Econometrics I

English Class from This Year!!

Too bad!! (for you and me)

Econometrics I —> Statistics

Econometrics II —> Econometrics

TA session: Tue, 3rd class (13:00 – 14:30), Room #4, 4/17 –, by Mr. Kinoshita (2nd year of the doctor course)

You can get this lecture note from:

`www2.econ.osaka-u.ac.jp/~tanizaki/class/2012`

Some Textbooks

- 『確率統計演習 1 確率』(国沢清典編，1966，培風館)
- 『確率統計演習 2 統計』(国沢清典編，1966，培風館)
- H. Tanizaki, 2004, *Computational Methods in Statistics and Econometrics* (STATISTICS: textbooks and monographs,

Vol.172), Mercel Dekker.

- R.V. Hogg, J.W. McKean and A.T. Craig, 2005, *Introduction to Mathematical Statistics* (Sixth edition), Pearson Prentice Hall.

More elementary statistics:

- Undergraduate, Tue., 3rd class, Room #5, Prof. Oya
- Graduate, Tue., 6th class, Room #1, Prof. Fukushige

1 Event and Probability

1.1 Event (事象)

We consider an **experiment (実験)** whose outcome is not known in advance but an event occurs with probability, which is sometimes called a **random experiment (無作為実験)**.
The **sample space (標本空間)** of an experiment is a set of all possible outcomes.

Each outcome of a sample space is called an **element (要素，元)** of the sample space or a **sample point (標本点)**, which represents each outcome obtained by the experiment.
An **event (事象)** is any collection of outcomes contained in the sample space, or equivalently a subset of the sample space.

An **elementary event (根元事象)** consists of exactly one element and a **compound event (複合事象)** consists of more than one element.

Sample space is denoted by Ω and **sample point** is given by ω.

Suppose that event A is a subset of sample space Ω.

Let ω be a sample point in event A.

Then, we say that a sample point ω is contained in a sample space A, which is denoted by $\omega \in A$.

The event which does not belong to event A is called the **complementary event (余事象)** of A, which is denoted by A^c.

The event which does not have any sample point is called the **empty event (空事象)**, denoted by ϕ.

Conversely, the event which includes all possible sample points is called the **whole event (全事象)**, represented by Ω, which is equivalent to a **sample space (標本空間)**.

Next, consider two events A and B.

The event which belongs to either event A or event B is called the **sum event (和事象)**, which is denoted by $A \cup B$.

The event which belongs to both event A and event B is called the **product event (積事象)**, denoted by $A \cap B$.

When $A \cap B = \phi$, we say that events A and B are **exclusive (排反)**.

Example 1.1: Consider an experiment of casting a die (サイコロ).

We have six sample points, which are denoted by $\omega_1 = \{1\}$, $\omega_2 = \{2\}$, $\omega_3 = \{3\}$, $\omega_4 = \{4\}$, $\omega_5 = \{5\}$ and $\omega_6 = \{6\}$, where ω_i represents the sample point that we have i.

In this experiment, the sample space is given by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$.

Let A be the event that we have even numbers and B be the event that we have multiples of three.

Then, we can write as $A = \{\omega_2, \omega_4, \omega_6\}$ and $B = \{\omega_3, \omega_6\}$.

The complementary event of A is given by $A^c = \{\omega_1, \omega_3, \omega_5\}$, which is the event that we have odd numbers.

The sum event of A and B is written as $A \cup B = \{\omega_2, \omega_3, \omega_4, \omega_6\}$, while the product event is $A \cap B = \{\omega_6\}$.

Since $A \cap A^c = \phi$, we have the fact that A and A^c are exclusive.

Example 1.2: Cast a coin three times. In this case, we have the following eight sample points:

$$\omega_1 = (\text{H,H,H}), \quad \omega_2 = (\text{H,H,T}), \quad \omega_3 = (\text{H,T,H}),$$
$$\omega_4 = (\text{H,T,T}), \quad \omega_5 = (\text{T,H,H}), \quad \omega_6 = (\text{T,H,T}),$$
$$\omega_7 = (\text{T,T,H}), \quad \omega_8 = (\text{T,T,T})$$

where H represents head (表) while T indicates tail (裏).

For example, (H,T,H) means that the first flip lands head, the second flip is tail and the third one is head.

Therefore, the sample space of this experiment can be written as:

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}.$$

Let A be the event that we have two heads, B be the event that we obtain at least one tail, C be the event that we have head in the second flip, and D be an event that we obtain tail in the third flip.

Then, the events A, B, C and D are give by:

$$A = \{\omega_2, \omega_3, \omega_5\}, \qquad B = \{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\},$$

$$C = \{\omega_1, \omega_2, \omega_5, \omega_6\}, \qquad D = \{\omega_2, \omega_4, \omega_6, \omega_8\}.$$

Since A is a subset of B, denoted by $A \subset B$, a sum event is $A \cup B = B$, while a product event is $A \cap B = A$.

Moreover, we obtain $C \cap D = \{\omega_2, \omega_6\}$ and $C \cup D = \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_8\}$.

1.2 Probability

Let $n(A)$ be the number of sample points in A.

We have $n(A) \leq n(B)$ when $A \subseteq B$.

Each sample point is equally likely to occur.

In the case of Example 1.1 (Section 1.1), each of the six possible outcomes has probability 1/6 and in Example 1.2 (Section 1.1), each of the eight possible outcomes has probability 1/8.

Thus, the probability which the event A occurs is defined as:

$$P(A) = \frac{n(A)}{n(\Omega)}.$$

In Example 1.1, $P(A) = 3/6$ and $P(A \cap B) = 1/6$ are obtained, because $n(\Omega) = 6$, $n(A) = 3$ and $n(A \cap B) = 1$.

