{\partial\lambda}\right)^2\right]} \\
&= \frac{1}{n\mathrm{E}\left[\left(\dfrac{X}{\lambda} - 1\right)^2\right]} = \frac{\lambda^2}{n\mathrm{E}[(X-\lambda)^2]} \\
&= \frac{\lambda^2}{n\mathrm{V}(X)} = \frac{\lambda^2}{n\lambda} = \frac{\lambda}{n}.
\end{aligned}$$

Therefore,

$$\mathrm{V}(\hat{\lambda}) = \frac{1}{n\mathrm{E}\left[\left(\dfrac{\partial \log f(X;\lambda)}{\partial\lambda}\right)^2\right]}.$$

That is, $\mathrm{V}(\hat{\lambda})$ is equal to the lower bound of the Cramer-Rao inequality. Therefore, $\hat{\lambda}$ is efficient.

(4) We show that $\hat{\lambda}$ is a consistent estimator of λ. Note as follows:

$$\mathrm{E}(\hat{\lambda}) = \lambda, \qquad \mathrm{V}(\hat{\lambda}) = \frac{\lambda}{n}.$$

In Chebyshev's inequality:

$$P(|\hat{\lambda} - \mathrm{E}(\hat{\lambda})| \geq \epsilon) \leq \frac{\mathrm{V}(\hat{\lambda})}{\epsilon^2},$$

$\mathrm{E}(\hat{\lambda})$ and $\mathrm{V}(\hat{\lambda})$ are substituted. Then, we have:

$$P(|\hat{\lambda} - \lambda| > \epsilon) < \frac{\lambda}{n\epsilon^2} \longrightarrow 0,$$

which implies that $\hat{\lambda}$ is consistent.

16 $X_1, X_2, \cdots, X_n$ are mutually independently distributed as normal random variables. Note that the normal density is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}.$$

Then, answer the following questions.

(1) Prove that the sample mean $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$ is normally distributed with mean μ and variance σ^2/n.

(2) Define:

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}.$$

Show that Z is normally distributed with mean zero and variance one.

(3) Consider the sample unbiased variance:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

The distribution of $(n-1)S^2/\sigma^2$ is known as a Chi-square distribution with $n-1$ degrees of freedom. Obtain mean and variance of S^2. Note that a Chi-square distribution with m degrees of freedom is:

$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(4) Prove that S^2 is an consistent estimator of σ^2.

[Answer]

(1) The distribution of the sample mean $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$ is derived using the moment generating function. Note that for $X \sim N(\mu, \sigma^2)$ the moment generating function $\phi(\theta)$ is:

$$\begin{aligned}
\phi(\theta) \equiv \mathrm{E}(e^{\theta X}) &= \int_{-\infty}^{\infty} e^{\theta x} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} e^{\theta x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2+\theta x}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(x^2-2(\mu+\sigma^2\theta)x+\mu^2\right)}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(x-(\mu+\sigma^2\theta)\right)^2+(\mu\theta+\frac{1}{2}\sigma^2\theta^2)}\mathrm{d}x \\
&= e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(x-(\mu+\sigma^2\theta)\right)^2}\mathrm{d}x = \exp\Big(\mu\theta + \frac{1}{2}\sigma^2\theta^2\Big).
\end{aligned}$$

In the integration above, $N(\mu + \sigma^2\theta, \sigma^2)$ is utilized. Therefore, we have:

$$\phi_i(\theta) = \exp\Big(\mu\theta + \frac{1}{2}\sigma^2\theta^2\Big).$$

Now, consider the moment generating function of $\overline{X}$, denoted by $\phi_{\overline{x}}(\theta)$:

$$\begin{aligned}
\phi_{\overline{x}}(\theta) \equiv \mathrm{E}(e^{\theta\overline{X}}) &= \mathrm{E}(e^{\theta\frac{1}{n}\sum_{i=1}^{n} X_i}) = \mathrm{E}(\prod_{i=1}^{n} e^{\frac{\theta}{n}X_i}) = \prod_{i=1}^{n} \mathrm{E}(e^{\frac{\theta}{n}X_i}) = \prod_{i=1}^{n} \phi_i(\frac{\theta}{n}) \\
&= \prod_{i=1}^{n} \exp\Big(\mu\frac{\theta}{n} + \frac{1}{2}\sigma^2(\frac{\theta}{n})^2\Big) = \exp\Big(\mu\theta + \frac{1}{2}\sigma^2\frac{\theta^2}{n}\Big) = \exp\Big(\mu\theta + \frac{1}{2}\frac{\sigma^2}{n}\theta^2\Big),
\end{aligned}$$

which is equivalent to the moment generating function of the normal distribution with mean μ and variance σ^2/n.

(2) We derive the distribution of Z, which is shown as:

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}.$$

From the question (1), the moment generating function of $\overline{X}$, denoted by $\phi_{\overline{x}}(\theta)$, is:

$$\phi_{\overline{x}}(\theta) \equiv \mathrm{E}(e^{\theta\overline{X}}) = \exp\Big(\mu\theta + \frac{1}{2}\frac{\sigma^2}{n}\theta^2\Big).$$

The moment generating function of Z, denoted by $\phi_z(\theta)$:

$$\begin{aligned}\phi_z(\theta) &\equiv \mathrm{E}(e^{\theta Z}) = \mathrm{E}\Big(\exp(\theta\frac{\overline{X}-\mu}{\sigma/\sqrt{n}})\Big)\\ &= \exp\Big(-\theta\frac{\mu}{\sigma/\sqrt{n}}\Big)\mathrm{E}\Big(\exp(\frac{\theta}{\sigma/\sqrt{n}}\overline{X})\Big)\\ &= \exp\Big(-\theta\frac{\mu}{\sigma/\sqrt{n}}\Big)\phi_{\overline{x}}(\frac{\theta}{\sigma/\sqrt{n}})\\ &= \exp\Big(-\theta\frac{\mu}{\sigma/\sqrt{n}}\Big)\exp\Big(\mu\frac{\theta}{\sigma/\sqrt{n}}+\frac{1}{2}\frac{\sigma^2}{n}\Big(\frac{\theta}{\sigma/\sqrt{n}}\Big)^2\Big) = \exp(\frac{1}{2}\theta^2),\end{aligned}$$

which is the moment generating function of $N(0,1)$.

