That is, test of $\beta_i = 0$:

Define: $s^2 = \dfrac{(y - X\hat{\beta})'(y - X\hat{\beta})}{n - k}$.

Then,

$$\frac{\dfrac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{G}}{s^2} = \frac{\hat{\beta}_i^2}{s^2 a_{ii}} \sim F(1, n - k),$$

where $R\hat{\beta} = \hat{\beta}_i$ and $a_{ii} = $ the i row and ith column of $(X'X)^{-1}$.

*) Recall that $Y \sim F(1, m)$ when $X \sim t(m)$ and $Y = X^2$.

Therefore, the test of $\beta_i = 0$ is given by:

$$\frac{\hat{\beta}_i}{s \sqrt{a_{ii}}} \sim t(n - k).$$

(b) Test of structural change (Part 1):

$$y_i = \begin{cases} x_i\beta_1 + u_i, & i = 1, 2, \cdots, m \\ x_i\beta_2 + u_i, & i = m + 1, m + 2, \cdots, n \end{cases}$$

Assume that $u_i \sim N(0, \sigma^2)$.

In a matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ \vdots & \vdots \\ x_m & 0 \\ 0 & x_{m+1} \\ 0 & x_{m+2} \\ \vdots & \vdots \\ 0 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ u_{m+1} \\ u_{m+2} \\ \vdots \\ u_n \end{pmatrix}$$

Moreover, rewriting,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u$$

Again, rewriting,

$$Y = X\beta + u$$

The null hypothesis is $H_0 : \beta_1 = \beta_2$.

Apply the F test, using $R = (I_k \quad -I_k)$ and $r = 0$.

In this case, $G = \text{rank}(R) = k$ and β is a $2k \times 1$ vector.

The distribution is $F(k, n - 2k)$.

(c) The hypothesis in which sum of the 1st and 2nd coefficients is equal to one:

$R = (1, 1, 0, \cdots, 0)$, $r = 1$

In this case, $G = \text{rank}(R) = 1$

The distribution of the test statistic is $F(1, n - k)$.

(d) Testing seasonality:

The regression model: The case of **quarterly data (四半期データ)**

$$y = \alpha + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + X\beta_0 + u$$

$D_j = 1$ in the jth quarter and 0 otherwise, i.e., D_j, $j = 1, 2, 3$, are seasonal dummy variables.

Testing seasonality $\implies H_0 : \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\beta = \begin{pmatrix} \alpha \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case, $G = \text{rank}(R) = 3$, and β is a $k \times 1$ vector.

The distribution of the test statistic is $F(3, n - k)$.

(e) Cobb-Douglas Production Function:

Let Q_i, K_i and L_i be production, capital stock and labor.

We estimate the following production function:

$$\log(Q_i) = \beta_1 + \beta_2 \log(K_i) + \beta_3 \log(L_i) + u_i.$$

We want to test a linear homogeneous (一次同次) production function, i.e., $\beta_2 + \beta_3 = 1$.

The null and alternative hypotheses are:

$$H_0 : \beta_2 + \beta_3 = 1,$$
$$H_1 : \beta_2 + \beta_3 \neq 1.$$

Then, set as follows:

$$R = (0 \quad 1 \quad 1), \qquad r = 1.$$

(f) Test of structural change (Part 2):

Test the structural change between time periods m and $m + 1$.

In the case where both the constant term and the slope are changed, the

regression model is as follows:

$$y_i = \alpha + \beta x_i + \gamma d_i + \delta d_i x_i + u_i,$$

where

$$d_i = \begin{cases} 0, & \text{for } i = 1, 2, \cdots, m, \\ 1, & \text{for } i = m + 1, m + 2, \cdots, n. \end{cases}$$

We consider testing the structural change at time $m + 1$.

The null and alternative hypotheses are as follows:

$$H_0 : \gamma = \delta = 0,$$
$$H_1 : \gamma \neq 0, \text{ or, } \delta \neq 0.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(g) Multiple regression model:

Consider the case of two explanatory variables:

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i.$$

We want to test the hypothesis that neither x_i nor z_i depends on y_i.

