Defining $C = D + (X'X)^{-1}X'$, $\mathrm{V}(\widetilde{\beta})$ is rewritten as:

$$\mathrm{V}(\widetilde{\beta}) = \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')'.$$

Moreover, because $\widetilde{\beta}$ is unbiased, we have the following:

$$CX = I_k = (D + (X'X)^{-1}X')X = DX + I_k.$$

Therefore, we have the following condition:

$$DX = 0.$$

Accordingly, $\mathrm{V}(\widetilde{\beta})$ is rewritten as:

$$\begin{aligned}\mathrm{V}(\widetilde{\beta}) &= \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')' \\ &= \sigma^2 (X'X)^{-1} + \sigma^2 DD' = \mathrm{V}(\hat{\beta}) + \sigma^2 DD'\end{aligned}$$

Thus, $\mathrm{V}(\widetilde{\beta}) - \mathrm{V}(\hat{\beta})$ is a positive definite matrix.

$\Longrightarrow$ $\mathrm{V}(\widetilde{\beta}_i) - \mathrm{V}(\hat{\beta}_i) > 0$

$\Longrightarrow$ $\hat{\beta}$ is a minimum variance (i.e., best) linear unbiased estimator of β.

Note as follows:

$\Longrightarrow$ A is positive definite when $d'Ad > 0$ except $d = 0$.

$\Longrightarrow$ The ith diagonal element of A, i.e., a_{ii}, is positive (choose d such that the ith element of d is one and the other elements are zeros).

***F* Distribution ($H_0 : \beta = 0$):**

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$.
   Therefore, $\dfrac{(\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)}{\sigma^2} \sim \chi^2(k)$.
2. **Proof:**
   Using $\hat{\beta} - \beta = (X'X)^{-1}X'u$, we obtain:

$$\begin{aligned}(\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) &= ((X'X)^{-1}X'u)'X'X(X'X)^{-1}X'u \\ &= u'X(X'X)^{-1}X'X(X'X)^{-1}X'u = u'X(X'X)^{-1}X'u\end{aligned}$$

Note that $X(X'X)^{-1}X'$ is symmetric and idempotent, i.e., $A'A = A$.

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(X(X'X)^{-1}X')\Big)$$

The degree of freedom is given by:

$$\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$$

Therefore, we obtain:

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k)$$

3. (*) Formula:

   Suppose that $X \sim N(0, I_k)$.

   If A is symmetric and idempotent, i.e., $A'A = A$, then $X'AX \sim \chi^2(\mathrm{tr}(A))$.

   Here, $X = \dfrac{1}{\sigma}u \sim N(0, I_n)$ from $u \sim N(0, \sigma^2 I_n)$, and $A = X(X'X)^{-1}X'$.

4. **Sum of Residuals:** e is rewritten as:

$$e = (I_n - X(X'X)^{-1}X')u.$$

Therefore, the sum of residuals is given by:

$$e'e = u'(I_n - X(X'X)^{-1}X')u.$$

Note that $I_n - X(X'X)^{-1}X'$ is symmetric and idempotent.

We obtain the following result:

$$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(I_n - X(X'X)^{-1}X')\Big),$$

where the trace is:

$$\mathrm{tr}(I_n - X(X'X)^{-1}X') = n - k.$$

Therefore, we have the following result:

$$\frac{e'e}{\sigma^2} = \frac{(n-k)s^2}{\sigma^2} \sim \chi^2(n-k),$$

where

$$s^2 = \frac{1}{n-k}e'e.$$

5. We show that $\hat\beta$ is independent of e.

Proof:

Because $u \sim N(0, \sigma^2 I_n)$, we show that $\mathrm{Cov}(e, \hat\beta) = 0$.

$$\begin{aligned}
\mathrm{Cov}(e,\hat\beta) &= \mathrm{E}(e(\hat\beta-\beta)') = \mathrm{E}\Big((I_n - X(X'X)^{-1}X')u((X'X)^{-1}X'u)'\Big) \\
&= \mathrm{E}\Big((I_n - X(X'X)^{-1}X')uu'X(X'X)^{-1}\Big) = (I_n - X(X'X)^{-1}X')\mathrm{E}(uu')X(X'X)^{-1} \\
&= (I_n - X(X'X)^{-1}X')(\sigma^2 I_n)X(X'X)^{-1} = \sigma^2(I_n - X(X'X)^{-1}X')X(X'X)^{-1} \\
&= \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}X'X(X'X)^{-1}) = \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}) = 0.
\end{aligned}$$

Therefore, $\hat\beta$ is independent of e.

