**Example: $\chi^2(1)$ Distribution:** Define $Y = X^2$, where $X \sim N(0,1)$. Then, $Y \sim \chi^2(1)$.

**proof:**

$$f_y(y) = \frac{1}{2\sqrt{y}}\Big(f_x(\sqrt{y}) + f_x(-\sqrt{y})\Big)$$
$$= \frac{1}{\sqrt{2\pi}} y^{-1/2} \exp\Big(-\frac{1}{2}y\Big)$$

which is $\chi^2(1)$ distribution, where

$$f_x(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}x^2\Big)$$
$$f_x(\sqrt{y}) = f_x(-\sqrt{y}) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}y\Big)$$



Note that the $\chi^2(n)$ distribution is:

$$f_x(x) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp(-\frac{x}{2}), \qquad x > 0,$$

where $\Gamma(\frac{n}{2}) = \sqrt{\pi}$



## 4.2 Multivariate Cases

**Bivariate Case:** Let $f_{xy}(x,y)$ be a joint probability density function of $X$ and $Y$. Let $X = \psi_1(U,V)$ and $Y = \psi_2(U,V)$ be a one-to-one transformation from $(X,Y)$ to $(U,V)$. Then, we obtain a joint probability density function of $U$ and $V$, denoted by $f_{uv}(u,v)$, as follows:

$$f_{uv}(u,v) = |J| f_{xy}\Big(\psi_1(u,v), \psi_2(u,v)\Big),$$



where $J$ is called the **Jacobian** of the transformation, which is defined as:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$



**Multivariate Case:** Let $f_x(x_1, x_2, \cdots, x_n)$ be a joint probability density function of $X_1$, $X_2$, $\cdots$ $X_n$. Suppose that a one-to-one transformation from $(X_1, X_2, \cdots, X_n)$ to $(Y_1, Y_2, \cdots, Y_n)$ is given by:

$$X_1 = \psi_1(Y_1, Y_2, \cdots, Y_n),$$
$$X_2 = \psi_2(Y_1, Y_2, \cdots, Y_n),$$
$$\vdots$$
$$X_n = \psi_n(Y_1, Y_2, \cdots, Y_n).$$



Then, we obtain a joint probability density function of $Y_1$, $Y_2$, $\cdots$, $Y_n$, denoted by $f_y(y_1, y_2, \cdots, y_n)$, as follows:

$$f_y(y_1, y_2, \cdots, y_n)$$
$$= |J| f_x\Big(\psi_1(y_1, \cdots, y_n), \psi_2(y_1, \cdots, y_n), \cdots, \psi_n(y_1, \cdots, y_n)\Big),$$

where $J$ is called the Jacobian of the transformation, which is defined as:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}.$$



**Example: Normal Distribution:** $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$. $X$ is independent of $Y$.

Then, $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

**Proof:**

The density functions of $X$ and $Y$ are:

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{1}{2\sigma_1^2}(x-\mu_1)^2\right)$$

$$f_y(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{1}{2\sigma_2^2}(y-\mu_2)^2\right)$$



The joint density of $X$ and $Y$ is:

$$\begin{aligned} f_{xy}(x,y) &= f_x(x) f_y(y) \\ &= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2\sigma_1^2}(x-\mu_1)^2 - \frac{1}{2\sigma_2^2}(y-\mu_2)^2\right) \end{aligned}$$

Define $U = X+Y$ and $V = Y$. We obtain the joint distribution of $U$ and $V$.

Using $X = U - V$ and $Y = V$, the Jacobian is:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$



The joint density function of $U$ and $V$, $f_{uv}(u,v)$, is given by:

$$\begin{aligned} f_{uv}(u,v) &= |J| f_{xy}(u-v, v) \\ &= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2\sigma_1^2}(u-v-\mu_1)^2 - \frac{1}{2\sigma_2^2}(v-\mu_2)^2\right) \end{aligned}$$

