5 Restricted OLS (制約付き最小二乗法)

1. Minimize $(y - X\beta)'(y - X\beta)$ subject to $R\beta = r$.

   Let L be the Lagrangian for the minimization problem.

$$L = (y - X\beta)'(y - X\beta) - 2\lambda'(R\beta - r)$$

   Let the solutions of β and λ for minimization be $\tilde\beta$ and $\tilde\lambda$.

$$\frac{\partial L}{\partial \beta} = -2X'(y - X\tilde\beta) - 2R'\tilde\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = -2(R\tilde\beta - r) = 0$$

From $\partial L/\partial\beta = 0$, we obtain:

$$\begin{aligned}\tilde\beta &= (X'X)^{-1}X'y + (X'X)^{-1}R'\tilde\lambda \\ &= \hat\beta + (X'X)^{-1}R'\tilde\lambda.\end{aligned}$$

Multiplying R from the left, we have:

$$R\tilde\beta = R\hat\beta + R(X'X)^{-1}R'\tilde\lambda.$$

Because $R\tilde\beta = r$ has to be satisfied, we have the following expression:

$$r = R\hat\beta + R(X'X)^{-1}R'\tilde\lambda.$$

Therefore, solving the above equation with respect to $\tilde\lambda$, we obtain:

$$\tilde\lambda = \Big(R(X'X)^{-1}R'\Big)^{-1}(r - R\hat\beta)$$

Substituting $\tilde\lambda$ into $\tilde\beta = \hat\beta + (X'X)^{-1}R'\tilde\lambda$, the restricted OLSE is given by:

$$\tilde\beta = \hat\beta + (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}(r - R\hat\beta).$$

(a) The expectation of $\tilde\beta$ is:

$$\begin{aligned}\mathrm{E}(\tilde\beta) &= \mathrm{E}(\hat\beta) + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\mathrm{E}(\hat\beta)) \\ &= \beta + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\beta)) = \beta,\end{aligned}$$

which shows that $\tilde\beta$ is unbiased.

(b) The variance of $\tilde\beta$ is as follows.

First, rewrite as follows:

$$\begin{aligned}(\tilde\beta - \beta) &= (\hat\beta - \beta) + (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}(R\beta - R\hat\beta) \\ &= (\hat\beta - \beta) - (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}(R\hat\beta - R\beta) \\ &= (\hat\beta - \beta) - (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R(\hat\beta - \beta) \\ &= \Big(I - (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R\Big)^{-1}(\hat\beta - \beta) \\ &= W(\hat\beta - \beta).\end{aligned}$$

Then, we obtain the following variance:

$$\begin{aligned}\mathrm{V}(\tilde\beta) &\equiv \mathrm{E}((\tilde\beta - \beta)(\tilde\beta - \beta)') = \mathrm{E}(W(\hat\beta - \beta)(\hat\beta - \beta)'W') \\ &= W\mathrm{E}((\hat\beta - \beta)(\hat\beta - \beta)')W' = W\mathrm{V}(\hat\beta)W' = \sigma^2 W(X'X)^{-1}W' \\ &= \sigma^2\Big(I - (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R\Big)(X'X)^{-1} \\ &\qquad \times\Big(I - (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R\Big)' \\ &= \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R(X'X)^{-1} \\ &= \mathrm{V}(\hat\beta) - \sigma^2(X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}R(X'X)^{-1}\end{aligned}$$

Thus, $\mathrm{V}(\hat\beta) - \mathrm{V}(\tilde\beta)$ is positive definite.

2. Another solution:

   (a) Again, write the first-order condition for minimization:

$$\begin{aligned}\frac{\partial L}{\partial \beta} &= -2X'(y - X\tilde\beta) - 2R'\tilde\lambda = 0, \\ \frac{\partial L}{\partial \lambda} &= -2(R\tilde\beta - r) = 0,\end{aligned}$$

   which can be written as:

$$\begin{aligned}X'X\tilde\beta - R'\tilde\lambda &= X'y, \\ R\tilde\beta &= r.\end{aligned}$$

Using the matrix form:

$$\begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix} \begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \begin{pmatrix} X'y \\ r \end{pmatrix}$$

The solutions of $\tilde{\beta}$ and $-\tilde{\lambda}$ are given by:

$$\begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix}^{-1} \begin{pmatrix} X'y \\ r \end{pmatrix}.$$

(b) Formula to the inverse matrix:

$$\begin{pmatrix} A & B \\ B' & D \end{pmatrix}^{-1} = \begin{pmatrix} E & F \\ F' & G \end{pmatrix},$$

where E, F and G are given by:

$$\begin{aligned} E &= (A - BD^{-1}B')^{-1} \\ &= A^{-1} + A^{-1}B(D - B'A^{-1}B)^{-1}B'A^{-1} \\ F &= -(A - BD^{-1}B')^{-1}BD^{-1} \\ &= -A^{-1}B(D - B'A^{-1}B)^{-1} \\ G &= (D - B'A^{-1}B)^{-1} \\ &= D^{-1} + D^{-1}B'(A - BD^{-1}B')^{-1}BD^{-1} \end{aligned}$$

(c) In this case, E and F correspond to:

$$\begin{aligned} E &= (X'X)^{-1} - (X'X)^{-1}R'(R(X'X)^{-1}R')R(X'X)^{-1} \\ F &= (X'X)^{-1}R'(R(X'X)^{-1}R'). \end{aligned}$$

