6. Suppose that the regression model is given by:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega).$$

In this case, when we use OLS, what happens?

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u$$

$$\mathrm{V}(\hat{\beta}) = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}$$

Compare GLS and OLS.

(a) Expectation:

$$\mathrm{E}(\hat{\beta}) = \beta, \quad \text{and} \quad \mathrm{E}(b) = \beta$$

Thus, both $\hat{\beta}$ and b are unbiased estimator.

(b) Variance:

$$\begin{aligned} \mathrm{V}(\hat{\beta}) &= \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1} \\ \mathrm{V}(b) &= \sigma^2(X'\Omega^{-1}X)^{-1} \end{aligned}$$

Which is more efficient, OLS or GLS?.

$$\begin{aligned} \mathrm{V}(\hat{\beta}) - \mathrm{V}(b) &= \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1} - \sigma^2(X'\Omega^{-1}X)^{-1} \\ &= \sigma^2\Big((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Big)\Omega \\ &\qquad \times\Big((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Big)' \\ &= \sigma^2 A\Omega A' \end{aligned}$$

Ω is the variance-covariance matrix of u, which is a positive definite matrix.

Therefore, except for $\Omega = I_n$, $A\Omega A'$ is also a positive definite matrix.

This implies that $\mathrm{V}(\hat{\beta}_i) - \mathrm{V}(b_i) > 0$ for the ith element of β.

Accordingly, b is more efficient than $\hat{\beta}$.

7. If $u \sim N(0, \sigma^2\Omega)$, then $b \sim N(\beta, \sigma^2(X'\Omega^{-1}X)^{-1})$.

Consider testing the hypothesis $H_0: R\beta = r$.

$R: \ G \times k, \quad \mathrm{rank}(R) = G \le k.$

$Rb \sim N(R\beta, \sigma^2 R(X'\Omega^{-1}X)^{-1}R').$

Therefore, the following quadratic form is distributed as:

$$\frac{(Rb - r)'(R(X'\Omega^{-1}X)^{-1}R')^{-1}(Rb - r)}{\sigma^2} \sim \chi^2(G)$$

8. Because $(y^\star - X^\star b)'(y^\star - X^\star b)/\sigma^2 \sim \chi^2(n-k)$, we obtain:

$$\frac{(y - Xb)'\Omega^{-1}(y - Xb)}{\sigma^2} \sim \chi^2(n-k)$$

9. Furthermore, from the fact that b is independent of $y - Xb$, the following F distribution can be derived:

$$\frac{(Rb - r)'(R(X'\Omega^{-1}X)^{-1}R')^{-1}(Rb - r)/G}{(y - Xb)'\Omega^{-1}(y - Xb)/n - k} \sim F(G, n-k)$$

10. Let b be the unrestricted GLSE and $\tilde{b}$ be the restricted GLSE.

Their residuals are given by e and $\tilde{e}$, respectively.

$$e = y - Xb, \qquad \tilde{e} = y - X\tilde{b}$$

Then, the F test statistic is written as follows:

$$\frac{(\tilde{e}'\Omega^{-1}\tilde{e} - e'\Omega^{-1}e)/G}{e'\Omega^{-1}e/(n-k)} \sim F(G, n-k)$$

8.1 Example: Mixed Estimation (Theil and Goldberger Model)

A generalization of the restricted OLS $\Longrightarrow$ Stochastic linear restriction:

$$r = R\beta + v, \qquad \mathrm{E}(v) = 0 \ \text{ and } \ \mathrm{V}(v) = \sigma^2\Psi$$

$$y = X\beta + u, \qquad \mathrm{E}(u) = 0 \ \text{ and } \ \mathrm{V}(u) = \sigma^2 I_n$$

Using a matrix form,

$$\begin{pmatrix} y \\ r \end{pmatrix} = \begin{pmatrix} X \\ R \end{pmatrix}\beta + \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathrm{E}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \text{ and } \ \mathrm{V}\begin{pmatrix} u \\ v \end{pmatrix} = \sigma^2\begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}$$

For estimation, we do not need normality assumption.

