8. **Theorem:** When $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed and the moment-generating function of X_i is given by $\phi(\theta)$ for all i, the moment-generating function of $\overline{X}$ is represented by $\Big(\phi(\dfrac{\theta}{n})\Big)^n$, where $\overline{X} = (1/n)\sum_{i=1}^n X_i$.

Proof:

Let $\phi_{\overline{x}}(\theta)$ be the moment-generating function of $\overline{X}$.

$$\phi_{\overline{x}}(\theta) = \mathrm{E}(e^{\theta\overline{X}}) = \mathrm{E}(e^{\frac{\theta}{n}\sum_{i=1}^n X_i}) = \prod_{i=1}^n \mathrm{E}(e^{\frac{\theta}{n}X_i})$$
$$= \prod_{i=1}^n \phi(\frac{\theta}{n}) = \Big(\phi(\frac{\theta}{n})\Big)^n.$$

Bernoulli Distribution: The probability function of Bernoulli random variable X is:

$$f(x) = p^x(1-p)^{1-x}, \qquad x = 0, 1$$

The moment-generating function of X is:

$$\phi(\theta) = pe^\theta + 1 - p$$

Mean: $\mathrm{E}(X) = \phi'(0) = p$

Variance: $\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \phi''(0) - p^2 = p(1-p)$

Binomial Distribution: For the binomial random variable, the moment-generating function $\phi(\theta)$ is known as:

$$\phi(\theta) = (pe^\theta + 1 - p)^n,$$

which is discussed in Example 1.5 (Section 3.1). Using the moment-generating function, we check whether $\mathrm{E}(X) = np$ and $\mathrm{V}(X) = np(1-p)$ are obtained when X is a binomial random variable.

The first- and the second-derivatives with respect to θ are

given by:

$$\phi'(\theta) = npe^\theta(pe^\theta + 1 - p)^{n-1},$$
$$\phi''(\theta) = npe^\theta(pe^\theta + 1 - p)^{n-1} + n(n-1)p^2e^{2\theta}(pe^\theta + 1 - p)^{n-2}.$$

Evaluating at $\theta = 0$, we have:

$$\mathrm{E}(X) = \phi'(0) = np, \quad \mathrm{E}(X^2) = \phi''(0) = np + n(n-1)p^2.$$

Therefore, $\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = np(1-p)$ can be derived. Thus, we can make sure that $\mathrm{E}(X)$ and $\mathrm{V}(X)$ are obtained from $\phi(\theta)$.

Poisson Distribution: The probability function of Poisson random variable X is:

$$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \cdots$$

The moment-generating function of X is:

$$\phi(\theta) = \sum_{x=0}^\infty e^{\theta x}e^{-\lambda}\frac{\lambda^x}{x!}$$
$$= \sum_{x=0}^\infty e^{-\lambda}e^{e^\theta\lambda}e^{-e^\theta\lambda}\frac{(e^\theta\lambda)^x}{x!}$$
$$= \exp(\lambda(e^\theta - 1))$$

Normal Distribution: When $X \sim N(\mu, \sigma^2)$, the moment-generating function of X is given by: $\phi(\theta) = \exp(\mu\theta + \frac{1}{2}\sigma^2\theta^2)$ from the previous example.

Obtain E(X) and V(X), using $\phi(\theta)$.

- $\mathrm{E}(X) = \phi'(0) = \mu$

from $\phi'(\theta) = (\mu + \sigma^2\theta)\exp(\mu\theta + \frac{1}{2}\sigma^2\theta^2)$.

- $\mathrm{E}(X^2) = \phi''(0) = \sigma^2 + \mu^2$

from $\phi''(\theta) = \sigma^2\exp(\mu\theta+\frac{1}{2}\sigma^2\theta^2)+(\mu+\sigma^2\theta)^2\exp(\mu\theta+\frac{1}{2}\sigma^2\theta^2)$.

- $\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = (\sigma^2 + \mu^2) - \mu^2 = \sigma^2$

Cauchy Distribution: Cauchy distribution: $f(x) = \dfrac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$.

$$\begin{aligned}\mathrm{E}(X) &= \int x f(x)\mathrm{d}x = \int \frac{x}{\pi(1+x^2)}\mathrm{d}x \\ &= \frac{1}{2\pi}[\log(1+x^2)]_{-\infty}^{\infty}\end{aligned}$$

$\Longrightarrow$ $\phi(\theta)$ does not exists.