The marginal density function of $U$ is:

$$\begin{aligned} f_u(u) &= \int f_{uv}(u,v)\mathrm{d}v \\ &= \int \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2\sigma_1^2}(u-v-\mu_1)^2 - \frac{1}{2\sigma_2^2}(v-\mu_2)^2\right)\mathrm{d}v \end{aligned}$$



$$\begin{aligned} &= \int \frac{1}{2\pi\sigma_1\sigma_2} \exp\Bigg( -\frac{1}{2\sigma_1^2}((v-\mu_2) - (u-\mu_1-\mu_2))^2 \\ &\qquad\qquad - \frac{1}{2\sigma_2^2}(v-\mu_2)^2\Bigg)\mathrm{d}v \\ &= \int \frac{1}{2\pi\sigma_1\sigma_2} \exp\Bigg( -\frac{1}{2/(1/\sigma_1^2 + 1/\sigma_2^2)}((v-\mu_2) \\ &\qquad -\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}(u-\mu_1-\mu_2))^2 - \frac{1}{2(\sigma_1^2+\sigma_2^2)}(u-\mu_1-\mu_2)^2\Bigg)\mathrm{d}v \end{aligned}$$



$$\begin{aligned} &= \int \frac{1}{\sqrt{2\pi/(1/\sigma_1^2 + 1/\sigma_2^2)}} \\ &\times \exp\left( -\frac{1}{2/(1/\sigma_1^2 + 1/\sigma_2^2)}((v-\mu_2) - \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}(u-\mu_1-\mu_2))^2\right) \\ &\times \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left( -\frac{1}{2(\sigma_1^2+\sigma_2^2)}(u-\mu_1-\mu_2)^2\right)\mathrm{d}v \end{aligned}$$

$$
\begin{aligned}
&= \int \frac{1}{\sqrt{2\pi/(1/\sigma_1^2 + 1/\sigma_2^2)}} \\
&\quad \times \exp\bigg( - \frac{1}{2/(1/\sigma_1^2 + 1/\sigma_2^2)}((v - \mu_2) \\
&\qquad - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}(u - \mu_1 - \mu_2))^2\bigg)\mathrm{d}v \\
&\quad \times \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\bigg( - \frac{1}{2(\sigma_1^2 + \sigma_2^2)}(u - \mu_1 - \mu_2)^2\bigg) \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\bigg( - \frac{1}{2(\sigma_1^2 + \sigma_2^2)}(u - \mu_1 - \mu_2)^2\bigg)
\end{aligned}
$$



**Example: $\chi^2$ Distribution:** $X \sim \chi^2(n)$ and $Y \sim \chi^2(m)$.

$X$ is independent of $Y$.

Then, $X + Y \sim \chi^2(n + m)$.

**Proof:**

The density functions of $X$ and $Y$ are:

$$
\begin{aligned}
f_x(x) &= \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp(-\frac{x}{2}), \quad x > 0 \\
f_y(y) &= \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} \exp(-\frac{y}{2}), \quad y > 0
\end{aligned}
$$



The joint density function of $X$ and $Y$ is:

$$
\begin{aligned}
f_{xy}(x, y) &= f_x(x) f_y(y) \\
&= \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp(-\frac{x}{2}) \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} \exp(-\frac{y}{2}) \\
&= C x^{\frac{n}{2}-1} y^{\frac{m}{2}-1} \exp(-\frac{x + y}{2})
\end{aligned}
$$

where $C = \dfrac{1}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}$.

From $X = U - V$ and $Y = V$, the Jacobian is:

$$
J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1
$$



The joint density function of $U$ and $V$, $f_{uv}(u, v)$, is given by:

$$
\begin{aligned}
f_{uv}(u, v) &= |J| f_{xy}(u - v, v) \\
&= C(u - v)^{\frac{n}{2}-1} v^{\frac{m}{2}-1} \exp(-\frac{u}{2})
\end{aligned}
$$