Applying GLS, we obtain:

$$\begin{aligned} b &= \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} y \\ r \end{pmatrix} \right) \\ &= \left(X'X + R'\Psi^{-1}R \right)^{-1} \left(X'y + R'\Psi^{-1}r \right). \end{aligned}$$

Mean and Variance of b: b is rewritten as follows:

$$\begin{aligned} b &= \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} y \\ r \end{pmatrix} \right) \\ &= \beta + \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

Threfore, the mean and variance are given by:

$$\mathrm{E}(b) = \beta \qquad \Longrightarrow \qquad b \text{ is unbiased.}$$

$$\begin{aligned} \mathrm{V}(b) &= \sigma^2 \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \\ &= \sigma^2 \left(X'X + R'\Psi^{-1}R \right)^{-1} \end{aligned}$$

9 Maximum Likelihood Estimation (MLE, 最尤法)

$\Longrightarrow$ Review of Last Semester

1. The distribution function of $\{X_i\}_{i=1}^n$ is $f(x;\theta)$, where $x = (x_1, x_2, \cdots, x_n)$ and $\theta = (\mu, \Sigma)$.

   Note that X is a vector of random variables and x is a vector of their realizations (i.e., observed data).

   Likelihood function $L(\cdot)$ is defined as $L(\theta; x) = f(x;\theta)$.

Note that $f(x;\theta) = \prod_{i=1}^n f(x_i;\theta)$ when $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed.

The maximum likelihood estimator (MLE) of θ is θ such that:

$$\max_{\theta} \ L(\theta; X). \qquad \Longleftrightarrow \qquad \max_{\theta} \ \log L(\theta; X).$$

MLE satisfies the following two conditions:

(a) $\dfrac{\partial \log L(\theta; X)}{\partial \theta} = 0.$

(b) $\dfrac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}$ is a negative definite matrix.

2. **Fisher's information matrix (フィッシャーの情報行列)** is defined as:

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}\Big),$$

   where we have the following equality:

$$-\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}\Big) = \mathrm{E}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta}\frac{\partial \log L(\theta; X)}{\partial \theta'}\Big) = \mathrm{V}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta}\Big)$$

   Proof of the above equality:

$$\int L(\theta; x)\mathrm{d}x = 1$$

Take a derivative with respect to θ.

$$\int \frac{\partial L(\theta; x)}{\partial \theta} \mathrm{d}x = 0$$

(We assume that (i) the domain of x does not depend on θ and (ii) the derivative $\dfrac{\partial L(\theta; x)}{\partial \theta}$ exists.)

Rewriting the above equation, we obtain:

$$\int \frac{\partial \log L(\theta; x)}{\partial \theta} L(\theta; x) \mathrm{d}x = 0,$$

i.e.,

$$\mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right) = 0.$$

Again, differentiating the above with respect to θ, we obtain:

$$\begin{aligned}
&\int \frac{\partial^2 \log L(\theta; x)}{\partial \theta \partial \theta'} L(\theta; x) \mathrm{d}x + \int \frac{\partial \log L(\theta; x)}{\partial \theta} \frac{\partial L(\theta; x)}{\partial' \theta} \mathrm{d}x \\
&\quad = \int \frac{\partial^2 \log L(\theta; x)}{\partial \theta \partial \theta'} L(\theta; x) \mathrm{d}x + \int \frac{\partial \log L(\theta; x)}{\partial \theta} \frac{\partial \log L(\theta; x)}{\partial \theta'} L(\theta; x) \mathrm{d}x \\
&\quad = \mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}\Big) + \mathrm{E}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta} \frac{\partial \log L(\theta; X)}{\partial \theta'}\Big) = 0.
\end{aligned}$$

Therefore, we can derive the following equality:

$$-\mathrm{E}\left(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}\right) = \mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial \theta} \frac{\partial \log L(\theta; X)}{\partial \theta'}\right) = \mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right),$$

where the second equality utilizes $\mathrm{E}\left(\dfrac{\partial \log L(\theta; X)}{\partial \theta}\right) = 0$.