Therefore, $\tilde{\beta}$ is derived as follows:

$$\begin{aligned} \tilde{\beta} &= EX'y + Fr \\ &= \hat{\beta} + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta}). \end{aligned}$$

(d) The variance is:

$$\mathrm{V}\begin{pmatrix} \tilde{\beta} \\ -\tilde{\lambda} \end{pmatrix} = \sigma^2 \begin{pmatrix} X'X & R' \\ R & 0 \end{pmatrix}^{-1}.$$

Therefore, $\mathrm{V}(\tilde{\beta})$ is:

$$\begin{aligned} \mathrm{V}(\tilde{\beta}) &= \sigma^2 E \\ &= \sigma^2\Big((X'X)^{-1} - (X'X)^{-1}R'(R(X'X)^{-1}R')R(X'X)^{-1}\Big) \end{aligned}$$

(e) Under the restriction: $R\beta = r$,

$$\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta}) = \sigma^2(X'X)^{-1}R'(R(X'X)^{-1}R')R(X'X)^{-1}$$

is positive definite.

6 F Distribution (Restricted OLS and Unrestricted OLS)

1. As mentioned above, under the null hypothesis $H_0 : R\beta = r$,

$$\frac{\dfrac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{G}}{\dfrac{(y - X\hat{\beta})'(y - X\hat{\beta})}{n - k}} \sim F(G, n - k),$$

where $G = \mathrm{Rank}(R)$.

The numerator is written as follows:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = (\hat{\beta} - \tilde{\beta})'(X'X)(\hat{\beta} - \tilde{\beta}).$$

Remember that

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta}).$$

Moreover, rewrite as follows:

$$
\begin{aligned}
(y - X\tilde{\beta})'(y - X\tilde{\beta}) =& (y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta}))'(y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta})) \\
=& (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) \\
& - (y - X\hat{\beta})'X(\tilde{\beta} - \hat{\beta}) - (\tilde{\beta} - \hat{\beta})'X'(y - X\hat{\beta}) \\
=& (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}).
\end{aligned}
$$

$X'(y - X\hat{\beta}) = X'e = 0$ is utilized.

Summarizing, we have following representation:

$$
\begin{aligned}
(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) =& (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) \\
=& (y - X\tilde{\beta})'(y - X\tilde{\beta}) - (y - X\hat{\beta})'(y - X\hat{\beta}) \\
=& \tilde{u}'\tilde{u} - e'e,
\end{aligned}
$$

where e and $\tilde{u}$ are the restricted residual and the unrestricted residual.

That is,

$$e = y - X\hat{\beta}, \qquad \text{and} \qquad \tilde{u} = y - X\tilde{\beta}.$$

Therefore, we obtain the following result:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} = \frac{(\tilde{u}'\tilde{u} - e'e)/G}{e'e/(n-k)}$$

7 Example: *F* Distribution (Restricted OLS and Unrestricted OLS)

Date file $\Longrightarrow$ cons99.txt (Next slide)

Each column denotes year, nominal housdehold expenditures (家計消費，10 billion yen), household disposable income (家計可処分所得，10 billion yen) and household expenditure deflator (家計消費デフレータ，1990=100) from the left.

1955	5430.1	6135.0	18.1	1970	37784.1	45913.2	35.2	1985	185335.1	220655.6	93.9
1956	5974.2	6828.4	18.3	1971	42571.6	51944.3	37.5	1986	193069.6	229938.8	94.8
1957	6686.3	7619.5	19.0	1972	49124.1	60245.4	39.7	1987	202072.8	235924.0	95.3
1958	7169.7	8153.3	19.1	1973	59366.1	74924.8	44.1	1988	212939.9	247159.7	95.8
1959	8019.3	9274.3	19.7	1974	71782.1	93833.2	53.3	1989	227122.2	263940.5	97.7
1960	9234.9	10776.5	20.5	1975	83591.1	108712.8	59.4	1990	243035.7	280133.0	100.0
1961	10836.2	12869.4	21.8	1976	94443.7	123540.9	65.2	1991	255531.8	297512.9	102.5
1962	12430.8	14701.4	23.2	1977	105397.8	135318.4	70.1	1992	265701.6	309256.6	104.5
1963	14506.6	17042.7	24.9	1978	115960.3	147244.2	73.5	1993	272075.3	317021.6	105.9
1964	16674.9	19709.9	26.0	1979	127600.9	157071.1	76.0	1994	279538.7	325655.7	106.7
1965	18820.5	22337.4	27.8	1980	138585.0	169931.5	81.6	1995	283245.4	331967.5	106.2
1966	21680.6	25514.5	29.0	1981	147103.4	181349.2	85.4	1996	291458.5	340619.1	106.0
1967	24914.0	29012.6	30.1	1982	157994.0	190611.5	87.7	1997	298475.2	345522.7	107.3
1968	28452.7	34233.6	31.6	1983	166631.6	199587.8	89.5				
1969	32705.2	39486.3	32.9	1984	175383.4	209451.9	91.8				

Estimate using TSP 5.0.

```
LINE  ************************************************
|     freq a;
|     smpl 1955 1997;
|     read(file='cons99.txt') year cons yd price;
|     rcons=cons/(price/100);
|     ryd=yd/(price/100);
|     d1=0.0;
|     smpl 1974 1997;
|     d1=1.0;
|     smpl 1956 1997;
|     d1ryd=d1*ryd;
|     olsq rcons c ryd;
|     olsq rcons c d1 ryd d1ryd;
|     end;
******************************************************
```