(3) First, as preliminaries, we derive mean and variance of the chi-square distribution with m degrees of freedom. The chi-square distribution with m degrees of freedom is:

$$f(x) = \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})}x^{\frac{m}{2}-1}e^{-\frac{x}{2}}, \quad \text{if } x > 0.$$

Therefore, the moment generating function $\phi_{\chi^2}(\theta)$ is:

$$\begin{aligned}\phi_{\chi^2}(\theta) &= \mathrm{E}(e^{\theta X}) = \int_0^\infty e^{\theta x}\frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})}x^{\frac{m}{2}-1}e^{-\frac{x}{2}}\mathrm{d}x\\ &= \int_0^\infty \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})}x^{\frac{m}{2}-1}e^{-\frac{1}{2}(1-2\theta)x}\mathrm{d}x\\ &= \int_0^\infty \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})}\Big(\frac{y}{1-2\theta}\Big)^{\frac{m}{2}-1}e^{-\frac{1}{2}y}\frac{1}{1-2\theta}\mathrm{d}x\\ &= \Big(\frac{1}{1-2\theta}\Big)^{\frac{m}{2}-1}\frac{1}{1-2\theta}\int_0^\infty \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})}y^{\frac{m}{2}-1}e^{-\frac{1}{2}y}\mathrm{d}x = (1-2\theta)^{-\frac{m}{2}}.\end{aligned}$$

In the fourth equality, use $y = (1-2\theta)x$. The first and second derivatives of the moment generating function is:

$$\phi'_{\chi^2}(\theta) = m(1-2\theta)^{-\frac{m}{2}-1}, \qquad \phi''_{\chi^2}(\theta) = m(m+2)(1-2\theta)^{-\frac{m}{2}-2}.$$

Therefore, we obtain:

$$\mathrm{E}(X) = \phi'_{\chi^2}(0) = m, \qquad \mathrm{E}(X^2) = \phi''_{\chi^2}(0) = m(m+2).$$

Thus, for the chi-square distribution with m degrees of freedom, mean is given by m and variance is:

$$V(X) = E(X^2) - (E(X))^2 = m(m+2) - m^2 = 2m.$$

Therefore, using $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$, we have:

$$E(\frac{(n-1)S^2}{\sigma^2}) = n-1, \qquad V(\frac{(n-1)S^2}{\sigma^2}) = 2(n-1),$$

which implies

$$\frac{n-1}{\sigma^2}E(S^2) = n-1, \qquad (\frac{n-1}{\sigma^2})^2 V(S^2) = 2(n-1).$$

Finally, mean and variance of S^2 are:

$$E(S^2) = \sigma^2, \qquad V(S^2) = \frac{2\sigma^4}{n-1}.$$

(4) We show that S^2 is a consistent estimator of σ^2. Chebyshev's inequality is utilized, which is:

$$P(|S^2 - E(S^2)| \geq \epsilon) \leq \frac{V(S^2)}{\epsilon^2}.$$

Substituting $E(S^2)$ and $V(S^2)$, we obtain:

$$P(|S^2 - \sigma^2| \geq \epsilon) \leq \frac{2\sigma^4}{(n-1)\epsilon^2} \longrightarrow 0.$$

Therefore, S^2 is consistent.

Statistical Tables

Table 1: Standard Normal Distribution — $Z \sim N(0, 1)$

$$\alpha = P(Z > z_\alpha) = \int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)\mathrm{d}x$$

z_α	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414
0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08692	.08534	.08379	.08226
1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003

Table 2: Chi-Square Distribution — $X \sim \chi^2(m)$

$$\alpha = P(X > \chi^2_\alpha) = \int_{\chi^2_\alpha}^{\infty} f(\chi^2)\mathrm{d}\chi^2$$

α / m	.995	.99	.975	.95	.90	.10	.05	.025	.010	.005
1	.0000393	.000157	.000982	.00393	.0158	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	4.61	5.99	7.38	9.21	10.60
3	.0717	.115	.216	.352	.584	6.25	7.81	9.35	11.34	12.84
4	.207	.297	.484	.711	1.06	7.78	9.49	11.14	13.28	14.86
5	.412	.554	.831	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	.676	.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.95
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.65
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.43	104.21
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	107.57	113.15	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81	140.17

Table 3: F Distribution — $F \sim F(m_1, m_2)$

$\alpha = P(F > F_\alpha) = \int_{F_\alpha}^{\infty} f(F)\mathrm{d}F$

m_1 =Degree of freedom in the numerator
m_2 =Degree of freedom in the denominator