$t(k)$ distrubution $\Longrightarrow$ $\mathrm{E}(X^k)$ does not exists.

Uniform Distribution: The density function is:

$$f(x) = \frac{1}{b-a}, \qquad a < x < b$$

The moment-generating function is:

$$\begin{aligned}\phi(\theta) &= \int_{-\infty}^{\infty} e^{\theta x} f(x)\mathrm{d}x = \int_a^b e^{\theta x}\frac{1}{b-a}\mathrm{d}x \\ &= \left[\frac{e^{\theta x}}{\theta(b-a)}\right]_a^b = \frac{e^{\theta b} - e^{\theta a}}{\theta(b-a)}\end{aligned}$$

$$\phi'(\theta) = \frac{\theta(be^{\theta b} - ae^{\theta a}) - (e^{\theta b} - e^{\theta a})}{\theta^2(b-a)}$$

Mean:

$$\begin{aligned}\mathrm{E}(X) &= \phi'(0) \longleftarrow \text{ Use L'Hospital's rule.} \\ &= \frac{a+b}{2}\end{aligned}$$

(*) $f(\theta) = \theta(be^{\theta b} - ae^{\theta a}) - (e^{\theta b} - e^{\theta a})$, $\quad g(\theta) = \theta^2(b-a)$

$f'(\theta) = \theta(b^2e^{\theta b} - a^2e^{\theta a})$, $\quad g'(\theta) = 2\theta(b-a)$

$$\lim_{\theta\to 0}\frac{f(\theta)}{g(\theta)} = \lim_{\theta\to 0}\frac{f'(\theta)}{g'(\theta)} = \lim_{\theta\to 0}\frac{\theta(b^2e^{\theta b} - a^2e^{\theta a})}{2\theta(b-a)} = \frac{a+b}{2}$$

(*) L'Hospital's rule

For two continuous functions $f(x)$ and $g(x)$,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}, \qquad \text{or} \qquad \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)},$$

L'Hospital's rule is used when we have:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\infty}{\infty} \text{ or } \frac{0}{0},$$

or

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{\infty}{\infty} \text{ or } \frac{0}{0}.$$

Variance: $\quad \mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$$\begin{aligned}\mathrm{E}(X^2) &= \phi''(0) \\ &= \frac{\theta^2(b^2e^{\theta b} - a^2e^{\theta a}) - 2\theta(be^{\theta b} - ae^{\theta a}) + 2(e^{\theta b} - e^{\theta a})}{\theta^3(b-a)}\end{aligned}$$

$$\begin{cases} f(\theta) = \theta^2(b^2e^{\theta b} - a^2e^{\theta a}) - 2\theta(be^{\theta b} - ae^{\theta a}) + 2(e^{\theta b} - e^{\theta a}) \\ g(\theta) = \theta^3(b-a) \end{cases}$$

$$\begin{cases} f'(\theta) = \theta^2(b^3e^{\theta b} - a^3e^{\theta a}) \\ g'(\theta) = 3\theta^2(b-a) \end{cases}$$

$$\phi''(0) = \lim_{\theta\to 0}\frac{f(\theta)}{g(\theta)} = \lim_{\theta\to 0}\frac{f'(\theta)}{g'(\theta)}$$
$$= \lim_{\theta\to 0}\frac{\theta^2(b^3e^{\theta b} - a^3e^{\theta a})}{3\theta^2(b-a)} = \frac{b^2+ba+a^2}{3}$$

$$\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$$
$$= \phi''(0) - (\phi'(0))^2 \longleftarrow \text{ L'Hospital's rule}$$
$$= \frac{b^2+ba+a^2}{3} - \Big(\frac{a+b}{2}\Big)^2 = \frac{(b-a)^2}{12}$$

Exponential Distribution: The exponential distribution is:

$$f(x) = \lambda e^{-\lambda x}, \quad 0 < x$$

The moment-generating function is:

$$\phi(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x)\mathrm{d}x = \int_0^{\infty} e^{\theta x}\lambda e^{-\lambda x}\mathrm{d}x$$
$$= \frac{\lambda}{\lambda-\theta}\int_0^{\infty}(\lambda-\theta)e^{-(\lambda-\theta)x}\mathrm{d}x = \frac{\lambda}{\lambda-\theta}$$

Use the exponential distribution with parameter $\lambda - \theta$ in the integration.