The marginal density function of $U$ is:

$$
\begin{aligned}
f_u(u) &= \int f_{uv}(u, v)\mathrm{d}v \\
&= C \exp(-\frac{u}{2}) \int_0^\infty (u - v)^{\frac{n}{2}-1} v^{\frac{m}{2}-1} \mathrm{d}v \\
&= C \exp(-\frac{u}{2}) \int_0^\infty (u - uw)^{\frac{n}{2}-1} (uw)^{\frac{m}{2}-1} u\mathrm{d}w
\end{aligned}
$$



$$
\begin{aligned}
&= C u^{\frac{n+m}{2}-1} \exp(-\frac{u}{2}) \int_0^1 (1 - w)^{\frac{n}{2}-1} w^{\frac{m}{2}-1} \mathrm{d}w \\
&= C B(\frac{n}{2}, \frac{m}{2}) u^{\frac{n+m}{2}-1} \exp(-\frac{u}{2}) \\
&= \frac{1}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}{\Gamma(\frac{n+m}{2})} u^{\frac{n+m}{2}-1} \exp(-\frac{u}{2}) \\
&= \frac{1}{2^{\frac{n+m}{2}}\Gamma(\frac{n+m}{2})} u^{\frac{n+m}{2}-1} \exp(-\frac{u}{2})
\end{aligned}
$$

Beta function $B(n, m)$ is:

$$
B(n, m) = \int_0^\infty (1 - x)^{n-1} x^{m-1} \mathrm{d}x = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n + m)}
$$



**Example: $t$ Distribution:** $X \sim N(0, 1)$ and $Y \sim \chi^2(n)$.

$X$ is independent of $Y$.

Then, $U = \dfrac{X}{\sqrt{Y/n}} \sim t(n)$

Note that the density function of $t(n)$ is:

$$
f_u(x) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \frac{1}{\sqrt{n\pi}} \bigg(1 + \frac{x^2}{n}\bigg)^{-\frac{n+1}{2}}
$$

**Proof:** The density functions of $X$ and $Y$ are:

$$f_x(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2), \qquad -\infty < x < \infty$$
$$f_y(y) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} \exp(-\frac{y}{2}), \qquad y > 0$$

The joint density functions of $X$ and $Y$, $f_{xy}(x, y)$, is:

$$\begin{aligned} f_{xy}(x, y) &= f_x(x) f_y(y) \\ &= \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2) \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} \exp(-\frac{y}{2}) \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} \exp(-\frac{y}{2} - \frac{1}{2}x^2) \end{aligned}$$



From $X = U\sqrt{\frac{V}{n}}$ and $Y = V$, the Jacobian is:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{v}{n}} & \frac{u}{2\sqrt{nv}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{v}{n}}$$

The joint density function of $U$ and $V$, $f_{uv}(u, v)$, is:

$$\begin{aligned} f_{uv}(u, v) &= |J| f_{xy}(u\sqrt{\frac{v}{n}}, v) \\ &= \sqrt{\frac{v}{n}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\frac{u^2 v}{n}) \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} v^{\frac{n}{2}-1} \exp(-\frac{v}{2}) \\ &= C v^{\frac{n-1}{2}} \exp\left(-\frac{v}{2}(1 + \frac{u^2}{n})\right) \end{aligned}$$



where $C = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{\pi}} \frac{1}{2^{\frac{n+1}{2}}\Gamma(\frac{n}{2})}$.

The marginal density function of $U$ is:

$$\begin{aligned} f_u(u) &= \int f_{uv}(u, v) \mathrm{d}v \\ &= C \int v^{\frac{n-1}{2}} \exp\left(-\frac{v}{2}(1 + \frac{u^2}{n})\right) \mathrm{d}v \\ &= C \int \left(w(1 + \frac{u^2}{n})^{-1}\right)^{\frac{n-1}{2}} \exp(-\frac{1}{2}w) \left(1 + \frac{u^2}{n}\right)^{-1} \mathrm{d}w \\ &= C\left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} \int w^{\frac{n+1}{2}-1} \exp(-\frac{1}{2}w) \mathrm{d}w \end{aligned}$$



$$\begin{aligned} &= C\left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} 2^{\frac{n+1}{2}} \Gamma(\frac{n+1}{2}) \\ &\times \int \frac{1}{2^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})} w^{\frac{n+1}{2}-1} \exp(-\frac{1}{2}w) \mathrm{d}w \\ &= C\left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} 2^{\frac{n+1}{2}} \Gamma(\frac{n+1}{2}) \\ &= \frac{1}{\sqrt{\pi}} \frac{1}{2^{\frac{n+1}{2}}\Gamma(\frac{n}{2})} \frac{1}{\sqrt{n}} 2^{\frac{n+1}{2}} \Gamma(\frac{n+1}{2}) \left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} \\ &= \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \frac{1}{\sqrt{n\pi}} \left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} \end{aligned}$$