m_2 \ m_1	α	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	.050	161.	200.	216.	225.	230.	234.	237.	239.	241.	242.	243.	244.	245.	245.
	.025	648.	800.	864.	900.	922.	937.	948.	957.	963.	969.	973.	977.	980.	983.
	.010	4052	5000	5403	5625	5764	5859	5928	5981	6022	6056	6083	6106	6126	6143
	.005	16211	20000	21615	22500	23056	23437	23715	23925	24091	24224	24334	24426	24505	24572
2	.050	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4
	.025	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4
	.010	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4
	.005	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.
3	.050	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71
	.025	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.4	14.3	14.3	14.3
	.010	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1	27.0	26.9
	.005	55.6	49.8	47.5	46.2	45.4	44.8	44.4	44.1	43.9	43.7	43.5	43.4	43.3	43.2
4	.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87
	.025	12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.79	8.75	8.72	8.68
	.010	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.5	14.4	14.3	14.2
	.005	31.3	26.3	24.3	23.2	22.5	22.0	21.6	21.4	21.1	21.0	20.8	20.7	20.6	20.5
5	.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64
	.025	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.57	6.52	6.49	6.46
	.010	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.96	9.89	9.82	9.77
	.005	22.8	18.3	16.5	15.6	14.9	14.5	14.2	14.0	13.8	13.6	13.5	13.4	13.3	13.2
6	.050	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.41	5.37	5.33	5.30
	.010	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72	7.66	7.60
	.005	18.6	14.5	12.9	12.0	11.5	11.1	10.8	10.6	10.4	10.3	10.1	10.0	9.95	9.88
7	.050	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	4.63	4.60
	.010	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	6.41	6.36
	.005	16.2	12.4	10.9	10.1	9.52	9.16	8.89	8.68	8.51	8.38	8.27	8.18	8.10	8.03
8	.050	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24
	.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	4.16	4.13
	.010	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	5.61	5.56
	.005	14.7	11.0	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.10	7.01	6.94	6.87
9	.050	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03
	.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	3.83	3.80
	.010	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	5.05	5.01
	.005	13.6	10.1	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.31	6.23	6.15	6.09
10	.050	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86
	.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.66	3.62	3.58	3.55
	.010	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71	4.65	4.60
	.005	12.8	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.75	5.66	5.59	5.53
11	.050	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74
	.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.47	3.43	3.39	3.36
	.010	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40	4.34	4.29
	.005	12.2	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.32	5.24	5.16	5.10
12	.050	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64
	.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.32	3.28	3.24	3.21
	.010	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16	4.10	4.05
	.005	11.8	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.99	4.91	4.84	4.77
13	.050	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55
	.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.20	3.15	3.12	3.08
	.010	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96	3.91	3.86
	.005	11.4	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.72	4.64	4.57	4.51

Table 3: F Distribution — $F \sim F(m_1, m_2)$: < Continued >

$$\alpha = \mathrm{P}(F > F_\alpha) = \int_{F_\alpha}^{\infty} f(F)\mathrm{d}F$$

m_1 =Degree of freedom in the numerator
m_2 =Degree of freedom in the denominator

m_1 / m_2	α	15	16	17	18	19	20	25	30	40	50	60	80	100	200
1	.050	246.	246.	247.	247.	248.	248.	249.	250.	251.	252.	252.	253.	253.	254.
	.025	985.	987.	989.	990.	992.	993.	998.	1001	1006	1008	1010	1012	1013	1016
	.010	6157	6170	6181	6192	6201	6209	6240	6261	6287	6303	6313	6326	6334	6350
	.005	24630	24681	24727	24767	24803	24836	24960	25044	25148	25211	25253	25306	25337	25401
2	.050	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5
	.025	39.4	39.4	39.4	39.4	39.4	39.4	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5
	.010	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5
	.005	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.	199.
3	.050	8.70	8.69	8.68	8.67	8.67	8.66	8.63	8.62	8.59	8.58	8.57	8.56	8.55	8.54
	.025	14.3	14.2	14.2	14.2	14.2	14.2	14.1	14.1	14.0	14.0	14.0	14.0	14.0	13.9
	.010	26.9	26.8	26.8	26.8	26.7	26.7	26.6	26.5	26.4	26.4	26.3	26.3	26.2	26.2
	.005	43.1	43.0	42.9	42.9	42.8	42.8	42.6	42.5	42.3	42.2	42.1	42.1	42.0	41.9
4	.050	5.86	5.84	5.83	5.82	5.81	5.80	5.77	5.75	5.72	5.70	5.69	5.67	5.66	5.65
	.025	8.66	8.63	8.61	8.59	8.58	8.56	8.50	8.46	8.41	8.38	8.36	8.33	8.32	8.29
	.010	14.2	14.2	14.1	14.1	14.0	14.0	13.9	13.8	13.7	13.7	13.7	13.6	13.6	13.5
	.005	20.4	20.4	20.3	20.3	20.2	20.2	20.0	19.9	19.8	19.7	19.6	19.5	19.5	19.4
5	.050	4.62	4.60	4.59	4.58	4.57	4.56	4.52	4.50	4.46	4.44	4.43	4.42	4.41	4.39
	.025	6.43	6.40	6.38	6.36	6.34	6.33	6.27	6.23	6.18	6.14	6.12	6.10	6.08	6.05
	.010	9.72	9.68	9.64	9.61	9.58	9.55	9.45	9.38	9.29	9.24	9.20	9.16	9.13	9.08
	.005	13.1	13.1	13.0	13.0	12.9	12.9	12.8	12.7	12.5	12.5	12.4	12.3	12.3	12.2
6	.050	3.94	3.92	3.91	3.90	3.88	3.87	3.83	3.81	3.77	3.75	3.74	3.72	3.71	3.69
	.025	5.27	5.24	5.22	5.20	5.18	5.17	5.11	5.07	5.01	4.98	4.96	4.93	4.92	4.88
	.010	7.56	7.52	7.48	7.45	7.42	7.40	7.30	7.23	7.14	7.09	7.06	7.01	6.99	6.93
	.005	9.81	9.76	9.71	9.66	9.62	9.59	9.45	9.36	9.24	9.17	9.12	9.06	9.03	8.95
7	.050	3.51	3.49	3.48	3.47	3.46	3.44	3.40	3.38	3.34	3.32	3.30	3.29	3.27	3.25
	.025	4.57	4.54	4.52	4.50	4.48	4.47	4.40	4.36	4.31	4.28	4.25	4.23	4.21	4.18
	.010	6.31	6.28	6.24	6.21	6.18	6.16	6.06	5.99	5.91	5.86	5.82	5.78	5.75	5.70
	.005	7.97	7.91	7.87	7.83	7.79	7.75	7.62	7.53	7.42	7.35	7.31	7.25	7.22	7.15
8	.050	3.22	3.20	3.19	3.17	3.16	3.15	3.11	3.08	3.04	3.02	3.01	2.99	2.97	2.95
	.025	4.10	4.08	4.05	4.03	4.02	4.00	3.94	3.89	3.84	3.81	3.78	3.76	3.74	3.71
	.010	5.52	5.48	5.44	5.41	5.38	5.36	5.26	5.20	5.12	5.07	5.03	4.99	4.96	4.91
	.005	6.81	6.76	6.72	6.68	6.64	6.61	6.48	6.40	6.29	6.22	6.18	6.12	6.09	6.02
9	.050	3.01	2.99	2.97	2.96	2.95	2.94	2.89	2.86	2.83	2.80	2.79	2.77	2.76	2.73
	.025	3.77	3.74	3.72	3.70	3.68	3.67	3.60	3.56	3.51	3.47	3.45	3.42	3.40	3.37
	.010	4.96	4.92	4.89	4.86	4.83	4.81	4.71	4.65	4.57	4.52	4.48	4.44	4.42	4.36
	.005	6.03	5.98	5.94	5.90	5.86	5.83	5.71	5.62	5.52	5.45	5.41	5.36	5.32	5.26
10	.050	2.85	2.83	2.81	2.80	2.79	2.77	2.73	2.70	2.66	2.64	2.62	2.60	2.59	2.56
	.025	3.52	3.50	3.47	3.45	3.44	3.42	3.35	3.31	3.26	3.22	3.20	3.17	3.15	3.12
	.010	4.56	4.52	4.49	4.46	4.43	4.41	4.31	4.25	4.17	4.12	4.08	4.04	4.01	3.96
	.005	5.47	5.42	5.38	5.34	5.31	5.27	5.15	5.07	4.97	4.90	4.86	4.81	4.77	4.71
11	.050	2.72	2.70	2.69	2.67	2.66	2.65	2.60	2.57	2.53	2.51	2.49	2.47	2.46	2.43
	.025	3.33	3.30	3.28	3.26	3.24	3.23	3.16	3.12	3.06	3.03	3.00	2.97	2.96	2.92
	.010	4.25	4.21	4.18	4.15	4.12	4.10	4.01	3.94	3.86	3.81	3.78	3.73	3.71	3.66
	.005	5.05	5.00	4.96	4.92	4.89	4.86	4.74	4.65	4.55	4.49	4.45	4.39	4.36	4.29
12	.050	2.62	2.60	2.58	2.57	2.56	2.54	2.50	2.47	2.43	2.40	2.38	2.36	2.35	2.32
	.025	3.18	3.15	3.13	3.11	3.09	3.07	3.01	2.96	2.91	2.87	2.85	2.82	2.80	2.76
	.010	4.01	3.97	3.94	3.91	3.88	3.86	3.76	3.70	3.62	3.57	3.54	3.49	3.47	3.41
	.005	4.72	4.67	4.63	4.59	4.56	4.53	4.41	4.33	4.23	4.17	4.12	4.07	4.04	3.97
13	.050	2.53	2.51	2.50	2.48	2.47	2.46	2.41	2.38	2.34	2.31	2.30	2.27	2.26	2.23
	.025	3.05	3.03	3.00	2.98	2.96	2.95	2.88	2.84	2.78	2.74	2.72	2.69	2.67	2.63
	.010	3.82	3.78	3.75	3.72	3.69	3.66	3.57	3.51	3.43	3.38	3.34	3.30	3.27	3.22
	.005	4.46	4.41	4.37	4.33	4.30	4.27	4.15	4.07	3.97	3.91	3.87	3.81	3.78	3.71