In this case, the null and alternative hypotheses are as follows:

$$H_0 : \beta = \gamma = 0,$$
$$H_1 : \beta \neq 0, \text{ or, } \gamma \neq 0.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Coefficient of Determination R^2 and F distribution:

● The regression model:

$$y_i = x_i\beta + u_i = \beta_1 + x_{2i}\beta_2 + u_i$$

where

$$x_i = (1 \quad x_{2i}), \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

$$x_i : 1 \times k, \qquad x_{2i} : 1 \times (k - 1), \qquad \beta : k \times 1, \qquad \beta_2 : (k - 1) \times 1$$

$$y = X\beta + u = i\beta_1 + X_2\beta_2$$

where the first column of X corresponds to a constant term, i.e.,

$$X = (i \quad X_2), \qquad i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

● The F distribution:

$$R = (0 \quad I_{k-1}), \qquad r = 0$$

where R is a $(k-1) \times k$ matrix and r is a $(k-1) \times 1$ vector.

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k)$$

Note as follows:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2,$$

where $M = I_n - \frac{1}{n} ii'$.

Note that M is symmetric and idempotent, i.e., $M'M = M$.

$$\begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix} = My$$

$R(X'X)^{-1}R'$ is given by:

$$R(X'X)^{-1}R' = (0 \quad I_{k-1}) \left(\begin{pmatrix} i' \\ X_2' \end{pmatrix} (i \quad X_2) \right)^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix}$$

$$= (0 \quad I_{k-1}) \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix}$$

(*) The inverse of a partitioned matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square nonsingular matrices.

$$A^{-1} = \begin{pmatrix} B_{11} & -B_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1} \end{pmatrix},$$

where $B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}$, or alternatively,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -B_{22}A_{21}A_{11}^{-1} & B_{22} \end{pmatrix},$$

where $B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$

Go back to the F distribution.

$$\begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'X_2 - X_2'i(i'i)^{-1}i'X_2)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'(I_n - \frac{1}{n}ii')X_2)^{-1} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix}$$

Therefore, we obtain:

$$(0 \quad I_{k-1}) \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix}$$

$$= (0 \quad I_{k-1}) \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = (X_2'MX_2)^{-1}.$$

Thus, under $H_0 : \beta_2 = 0$, we obtain the following result:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)}$$

$$= \frac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k)$$

● Coefficient of Determination R^2:

Define e as $e = y - X\hat{\beta}$. The coefficient of determinant, R^2, is

$$R^2 = 1 - \frac{e'e}{y'My},$$

where $M = I_n - \frac{1}{n}ii'$, I_n is a $n \times n$ identity matrix and i is a $n \times 1$ vector consisting of 1, i.e., $i = (1, 1, \cdots, 1)'$.

$$Me = My - MX\hat{\beta}.$$

When $X = (i \quad X_2)$ and $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$,

$$Me = e,$$

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because $i'e = 0$, and

$$MX = M(i \quad X_2) = (Mi \quad MX_2) = (0 \quad MX_2)$$

because $Mi = 0$.

$$MX\hat{\beta} = (0 \quad MX_2)\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = MX_2\hat{\beta}_2$$

Thus,

$$My = MX\hat{\beta} + Me \quad \Longrightarrow \quad My = MX_2\hat{\beta}_2 + e$$

Therefore, $y'My$ is given by: $\quad y'My = \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 + e'e,$

because $X_2'e = 0$ and $Me = e$.

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The coefficient of determinant, R^2, is rewritten as:

$$R^2 = 1 - \frac{e'e}{y'My} \quad \Longrightarrow \quad e'e = (1 - R^2)y'My$$

$$R^2 = \frac{y'My - e'e}{y'My} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2}{y'My} \quad \Longrightarrow \quad \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 = R^2 y'My$$

Therefore,

$$\frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} = \frac{R^2 y'My/(k-1)}{(1-R^2)y'My/(n-k)}$$

$$= \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F(k-1, n-k)$$

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