1. Mean: $\mathrm{E}(X) = \phi'(0)$

$$\phi'(\theta) = \frac{\lambda}{(\lambda-\theta)^2}$$
$$\mathrm{E}(X) = \phi'(0) = \frac{1}{\lambda}$$

2. Variance: $\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$$\mathrm{E}(X^2) = \phi''(0) \qquad \phi''(\theta) = 2\frac{\lambda}{(\lambda-\theta)^3}$$
$$\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \phi''(0) - (\phi'(0))^2$$
$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

χ^2 **Distribution:** The density function is:

$$f(x) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}\exp(-\frac{x}{2}), \quad 0 < x$$

The moment-generating function is:

$$\phi(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x)\mathrm{d}x$$
$$= \int_0^{\infty} e^{\theta x}\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}\exp(-\frac{x}{2})\mathrm{d}x$$
$$= \int_0^{\infty}\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}\exp\left(-\frac{1}{2}(1-2\theta)x\right)\mathrm{d}x$$
$$= \int_0^{\infty}\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\left(\frac{y}{1-2\theta}\right)^{\frac{n}{2}-1}\exp(-\frac{1}{2}y)\frac{1}{1-2\theta}\mathrm{d}y$$

$$= \left(\frac{1}{1-2\theta}\right)^{\frac{n}{2}}\int_0^{\infty}\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}y^{\frac{n}{2}-1}\exp(-\frac{1}{2}y)\mathrm{d}y$$
$$= \Big(\frac{1}{1-2\theta}\Big)^{\frac{n}{2}}$$

Use integration by substitution by $y = (1-2\theta)x$

$\dfrac{\mathrm{d}x}{\mathrm{d}y} = (1-2\theta)^{-1}$

Use the $\chi^2(n)$ distribution in the integration.

1. Mean: $\mathrm{E}(X) = \phi'(0)$

$$\phi'(\theta) = (-\tfrac{n}{2})(-2)(1-2\theta)^{-\frac{n}{2}-1}$$
$$\mathrm{E}(X) = \phi'(0) = n$$

2. Variance: $\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$$\mathrm{E}(X^2) = \phi''(0)$$
$$\phi''(\theta) = (-\tfrac{n}{2})(-\tfrac{n}{2}-1)(-2)^2(1-2\theta)^{-\frac{n}{2}-1}$$
$$\mathrm{V}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \phi''(0) - (\phi'(0))^2$$
$$= n(n+2) - n^2 = 2n$$

Sum of Bernoulli Random Variables: $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as Bernoulli random variable with parameter p.

Then, the probability function of $Y = X_1 + X_2 + \cdots + X_n$ is $B(n, p)$.

Proof: The moment-generating function of X_i, $\phi_i(\theta)$, is:

$$\phi_i(\theta) = pe^{\theta} + 1 - p$$

The moment-generating function of Y, $\phi_y(\theta)$, is:

$$\begin{aligned}\phi_y(\theta) &= \mathrm{E}(e^{\theta Y}) = \mathrm{E}(e^{\theta(X_1+X_2+\cdots+X_n)}) \\ &= \mathrm{E}(e^{\theta X_1})\mathrm{E}(e^{\theta X_2})\cdots\mathrm{E}(e^{\theta X_n}) = \phi_1(\theta)\phi_2(\theta)\cdots\phi_n(\theta) \\ &= \Big(\phi(\theta)\Big)^n = (pe^{\theta} + 1 - p)^n,\end{aligned}$$

which is the moment-generating function of $B(n, p)$.

Note:

In the third equality, $X_1, X_2, \cdots, X_n$ are mutually independent.

In the fifth equality, $X_1, X_2, \cdots, X_n$ are identically distributed.

Sum of Two Normal Random Variables: $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$. X is independent of Y.

Then, $aX + bY \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$, where a are b are constant.

Proof: Suppose tha the moment-generating functions of X and Y are given by $\phi_x(\theta)$ and $\phi_y(\theta)$.

$$\begin{aligned}\phi_x(\theta) &= \exp\Big(\mu_1\theta + \frac{1}{2}\sigma_1^2\theta^2\Big) \\ \phi_y(\theta) &= \exp\Big(\mu_2\theta + \frac{1}{2}\sigma_2^2\theta^2\Big)\end{aligned}$$