Table 3: F Distribution — $F \sim F(m_1, m_2)$: < Continued >

$$\alpha = \mathrm{P}(F > F_\alpha) = \int_{F_\alpha}^{\infty} f(F)\mathrm{d}F$$

m_1 =Degree of freedom in the numerator
m_2 =Degree of freedom in the denominator

m_2 \ m_1	α	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	.050	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48
14	.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.09	3.05	3.01	2.98
	.010	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.86	3.80	3.75	3.70
	.005	11.1	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.51	4.43	4.36	4.30
	.050	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42
15	.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	3.01	2.96	2.92	2.89
	.010	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67	3.61	3.56
	.005	10.8	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.33	4.25	4.18	4.12
	.050	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37
16	.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.93	2.89	2.85	2.82
	.010	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55	3.50	3.45
	.005	10.6	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.18	4.10	4.03	3.97
	.050	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33
17	.025	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.87	2.82	2.79	2.75
	.010	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46	3.40	3.35
	.005	10.4	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	4.05	3.97	3.90	3.84
	.050	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29
18	.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.81	2.77	2.73	2.70
	.010	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37	3.32	3.27
	.005	10.2	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.94	3.86	3.79	3.73
	.050	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26
19	.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.76	2.72	2.68	2.65
	.010	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30	3.24	3.19
	.005	10.1	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.84	3.76	3.70	3.64
	.050	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.23
20	.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.72	2.68	2.64	2.60
	.010	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23	3.18	3.13
	.005	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.76	3.68	3.61	3.55
	.050	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.22	2.20
21	.025	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.68	2.64	2.60	2.56
	.010	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.24	3.17	3.12	3.07
	.005	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.68	3.60	3.54	3.48
	.050	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17
22	.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.65	2.60	2.56	2.53
	.010	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12	3.07	3.02
	.005	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.61	3.54	3.47	3.41
	.050	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24	2.20	2.18	2.15
23	.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.62	2.57	2.53	2.50
	.010	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.14	3.07	3.02	2.97
	.005	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.55	3.47	3.41	3.35
	.050	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18	2.15	2.13
24	.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.59	2.54	2.50	2.47
	.010	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03	2.98	2.93
	.005	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.50	3.42	3.35	3.30
	.050	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.14	2.11
25	.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.56	2.51	2.48	2.44
	.010	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	3.06	2.99	2.94	2.89
	.005	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.45	3.37	3.30	3.25
	.050	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04
30	.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.46	2.41	2.37	2.34
	.010	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84	2.79	2.74
	.005	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.25	3.18	3.11	3.06