Use integration by substitution by $w = v(1 + \frac{u^2}{n})$.
Note that $f(w) = \frac{1}{2^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})} w^{\frac{n+1}{2}-1} \exp(-\frac{1}{2}w)$ is the density function of $\chi^2(n + 1)$.



**Example: Cauchy Distribution:** $X \sim N(0, 1)$ and $Y \sim N(0, 1)$. $X$ is independent of $Y$.
Then, $U = \frac{X}{Y}$ is Cauchy.

Note that the density function of $U$, $f_u(u)$, is:

$$f(u) = \frac{1}{\pi(1 + u^2)}$$

**Proof:** The density functions of $X$ and $Y$ are:

$$f_x(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2), \qquad -\infty < x < \infty$$
$$f_y(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2), \qquad -\infty < y < \infty$$

The joint density function of $X$ and $Y$ is:

$$\begin{aligned} f_{xy}(x,y) &= f_x(x)f_y(y) \\ &= \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2) \\ &= \frac{1}{2\pi} \exp(-\frac{1}{2}(x^2+y^2)) \end{aligned}$$



Transformation of variables by $u = \dfrac{x}{y}$ and $v = y$.

From $x = uv$ and $y = v$, the Jacobian is:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

The joint density function of $U$ and $V$, $f_{uv}(u,v)$, is:

$$\begin{aligned} f_{uv}(u,v) &= |J| f_{xy}(uv, v) \\ &= |v| \frac{1}{2\pi} \exp(-\frac{1}{2}v^2(1+u^2)) \end{aligned}$$



The marginal density function of $U$ is:

$$\begin{aligned} f_u(u) &= \int f_{uv}(u,v)\mathrm{d}v \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |v| \exp(-\frac{1}{2}v^2(1+u^2))\mathrm{d}v \\ &= \frac{1}{\pi}\int_{0}^{\infty} v \exp(-\frac{1}{2}v^2(1+u^2))\mathrm{d}v \\ &= \frac{1}{\pi}\left[-\frac{1}{1+u^2}\exp(-\frac{1}{2}v^2(1+u^2))\right]_{v=0}^{\infty} \\ &= \frac{1}{\pi(1+u^2)} \end{aligned}$$



# 5 Moment-Generating Function (積率母関数)

## 5.1 Univariate Case

As discussed in Section 3.1, the moment-generating function is defined as $\phi(\theta) = \mathrm{E}(e^{\theta X})$.

For a random variable $X$, $\mu'_n \equiv \mathrm{E}(X^n)$ is called the ***n*th moment (*n* 次の積率)** of $X$.



1. **Theorem:** $\phi^{(n)}(0) = \mu'_n \equiv \mathrm{E}(X^n)$.

   **Proof:**

   First, from the definition of the moment-generating function, $\phi(\theta)$ is written as:

   $$\phi(\theta) = \mathrm{E}(e^{\theta X}) = \int_{-\infty}^{\infty} e^{\theta x} f(x)\,\mathrm{d}x.$$

   The $n$th derivative of $\phi(\theta)$, denoted by $\phi^{(n)}(\theta)$, is:

   $$\phi^{(n)}(\theta) = \int_{-\infty}^{\infty} x^n e^{\theta x} f(x)\,\mathrm{d}x.$$



Evaluating $\phi^{(n)}(\theta)$ at $\theta = 0$, we obtain:

$$\phi^{(n)}(0) = \int_{-\infty}^{\infty} x^n f(x)\,\mathrm{d}x = \mathrm{E}(X^n) \equiv \mu'_n,$$

where the second equality comes from the definition of the mathematical expectation.