The moment-generating function of $W = aX + bY$ is:

$$\begin{aligned}\phi_w(\theta) &= \mathrm{E}(e^{\theta W}) = \mathrm{E}(e^{\theta(aX+bY)}) = \mathrm{E}(e^{a\theta X})\mathrm{E}(e^{b\theta Y}) = \phi_x(a\theta)\phi_y(b\theta) \\ &= \exp\Big(\mu_1(a\theta) + \frac{1}{2}\sigma_1^2(a\theta)^2\Big) \times \exp\Big(\mu_2(b\theta) + \frac{1}{2}\sigma_2^2(b\theta)^2\Big) \\ &= \exp\Big((a\mu_1 + b\mu_2)\theta + \frac{1}{2}(a^2\sigma_1^2 + b^2\sigma_2^2)\theta^2\Big)\end{aligned}$$

which is the moment-generating function of normal distribution with mean $a\mu_1 + b\mu_2$ and variance $a^2\sigma^2 + b^2\sigma_2^2$.

Therefore, $aX + bY \sim N(a\mu_1 + b\mu_2, a^2\sigma^2 + b^2\sigma_2^2)$

Sum of Two χ^2 Random Variables: $X \sim \chi^2(n)$ and $Y \sim \chi^2(m)$. X is independent of Y.

Then, $Z = X + Y \sim \chi^2(n + m)$

Proof:

Let $\phi_x(\theta)$ and $\phi_y(\theta)$ be the moment-generating functions of X and Y.

$\phi_x(\theta)$ and $\phi_y(\theta)$ are given by:

$$\phi_x(\theta) = \Big(\frac{1}{1-2\theta}\Big)^{\frac{n}{2}}, \qquad \phi_y(\theta) = \Big(\frac{1}{1-2\theta}\Big)^{\frac{m}{2}}.$$

The moment-generating function of $Z = X + Y$ is:

$$\begin{aligned}\phi_z(\theta) &\equiv \mathrm{E}(e^{\theta Z}) = \mathrm{E}(e^{\theta(X+Y)}) = \mathrm{E}(e^{\theta X})\mathrm{E}(e^{\theta Y}) = \phi_x(\theta)\phi_y(\theta) \\ &= \Big(\frac{1}{1-2\theta}\Big)^{\frac{n}{2}}\Big(\frac{1}{1-2\theta}\Big)^{\frac{m}{2}} = \Big(\frac{1}{1-2\theta}\Big)^{\frac{n+m}{2}}\end{aligned}$$

which is the moment-generating function of $\chi^2(n + m)$ distribution. Therefore, $Z \sim \chi^2(n + m)$.

Note:

In the third equality, X and Y are independent.

5.2 Multivariate Cases

Bivariate Case: As discussed in Section 3.2, for two random variables X and Y, the moment-generating function is defined as $\phi(\theta_1, \theta_2) = \mathrm{E}(e^{\theta_1 X + \theta_2 Y})$. Some useful and important theorems and remarks are shown as follows.

1. **Theorem:** Consider two random variables X and Y. Let $\phi(\theta_1, \theta_2)$ be the moment-generating function of X and Y. Then, we have the following result:

$$\frac{\partial^{j+k}\phi(0,0)}{\partial\theta_1^j \partial\theta_2^k} = \mathrm{E}(X^j Y^k).$$

Proof:

Let $f_{xy}(x, y)$ be the probability density function of X and Y. From the definition, $\phi(\theta_1, \theta_2)$ is written as:

$$\phi(\theta_1, \theta_2) = \mathrm{E}(e^{\theta_1 X + \theta_2 Y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\theta_1 x + \theta_2 y} f_{xy}(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Taking the jth derivative of $\phi(\theta_1, \theta_2)$ with respect to θ_1 and at the same time the kth derivative with respect to θ_2, we have the following expression:

$$\frac{\partial^{j+k}\phi(\theta_1,\theta_2)}{\partial\theta_1^j \partial\theta_2^k} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k e^{\theta_1 x + \theta_2 y} f_{xy}(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Evaluating the above equation at $(\theta_1, \theta_2) = (0, 0)$, we can easily obtain:

$$\frac{\partial^{j+k}\phi(0,0)}{\partial\theta_1^j \partial\theta_2^k} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{xy}(x, y)\,\mathrm{d}x\,\mathrm{d}y \equiv \mathrm{E}(X^j Y^k).$$

2. **Remark:** Let (X_i, Y_i) be a pair of random variables. Suppose that the moment-generating function of (X_1, Y_1) is equivalent to that of (X_2, Y_2). Then, (X_1, Y_1) has the same distribution function as (X_2, Y_2).