Table 3: F Distribution — $F \sim F(m_1, m_2)$: < Continued >

$$\alpha = \mathrm{P}(F > F_\alpha) = \int_{F_\alpha}^{\infty} f(F)\mathrm{d}F$$

m_1 =Degree of freedom in the numerator
m_2 =Degree of freedom in the denominator

m_2 \ m_1	α	15	16	17	18	19	20	25	30	40	50	60	80	100	200
14	.050	2.46	2.44	2.43	2.41	2.40	2.39	2.34	2.31	2.27	2.24	2.22	2.20	2.19	2.16
	.025	2.95	2.92	2.90	2.88	2.86	2.84	2.78	2.73	2.67	2.64	2.61	2.58	2.56	2.53
	.010	3.66	3.62	3.59	3.56	3.53	3.51	3.41	3.35	3.27	3.22	3.18	3.14	3.11	3.06
	.005	4.25	4.20	4.16	4.12	4.09	4.06	3.94	3.86	3.76	3.70	3.66	3.60	3.57	3.50
15	.050	2.40	2.38	2.37	2.35	2.34	2.33	2.28	2.25	2.20	2.18	2.16	2.14	2.12	2.10
	.025	2.86	2.84	2.81	2.79	2.77	2.76	2.69	2.64	2.59	2.55	2.52	2.49	2.47	2.44
	.010	3.52	3.49	3.45	3.42	3.40	3.37	3.28	3.21	3.13	3.08	3.05	3.00	2.98	2.92
	.005	4.07	4.02	3.98	3.95	3.91	3.88	3.77	3.69	3.59	3.52	3.48	3.43	3.39	3.33
16	.050	2.35	2.33	2.32	2.30	2.29	2.28	2.23	2.19	2.15	2.12	2.11	2.08	2.07	2.04
	.025	2.79	2.76	2.74	2.72	2.70	2.68	2.61	2.57	2.51	2.47	2.45	2.42	2.40	2.36
	.010	3.41	3.37	3.34	3.31	3.28	3.26	3.17	3.10	3.02	2.97	2.93	2.89	2.86	2.81
	.005	3.92	3.87	3.83	3.80	3.76	3.73	3.62	3.54	3.44	3.37	3.33	3.28	3.25	3.18
17	.050	2.31	2.29	2.27	2.26	2.24	2.23	2.18	2.15	2.10	2.08	2.06	2.03	2.02	1.99
	.025	2.72	2.70	2.67	2.65	2.63	2.62	2.55	2.50	2.44	2.41	2.38	2.35	2.33	2.29
	.010	3.31	3.27	3.24	3.21	3.19	3.16	3.07	3.00	2.92	2.87	2.83	2.79	2.76	2.71
	.005	3.79	3.75	3.71	3.67	3.64	3.61	3.49	3.41	3.31	3.25	3.21	3.15	3.12	3.05
18	.050	2.27	2.25	2.23	2.22	2.20	2.19	2.14	2.11	2.06	2.04	2.02	1.99	1.98	1.95
	.025	2.67	2.64	2.62	2.60	2.58	2.56	2.49	2.44	2.38	2.35	2.32	2.29	2.27	2.23
	.010	3.23	3.19	3.16	3.13	3.10	3.08	2.98	2.92	2.84	2.78	2.75	2.71	2.68	2.62
	.005	3.68	3.64	3.60	3.56	3.53	3.50	3.38	3.30	3.20	3.14	3.10	3.04	3.01	2.94
19	.050	2.23	2.21	2.20	2.18	2.17	2.16	2.11	2.07	2.03	2.00	1.98	1.96	1.94	1.91
	.025	2.62	2.59	2.57	2.55	2.53	2.51	2.44	2.39	2.33	2.30	2.27	2.24	2.22	2.18
	.010	3.15	3.12	3.08	3.05	3.03	3.00	2.91	2.84	2.76	2.71	2.67	2.63	2.60	2.55
	.005	3.59	3.54	3.50	3.46	3.43	3.40	3.29	3.21	3.11	3.04	3.00	2.95	2.91	2.85
20	.050	2.20	2.18	2.17	2.15	2.14	2.12	2.07	2.04	1.99	1.97	1.95	1.92	1.91	1.88
	.025	2.57	2.55	2.52	2.50	2.48	2.46	2.40	2.35	2.29	2.25	2.22	2.19	2.17	2.13
	.010	3.09	3.05	3.02	2.99	2.96	2.94	2.84	2.78	2.69	2.64	2.61	2.56	2.54	2.48
	.005	3.50	3.46	3.42	3.38	3.35	3.32	3.20	3.12	3.02	2.96	2.92	2.86	2.83	2.76
21	.050	2.18	2.16	2.14	2.12	2.11	2.10	2.05	2.01	1.96	1.94	1.92	1.89	1.88	1.84
	.025	2.53	2.51	2.48	2.46	2.44	2.42	2.36	2.31	2.25	2.21	2.18	2.15	2.13	2.09
	.010	3.03	2.99	2.96	2.93	2.90	2.88	2.79	2.72	2.64	2.58	2.55	2.50	2.48	2.42
	.005	3.43	3.38	3.34	3.31	3.27	3.24	3.13	3.05	2.95	2.88	2.84	2.79	2.75	2.68
22	.050	2.15	2.13	2.11	2.10	2.08	2.07	2.02	1.98	1.94	1.91	1.89	1.86	1.85	1.82
	.025	2.50	2.47	2.45	2.43	2.41	2.39	2.32	2.27	2.21	2.17	2.14	2.11	2.09	2.05
	.010	2.98	2.94	2.91	2.88	2.85	2.83	2.73	2.67	2.58	2.53	2.50	2.45	2.42	2.36
	.005	3.36	3.32	3.27	3.24	3.21	3.18	3.06	2.98	2.88	2.82	2.77	2.72	2.69	2.62
23	.050	2.13	2.11	2.09	2.08	2.06	2.05	2.00	1.96	1.91	1.88	1.86	1.84	1.82	1.79
	.025	2.47	2.44	2.42	2.39	2.37	2.36	2.29	2.24	2.18	2.14	2.11	2.08	2.06	2.01
	.010	2.93	2.89	2.86	2.83	2.80	2.78	2.69	2.62	2.54	2.48	2.45	2.40	2.37	2.32
	.005	3.30	3.25	3.21	3.18	3.15	3.12	3.00	2.92	2.82	2.76	2.71	2.66	2.62	2.56
24	.050	2.11	2.09	2.07	2.05	2.04	2.03	1.98	1.94	1.89	1.86	1.84	1.82	1.80	1.77
	.025	2.44	2.41	2.39	2.36	2.35	2.33	2.26	2.21	2.15	2.11	2.08	2.05	2.02	1.98
	.010	2.89	2.85	2.82	2.79	2.76	2.74	2.64	2.58	2.49	2.44	2.40	2.36	2.33	2.27
	.005	3.25	3.20	3.16	3.12	3.09	3.06	2.95	2.87	2.77	2.70	2.66	2.60	2.57	2.50
25	.050	2.09	2.07	2.05	2.04	2.02	2.01	1.96	1.92	1.87	1.84	1.82	1.80	1.78	1.75
	.025	2.41	2.38	2.36	2.34	2.32	2.30	2.23	2.18	2.12	2.08	2.05	2.02	2.00	1.95
	.010	2.85	2.81	2.78	2.75	2.72	2.70	2.60	2.54	2.45	2.40	2.36	2.32	2.29	2.23
	.005	3.20	3.15	3.11	3.08	3.04	3.01	2.90	2.82	2.72	2.65	2.61	2.55	2.52	2.45
30	.050	2.01	1.99	1.98	1.96	1.95	1.93	1.88	1.84	1.79	1.76	1.74	1.71	1.70	1.66
	.025	2.31	2.28	2.26	2.23	2.21	2.20	2.12	2.07	2.01	1.97	1.94	1.90	1.88	1.84
	.010	2.70	2.66	2.63	2.60	2.57	2.55	2.45	2.39	2.30	2.25	2.21	2.16	2.13	2.07
	.005	3.01	2.96	2.92	2.89	2.85	2.82	2.71	2.63	2.52	2.46	2.42	2.36	2.32	2.25