6. Therefore, we obtain the following distribution:

$$\frac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2} = \frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k),$$

$$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2(n-k)$$

$\hat\beta$ is independent of e.

Accordingly, we can derive:

$$\frac{\dfrac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2}\Big/k}{\dfrac{e'e}{\sigma^2}\Big/(n-k)} = \frac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)/k}{s^2} \sim F(k, n-k)$$

Note as follows:

$$\frac{\dfrac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2}\Big/k}{\dfrac{e'e}{\sigma^2}\Big/(n-k)} = \frac{u'X(X'X)^{-1}X'u/k}{u'(I_n - X(X'X)^{-1}X')u/(n-k)} \sim F(k, n-k),$$

because $X(X'X)^{-1}X'(I_n - X(X'X)^{-1}X') = 0$.

(*) Formula:

When $X \sim N(0, I_n)$, A and B are $n \times n$ symmetric idempotent matrices, $\mathrm{Rank}(A) = \mathrm{tr}(A) = G$, $\mathrm{Rank}(B) = \mathrm{tr}(B) = K$ and $AB = 0$, then $\dfrac{X'AX/G}{X'BX/K} \sim F(G, K)$.

Coefficient of Determination (決定係数), R^2:

1. Definition of the Coefficient of Determination, R^2: $R^2 = 1 - \dfrac{\sum_{i=1}^n e_i^2}{\sum_{i=1}^n (y_i - \overline{y})^2}$
2. Numerator: $\displaystyle\sum_{i=1}^n e_i^2 = e'e$
3. Denominator: $\displaystyle\sum_{i=1}^n (y_i - \overline{y})^2 = y'(I_n - \frac{1}{n}ii')'(I_n - \frac{1}{n}ii')y = y'(I_n - \frac{1}{n}ii')y$

(*) Remark

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_T - \overline{y} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} - \begin{pmatrix} \overline{y} \\ \overline{y} \\ \vdots \\ \overline{y} \end{pmatrix} = y - \frac{1}{n} ii'y = (I_n - \frac{1}{n} ii')y,$$

where $i = (1, 1, \cdots, 1)'$.

4. In a matrix form, we can rewrite as: $R^2 = 1 - \dfrac{e'e}{y'(I_n - \frac{1}{n}ii')y}$

F Distribution and Coefficient of Determination:

$\Longrightarrow$ This will be discussed later.

Testing Linear Restrictions (F Distribution):

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$.

   Consider testing the hypothesis $H_0 : R\beta = r$.

   $R: \ G \times k, \qquad \text{rank}(R) = G \leq k.$

   $R\hat{\beta} \sim N(R\beta, \sigma^2 R(X'X)^{-1}R')$.

Therefore, $\dfrac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2(G)$.

Note that $R\beta = r$.

(a) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the mean is:

$\mathrm{E}(R\hat{\beta}) = R\mathrm{E}(\hat{\beta}) = R\beta$.

(b) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the variance is:

$$\begin{aligned} \mathrm{V}(R\hat{\beta}) &= \mathrm{E}((R\hat{\beta} - R\beta)(R\hat{\beta} - R\beta)') = \mathrm{E}(R(\hat{\beta} - \beta)(\hat{\beta} - \beta)'R') \\ &= R\mathrm{E}((\hat{\beta} - \beta)(\hat{\beta} - \beta)')R' = R\mathrm{V}(\hat{\beta})R' = \sigma^2 R(X'X)^{-1}R'. \end{aligned}$$