2. **Remark:** The moment-generating function is a weighted sum of all the moments.

$$\phi(\theta) = \mathrm{E}(e^{\theta X}) = \mathrm{E}\Big(\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\theta^n\Big)$$
$$= \mathrm{E}\Big(\sum_{n=0}^{\infty} \frac{1}{n!} X^n \theta^n\Big) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{E}(X^n)\theta^n$$

where $f(\theta) = e^{\theta X}$. $\qquad f(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\theta^n$

Note that $f^{(n)}(\theta) = X^n e^{\theta X}$.



3. **Remark:** $\phi(\theta)$ does not exist, if $\mathrm{E}(X^n)$ for some $n$ does not exist.

$\phi(\theta)$ is finite. $\iff$ All the moments exist.



4. **Remark:** Let $X$ and $Y$ be two random variables. Suppose that both moment-generating functions exist. When the moment-generating function of $X$ is equivalent to that of $Y$, we have the fact that $X$ has the same distribution as $Y$.

$$\phi_x(\theta) = \phi_y(\theta) \iff \mathrm{E}(X^n) = \mathrm{E}(Y^n) \text{ for all } n$$
$$\iff f_x(t) = f_y(t)$$



5. **Theorem:** Let $\phi(\theta)$ be the moment-generating function of $X$. Then, the moment-generating function of $Y$, where $Y = aX + b$, is given by $e^{b\theta}\phi(a\theta)$.

**Proof:**

Let $\phi_y(\theta)$ be the moment-generating function of $Y$. Then, $\phi_y(\theta)$ is rewritten as follows:

$$\phi_y(\theta) = \mathrm{E}(e^{\theta Y}) = \mathrm{E}(e^{\theta(aX+b)}) = e^{b\theta}\mathrm{E}(e^{a\theta X}) = e^{b\theta}\phi(a\theta).$$

$\phi(\theta)$ represents the moment-generating function of $X$.



6. **Theorem:** Let $\phi_1(\theta), \phi_2(\theta), \cdots, \phi_n(\theta)$ be the moment-generating functions of $X_1, X_2, \cdots, X_n$, which are mutually independently distributed random variables. Define $Y = X_1 + X_2 + \cdots + X_n$. Then, the moment-generating function of $Y$ is given by $\phi_1(\theta)\phi_2(\theta)\cdots\phi_n(\theta)$, i.e.,

$$\phi_y(\theta) = \mathrm{E}(e^{\theta Y}) = \phi_1(\theta)\phi_2(\theta)\cdots\phi_n(\theta),$$

where $\phi_y(\theta)$ represents the moment-generating function of $Y$.



**Proof:**

The moment-generating function of $Y$, i.e., $\phi_y(\theta)$, is rewritten as:

$$\phi_y(\theta) = \mathrm{E}(e^{\theta Y}) = \mathrm{E}(e^{\theta(X_1+X_2+\cdots+X_n)})$$
$$= \mathrm{E}(e^{\theta X_1})\mathrm{E}(e^{\theta X_2})\cdots\mathrm{E}(e^{\theta X_n})$$
$$= \phi_1(\theta)\phi_2(\theta)\cdots\phi_n(\theta).$$

The third equality holds because $X_1, X_2, \cdots, X_n$ are mutually independently distributed random variables.

7. **Theorem:** When $X_1$, $X_2$, $\cdots$, $X_n$ are mutually independently and identically distributed and the moment-generating function of $X_i$ is given by $\phi(\theta)$ for all $i$, the moment-generating function of $Y$ is represented by $\left(\phi(\theta)\right)^n$, where $Y = X_1 + X_2 + \cdots + X_n$.



**Proof:**

Using the above theorem, we have the following:

$$\begin{aligned}\phi_y(\theta) &= \phi_1(\theta)\phi_2(\theta)\cdots\phi_n(\theta) \\ &= \phi(\theta)\phi(\theta)\cdots\phi(\theta) = \left(\phi(\theta)\right)^n.\end{aligned}$$

Note that $\phi_i(\theta) = \phi(\theta)$ for all $i$.