Table 3: F Distribution — $F \sim F(m_1, m_2)$: < Continued >

$$\alpha = \mathrm{P}(F > F_\alpha) = \int_{F_\alpha}^{\infty} f(F)\mathrm{d}F$$

m_1 =Degree of freedom in the numerator
m_2 =Degree of freedom in the denominator

m_2 \ m_1	α	1	2	3	4	5	6	7	8	9	10	11	12	13	14
35	.050	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.08	2.04	2.01	1.99
	.025	5.48	4.11	3.52	3.18	2.96	2.80	2.68	2.58	2.50	2.44	2.39	2.34	2.30	2.27
	.010	7.42	5.27	4.40	3.91	3.59	3.37	3.20	3.07	2.96	2.88	2.80	2.74	2.69	2.64
	.005	8.98	6.19	5.09	4.48	4.09	3.81	3.61	3.45	3.32	3.21	3.12	3.05	2.98	2.93
40	.050	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00	1.97	1.95
	.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.33	2.29	2.25	2.21
	.010	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66	2.61	2.56
	.005	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	3.03	2.95	2.89	2.83
45	.050	4.06	3.20	2.81	2.58	2.42	2.31	2.22	2.15	2.10	2.05	2.01	1.97	1.94	1.92
	.025	5.38	4.01	3.42	3.09	2.86	2.70	2.58	2.49	2.41	2.35	2.29	2.25	2.21	2.17
	.010	7.23	5.11	4.25	3.77	3.45	3.23	3.07	2.94	2.83	2.74	2.67	2.61	2.55	2.51
	.005	8.71	5.97	4.89	4.29	3.91	3.64	3.43	3.28	3.15	3.04	2.96	2.88	2.82	2.76
50	.050	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99	1.95	1.92	1.89
	.025	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38	2.32	2.26	2.22	2.18	2.14
	.010	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.79	2.70	2.63	2.56	2.51	2.46
	.005	8.63	5.90	4.83	4.23	3.85	3.58	3.38	3.22	3.09	2.99	2.90	2.82	2.76	2.70
60	.050	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92	1.89	1.86
	.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.22	2.17	2.13	2.09
	.010	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50	2.44	2.39
	.005	8.49	5.80	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.82	2.74	2.68	2.62
70	.050	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.93	1.89	1.86	1.84
	.025	5.25	3.89	3.31	2.97	2.75	2.59	2.47	2.38	2.30	2.24	2.18	2.14	2.10	2.06
	.010	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.51	2.45	2.40	2.35
	.005	8.40	5.72	4.66	4.08	3.70	3.43	3.23	3.08	2.95	2.85	2.76	2.68	2.62	2.56
80	.050	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91	1.88	1.84	1.82
	.025	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.35	2.28	2.21	2.16	2.11	2.07	2.03
	.010	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.48	2.42	2.36	2.31
	.005	8.33	5.67	4.61	4.03	3.65	3.39	3.19	3.03	2.91	2.80	2.72	2.64	2.58	2.52
90	.050	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.90	1.86	1.83	1.80
	.025	5.20	3.84	3.26	2.93	2.71	2.55	2.43	2.34	2.26	2.19	2.14	2.09	2.05	2.02
	.010	6.93	4.85	4.01	3.54	3.23	3.01	2.84	2.72	2.61	2.52	2.45	2.39	2.33	2.29
	.005	8.28	5.62	4.57	3.99	3.62	3.35	3.15	3.00	2.87	2.77	2.68	2.61	2.54	2.49
100	.050	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89	1.85	1.82	1.79
	.025	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	2.12	2.08	2.04	2.00
	.010	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.43	2.37	2.31	2.27
	.005	8.24	5.59	4.54	3.96	3.59	3.33	3.13	2.97	2.85	2.74	2.66	2.58	2.52	2.46
150	.050	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.85	1.82	1.79	1.76
	.025	5.13	3.78	3.20	2.87	2.65	2.49	2.37	2.28	2.20	2.13	2.08	2.03	1.99	1.95
	.010	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53	2.44	2.37	2.31	2.25	2.20
	.005	8.12	5.49	4.45	3.88	3.51	3.25	3.05	2.89	2.77	2.67	2.58	2.51	2.44	2.38
200	.050	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80	1.77	1.74
	.025	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18	2.11	2.06	2.01	1.97	1.93
	.010	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27	2.22	2.17
	.005	8.06	5.44	4.41	3.84	3.47	3.21	3.01	2.86	2.73	2.63	2.54	2.47	2.40	2.35
500	.050	3.86	3.01	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.81	1.77	1.74	1.71
	.025	5.05	3.72	3.14	2.81	2.59	2.43	2.31	2.22	2.14	2.07	2.02	1.97	1.93	1.89
	.010	6.69	4.65	3.82	3.36	3.05	2.84	2.68	2.55	2.44	2.36	2.28	2.22	2.17	2.12
	.005	7.95	5.35	4.33	3.76	3.40	3.14	2.94	2.79	2.66	2.56	2.48	2.40	2.34	2.28
∞	.050	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75	1.72	1.69
	.025	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.99	1.94	1.90	1.87
	.010	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18	2.13	2.08
	.005	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.43	2.36	2.29	2.24

