2. If $X \sim N(\mu, \Sigma)$, then $(X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2(k)$.

   Note that $X'X \sim \chi^2(k)$ when $X \sim N(0, I_k)$.

3. X: $n \times 1$, Y: $m \times 1$, $X \sim N(\mu_x, \Sigma_x)$, $Y \sim N(\mu_y, \Sigma_y)$

   X is independent of Y, i.e., $\mathrm{E}\big((X-\mu_x)(Y-\mu_y)'\big) = 0$ in the case of normal random variables.

$$\frac{(X-\mu_x)'\Sigma_x^{-1}(X-\mu_x)/n}{(Y-\mu_y)'\Sigma_y^{-1}(Y-\mu_y)/m} \sim F(n,m)$$

4. If $X \sim N(0, \sigma^2 I_n)$ and A is a symmetric idempotent $n \times n$ matrix of rank G, then $X'AX/\sigma^2 \sim \chi^2(G)$.

   Note that $X'AX = (AX)'(AX)$ and $\mathrm{rank}(A) = \mathrm{tr}(A)$ because A is idempotent.

5. If $X \sim N(0, \sigma^2 I_n)$, A and B are symmetric idempotent $n \times n$ matrices of rank G and K, and $AB = 0$, then

$$\frac{X'AX}{G\sigma^2}\Big/\frac{X'BX}{K\sigma^2} = \frac{X'AX/G}{X'BX/K} \sim F(G,K).$$

3 Multiple Regression Model (重回帰モデル)

Up to now, only one independent variable, i.e., x_i, is taken into the regression model.

In this section, we extend it to more independent variables, which is called the **multiple regression (重回帰)**.

We consider the following regression model:

$$\begin{aligned} y_i &= \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_k x_{i,k} + u_i \\ &= (x_{i,1}, x_{i,2}, \cdots, x_{i,k}) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + u_i \\ &= x_i\beta + u_i, \end{aligned}$$

for $i = 1, 2, \cdots, n$,

where x_i and β denote a $1 \times k$ vector of the independent variables and a $k \times 1$ vector

of the unknown parameters to be estimated, which are represented as:

$$x_i = (x_{i,1}, x_{i,2}, \cdots, x_{i,k}), \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}.$$

$x_{i,j}$ denotes the ith observation of the jth independent variable.

The case of $k = 2$ and $x_{i,1} = 1$ for all i is exactly equivalent to (1).

Therefore, the matrix form above is a generalization of (1).

Writing all the equations for $i = 1, 2, \cdots, n$, we have:

$$\begin{aligned} y_1 &= \beta_1 x_{1,1} + \beta_2 x_{1,2} + \cdots + \beta_k x_{1,k} + u_1 = x_1\beta + u_1, \\ y_2 &= \beta_1 x_{2,1} + \beta_2 x_{2,2} + \cdots + \beta_k x_{2,k} + u_2 = x_2\beta + u_2, \\ &\vdots \\ y_n &= \beta_1 x_{n,1} + \beta_2 x_{n,2} + \cdots + \beta_k x_{n,k} + u_n = x_n\beta + u_n, \end{aligned}$$

which is rewritten as:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$
$$= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \beta + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

Again, the above equation is compactly rewritten as:

$$y = X\beta + u, \tag{18}$$

where y, X and u are denoted by:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \qquad X = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}.$$

Utilizing the matrix form (18), we derive the ordinary least squares estimator of β, denoted by $\hat{\beta}$.

In (18), replacing β by $\hat{\beta}$, we have the following equation:

$$y = X\hat{\beta} + e,$$

where e denotes a $n \times 1$ vector of the residuals.

The ith element of e is given by e_i.

The sum of squared residuals is written as follows:

$$S(\hat{\beta}) = \sum_{i=1}^{n} e_i^2 = e'e = (y - X\hat{\beta})'(y - X\hat{\beta}) = (y' - \hat{\beta}'X')(y - X\hat{\beta})$$
$$= y'y - y'X\hat{\beta} - \hat{\beta}'X'y + \hat{\beta}'X'X\hat{\beta} = y'y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}.$$

In the last equality, note that $\hat{\beta}'X'y = y'X\hat{\beta}$ because both are scalars.

To minimize $S(\hat{\beta})$ with respect to $\hat{\beta}$, we set the first derivative of $S(\hat{\beta})$ equal to zero, i.e.,

$$\frac{\partial S(\hat{\beta})}{\partial \hat{\beta}} = -2X'y + 2X'X\hat{\beta} = 0.$$

Solving the equation above with respect to $\hat{\beta}$, the **ordinary least squares estimator (OLS, 最小自乗推定量)** of β is given by:

$$\hat{\beta} = (X'X)^{-1}X'y. \tag{19}$$

Thus, the ordinary least squares estimator is derived in the matrix form.

(*) Remark

The second order condition for minimization:

$$\frac{\partial^2 S(\hat{\beta})}{\partial \hat{\beta} \partial \hat{\beta}'} = 2X'X$$

is a positive definite matrix.

Set $c = Xd$.

For any $d \neq 0$, we have $c'c = d'X'Xd > 0$.