Similarly, in Example 1.2, we have $P(C) = 4/8$, $P(A \cap B) = P(A) = 3/8$ and so on.

Note that we obtain $P(A) \leq P(B)$ because of $A \subseteq B$.

It is known that we have the following three properties on probability:

$$0 \leq P(A) \leq 1, \tag{1}$$

$$P(\Omega) = 1, \tag{2}$$

$$P(\phi) = 0. \tag{3}$$

$\phi \subseteq A \subseteq \Omega$ implies $n(\phi) \leq n(A) \leq n(\Omega)$.

Therefore, we have:

$$\frac{n(\phi)}{n(\Omega)} \leq \frac{n(A)}{n(\Omega)} \leq \frac{n(\Omega)}{n(\Omega)} = 1.$$

Dividing by $n(\Omega)$, we obtain:

$$P(\phi) \leq P(A) \leq P(\Omega) = 1.$$

Because ϕ has no sample point, the number of the sample point is given by $n(\phi) = 0$ and accordingly we have $P(\phi) = 0$.

Therefore, $0 \leq P(A) \leq 1$ is obtained as in (1).

When events A and B are exclusive, i.e., when $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$ holds.
Moreover, since A and A^c are exclusive, $P(A^c) = 1 - P(A)$ is obtained.
Note that $P(A \cup A^c) = P(\Omega) = 1$ holds.
Generally, unless A and B are not exclusive, we have the following formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

which is known as the **addition rule (加法定理)**.

In Example 1.1, each probability is given by $P(A \cup B) = 2/3$, $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/6$.
Thus, in the example we can verify that the above addition rule holds.

The probability which event A occurs, given that event B has occurred, is called the **conditional probability (条件付確率)**, i.e.,

$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{P(A \cap B)}{P(B)},$$

or equivalently,

$$P(A \cap B) = P(A|B)P(B),$$

which is called the **multiplication rule (乗法定理)**.

When event A is **independent (独立)** of event B, we have $P(A \cap B) = P(A)P(B)$, which implies that $P(A|B) = P(A)$.

Conversely, $P(A \cap B) = P(A)P(B)$ implies that A is independent of B.

In Example 1.2, because of $P(A \cap C) = 1/4$ and $P(C) = 1/2$, the conditional probability $P(A|C) = 1/2$ is obtained.
From $P(A) = 3/8$, we have $P(A \cap C) \neq P(A)P(C)$.
Therefore, A is not independent of C.
As for C and D, since we have $P(C) = 1/2$, $P(D) = 1/2$ and $P(C \cap D) = 1/4$, we can show that C is independent of D.

2 Random Variable and Distribution

2.1 Univariate Random Variable and Distribution

The **random variable (確率変数)** X is defined as the real value function on sample space Ω.
Since X is a function of a sample point ω, it is written as $X = X(\omega)$.
Suppose that $X(\omega)$ takes a real value on the interval I.

That is, X depends on a set of the sample point ω, i.e., $\{\omega; X(\omega) \in I\}$, which is simply written as $\{X \in I\}$.

In Example 1.1 (Section 1.1), suppose that X is a random variable which takes the number of spots up on the die.

Then, X is a function of ω and takes the following values:

$$X(\omega_1) = 1, \quad X(\omega_2) = 2, \quad X(\omega_3) = 3,$$
$$X(\omega_4) = 4, \quad X(\omega_5) = 5, \quad X(\omega_6) = 6.$$

Next, suppose that X is a random variable which takes 1 for odd numbers and 0 for even numbers on the die.

Then, X is a function of ω and takes the following values:

$$X(\omega_1) = 1, \quad X(\omega_2) = 0, \quad X(\omega_3) = 1,$$
$$X(\omega_4) = 0, \quad X(\omega_5) = 1, \quad X(\omega_6) = 0.$$

In Example 1.2 (Section 1.1), suppose that X is a random variable which takes the number of heads.

Depending on the sample point ω_i, X takes the following values:

$$X(\omega_1) = 3, \quad X(\omega_2) = 2, \quad X(\omega_3) = 2, \quad X(\omega_4) = 1,$$
$$X(\omega_5) = 2, \quad X(\omega_6) = 1, \quad X(\omega_7) = 1, \quad X(\omega_8) = 0.$$

Thus, the random variable depends on a sample point.

There are two kinds of random variables.

One is a **discrete random variable (離散型確率変数)**, while another is a **continuous random variable (連続型確率変数)**.

Discrete Random Variable (離散型確率変数) and Probability Function (確率関数): Suppose that the discrete random variable X takes $x_1, x_2, \cdots$, where $x_1 < x_2 < \cdots$ is assumed.

Consider the probability that X takes x_i, i.e., $P(X = x_i) = p_i$, which is a function of x_i.

That is, a function of x_i, say $f(x_i)$, is associated with $P(X = x_i) = p_i$.

The function $f(x_i)$ represents the probability in the case where X takes x_i.

Therefore, we have the following relation:

$$P(X = x_i) = p_i = f(x_i), \quad i = 1, 2, \cdots,$$

where $f(x_i)$ is called the **probability function (確率関数)** of X.

More formally, the function $f(x_i)$ which has the following properties is defined as the probability function.

$$f(x_i) \geq 0, \quad i = 1, 2, \cdots,$$
$$\sum_i f(x_i) = 1.$$

Furthermore, for a set A, we can write a probability as the following equation:

$$P(X \in A) = \sum_{x_i \in A} f(x_i).$$