Table 3: F Distribution — $F \sim F(m_1, m_2)$: < Continued >

$$\alpha = \mathrm{P}(F > F_\alpha) = \int_{F_\alpha}^{\infty} f(F)\mathrm{d}F$$

m_1 =Degree of freedom in the numerator
m_2 =Degree of freedom in the denominator

m_2 \ m_1	α	15	16	17	18	19	20	25	30	40	50	60	80	100	200
35	.050	1.96	1.94	1.92	1.91	1.89	1.88	1.82	1.79	1.74	1.70	1.68	1.65	1.63	1.60
	.025	2.24	2.21	2.18	2.16	2.14	2.12	2.05	2.00	1.93	1.89	1.86	1.82	1.80	1.75
	.010	2.60	2.56	2.53	2.50	2.47	2.44	2.35	2.28	2.19	2.14	2.10	2.05	2.02	1.96
	.005	2.88	2.83	2.79	2.76	2.72	2.69	2.58	2.50	2.39	2.33	2.28	2.22	2.19	2.11
40	.050	1.92	1.90	1.89	1.87	1.85	1.84	1.78	1.74	1.69	1.66	1.64	1.61	1.59	1.55
	.025	2.18	2.15	2.13	2.11	2.09	2.07	1.99	1.94	1.88	1.83	1.80	1.76	1.74	1.69
	.010	2.52	2.48	2.45	2.42	2.39	2.37	2.27	2.20	2.11	2.06	2.02	1.97	1.94	1.87
	.005	2.78	2.74	2.70	2.66	2.63	2.60	2.48	2.40	2.30	2.23	2.18	2.12	2.09	2.01
45	.050	1.89	1.87	1.86	1.84	1.82	1.81	1.75	1.71	1.66	1.63	1.60	1.57	1.55	1.51
	.025	2.14	2.11	2.09	2.07	2.05	2.03	1.95	1.90	1.83	1.79	1.76	1.72	1.69	1.64
	.010	2.46	2.43	2.39	2.36	2.34	2.31	2.21	2.14	2.05	2.00	1.96	1.91	1.88	1.81
	.005	2.71	2.66	2.62	2.59	2.56	2.53	2.41	2.33	2.22	2.16	2.11	2.05	2.01	1.93
50	.050	1.87	1.85	1.83	1.81	1.80	1.78	1.73	1.69	1.63	1.60	1.58	1.54	1.52	1.48
	.025	2.11	2.08	2.06	2.03	2.01	1.99	1.92	1.87	1.80	1.75	1.72	1.68	1.66	1.60
	.010	2.42	2.38	2.35	2.32	2.29	2.27	2.17	2.10	2.01	1.95	1.91	1.86	1.82	1.76
	.005	2.65	2.61	2.57	2.53	2.50	2.47	2.35	2.27	2.16	2.10	2.05	1.99	1.95	1.87
60	.050	1.84	1.82	1.80	1.78	1.76	1.75	1.69	1.65	1.59	1.56	1.53	1.50	1.48	1.44
	.025	2.06	2.03	2.01	1.98	1.96	1.94	1.87	1.82	1.74	1.70	1.67	1.63	1.60	1.54
	.010	2.35	2.31	2.28	2.25	2.22	2.20	2.10	2.03	1.94	1.88	1.84	1.78	1.75	1.68
	.005	2.57	2.53	2.49	2.45	2.42	2.39	2.27	2.19	2.08	2.01	1.96	1.90	1.86	1.78
70	.050	1.81	1.79	1.77	1.75	1.74	1.72	1.66	1.62	1.57	1.53	1.50	1.47	1.45	1.40
	.025	2.03	2.00	1.97	1.95	1.93	1.91	1.83	1.78	1.71	1.66	1.63	1.59	1.56	1.50
	.010	2.31	2.27	2.23	2.20	2.18	2.15	2.05	1.98	1.89	1.83	1.78	1.73	1.70	1.62
	.005	2.51	2.47	2.43	2.39	2.36	2.33	2.21	2.13	2.02	1.95	1.90	1.84	1.80	1.71
80	.050	1.79	1.77	1.75	1.73	1.72	1.70	1.64	1.60	1.54	1.51	1.48	1.45	1.43	1.38
	.025	2.00	1.97	1.95	1.93	1.90	1.88	1.81	1.75	1.68	1.63	1.60	1.55	1.53	1.47
	.010	2.27	2.23	2.20	2.17	2.14	2.12	2.01	1.94	1.85	1.79	1.75	1.69	1.65	1.58
	.005	2.47	2.43	2.39	2.35	2.32	2.29	2.17	2.08	1.97	1.90	1.85	1.79	1.75	1.66
90	.050	1.78	1.76	1.74	1.72	1.70	1.69	1.63	1.59	1.53	1.49	1.46	1.43	1.41	1.36
	.025	1.98	1.95	1.93	1.91	1.88	1.86	1.79	1.73	1.66	1.61	1.58	1.53	1.50	1.44
	.010	2.24	2.21	2.17	2.14	2.11	2.09	1.99	1.92	1.82	1.76	1.72	1.66	1.62	1.55
	.005	2.44	2.39	2.35	2.32	2.28	2.25	2.13	2.05	1.94	1.87	1.82	1.75	1.71	1.62
100	.050	1.77	1.75	1.73	1.71	1.69	1.68	1.62	1.57	1.52	1.48	1.45	1.41	1.39	1.34
	.025	1.97	1.94	1.91	1.89	1.87	1.85	1.77	1.71	1.64	1.59	1.56	1.51	1.48	1.42
	.010	2.22	2.19	2.15	2.12	2.09	2.07	1.97	1.89	1.80	1.74	1.69	1.63	1.60	1.52
	.005	2.41	2.37	2.33	2.29	2.26	2.23	2.11	2.02	1.91	1.84	1.79	1.72	1.68	1.59
150	.050	1.73	1.71	1.69	1.67	1.66	1.64	1.58	1.54	1.48	1.44	1.41	1.37	1.34	1.29
	.025	1.92	1.89	1.87	1.84	1.82	1.80	1.72	1.67	1.59	1.54	1.50	1.45	1.42	1.35
	.010	2.16	2.12	2.09	2.06	2.03	2.00	1.90	1.83	1.73	1.66	1.62	1.56	1.52	1.43
	.005	2.33	2.29	2.25	2.21	2.18	2.15	2.03	1.94	1.83	1.76	1.70	1.63	1.59	1.49
200	.050	1.72	1.69	1.67	1.66	1.64	1.62	1.56	1.52	1.46	1.41	1.39	1.35	1.32	1.26
	.025	1.90	1.87	1.84	1.82	1.80	1.78	1.70	1.64	1.56	1.51	1.47	1.42	1.39	1.32
	.010	2.13	2.09	2.06	2.03	2.00	1.97	1.87	1.79	1.69	1.63	1.58	1.52	1.48	1.39
	.005	2.30	2.25	2.21	2.18	2.14	2.11	1.99	1.91	1.79	1.71	1.66	1.59	1.54	1.44
500	.050	1.69	1.66	1.64	1.62	1.61	1.59	1.53	1.48	1.42	1.38	1.35	1.30	1.28	1.21
	.025	1.86	1.83	1.80	1.78	1.76	1.74	1.65	1.60	1.52	1.46	1.42	1.37	1.34	1.25
	.010	2.07	2.04	2.00	1.97	1.94	1.92	1.81	1.74	1.63	1.57	1.52	1.45	1.41	1.31
	.005	2.23	2.19	2.14	2.11	2.07	2.04	1.92	1.84	1.72	1.64	1.58	1.51	1.46	1.35
∞	.050	1.67	1.64	1.62	1.60	1.59	1.57	1.51	1.46	1.39	1.35	1.32	1.27	1.24	1.17
	.025	1.83	1.80	1.78	1.75	1.73	1.71	1.63	1.57	1.48	1.43	1.39	1.33	1.30	1.21
	.010	2.04	2.00	1.97	1.93	1.90	1.88	1.77	1.70	1.59	1.52	1.47	1.40	1.36	1.25
	.005	2.19	2.14	2.10	2.06	2.03	2.00	1.88	1.79	1.67	1.59	1.53	1.45	1.40	1.28

Table 4: t Distribution — $T \sim t(m)$

$$\alpha = P(T > t_\alpha) = \int_{t_\alpha}^{\infty} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} \frac{1}{\sqrt{m\pi}} \frac{1}{(1 + x^2/m)^{(m+1)/2}} \mathrm{d}x$$

m \ α	.10	.05	.025	.01	.005
1	3.0777	6.3137	12.7062	31.8210	63.6559
2	1.8856	2.9200	4.3027	6.9645	9.9250
3	1.6377	2.3534	3.1824	4.5407	5.8408
4	1.5332	2.1318	2.7765	3.7469	4.6041
5	1.4759	2.0150	2.5706	3.3649	4.0321
6	1.4398	1.9432	2.4469	3.1427	3.7074
7	1.4149	1.8946	2.3646	2.9979	3.4995
8	1.3968	1.8595	2.3060	2.8965	3.3554
9	1.3830	1.8331	2.2622	2.8214	3.2498
10	1.3722	1.8125	2.2281	2.7638	3.1693
11	1.3634	1.7959	2.2010	2.7181	3.1058
12	1.3562	1.7823	2.1788	2.6810	3.0545
13	1.3502	1.7709	2.1604	2.6503	3.0123
14	1.3450	1.7613	2.1448	2.6245	2.9768
15	1.3406	1.7531	2.1315	2.6025	2.9467
16	1.3368	1.7459	2.1199	2.5835	2.9208
17	1.3334	1.7396	2.1098	2.5669	2.8982
18	1.3304	1.7341	2.1009	2.5524	2.8784
19	1.3277	1.7291	2.0930	2.5395	2.8609
20	1.3253	1.7247	2.0860	2.5280	2.8453
21	1.3232	1.7207	2.0796	2.5176	2.8314
22	1.3212	1.7171	2.0739	2.5083	2.8188
23	1.3195	1.7139	2.0687	2.4999	2.8073
24	1.3178	1.7109	2.0639	2.4922	2.7970
25	1.3163	1.7081	2.0595	2.4851	2.7874
26	1.3150	1.7056	2.0555	2.4786	2.7787
27	1.3137	1.7033	2.0518	2.4727	2.7707
28	1.3125	1.7011	2.0484	2.4671	2.7633
29	1.3114	1.6991	2.0452	2.4620	2.7564
30	1.3104	1.6973	2.0423	2.4573	2.7500
31	1.3095	1.6955	2.0395	2.4528	2.7440
32	1.3086	1.6939	2.0369	2.4487	2.7385
33	1.3077	1.6924	2.0345	2.4448	2.7333
34	1.3070	1.6909	2.0322	2.4411	2.7284
35	1.3062	1.6896	2.0301	2.4377	2.7238
40	1.3031	1.6839	2.0211	2.4233	2.7045
50	1.2987	1.6759	2.0086	2.4033	2.6778
60	1.2958	1.6706	2.0003	2.3901	2.6603
70	1.2938	1.6669	1.9944	2.3808	2.6479
80	1.2922	1.6641	1.9901	2.3739	2.6387
90	1.2910	1.6620	1.9867	2.3685	2.6316
100	1.2901	1.6602	1.9840	2.3642	2.6259
∞	1.2816	1.6449	1.9600	2.3264	2.5758