11 Consistency and Asymptotic Normality of OLSE

Regression model:

$$y = X\beta + u, \qquad u \sim (0, \sigma^2 I_n)$$

Consistency:

1. Let $\hat\beta_n = (X'X)^{-1}X'y$ be the OLS with sample size n.

   Consistency: As n is large, $\hat\beta_n$ converges to β.

2. Assume the stationarity assumption for X, i.e.,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

Then, we have the following result:

$$\frac{1}{n}X'u \longrightarrow 0.$$

Proof:

According to Chebyshev's inequality, for $g(x) \geq 0$,

$$P(g(X) \geq k) \leq \frac{\mathrm{E}(g(X))}{k},$$

where k is a positive constant.

Set $g(X) = X'X$, and X is replaced by $\frac{1}{n}X'u$.

Apply Chebyshev's inequality.

$$\mathrm{E}\Big((\frac{1}{n}X'u)'\frac{1}{T}X'u\Big) = \frac{1}{n^2}\mathrm{E}\Big(u'XX'u\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(u'XX'u)\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(XX'uu')\Big)$$
$$= \frac{1}{n^2}\mathrm{tr}\Big(XX'\mathrm{E}(uu')\Big) = \frac{\sigma^2}{n^2}\mathrm{tr}(XX') = \frac{\sigma^2}{n^2}\mathrm{tr}(X'X) = \frac{\sigma^2}{n}\mathrm{tr}(\frac{1}{n}X'X).$$

Therefore,

$$P\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u \geq k\Big) \leq \frac{\sigma^2}{nk}\mathrm{tr}(\frac{1}{n}X'X) \longrightarrow 0 \times \mathrm{tr}(M_{xx}) = 0$$

Note that from the assumption,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

Therefore, we have:

$$(\frac{1}{n}X'u)'\frac{1}{n}X'u \longrightarrow 0,$$

which implies:

$$\frac{1}{n}X'u \longrightarrow 0,$$

because $(\frac{1}{n}X'u)'\frac{1}{n}X'u$ indicates a quadratic form.

3. Note that

$$\frac{1}{n}X'X \longrightarrow M_{xx}$$

results in

$$(\frac{1}{n}X'X)^{-1} \longrightarrow M_{xx}^{-1}$$

$\Longrightarrow$ Slutsky's Theorem

(*) **Slutsky's Theorem** $\quad g(\hat\theta) \longrightarrow g(\theta)$, when $\hat\theta \longrightarrow \theta$.

4. OLS is given by:

$$\hat\beta_n = \beta + (X'X)^{-1}X'u$$

$$= \beta + (\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'u).$$

Therefore,

$$\hat\beta_n \longrightarrow \beta + M_{xx}^{-1} \times 0 = \beta$$

Thus, OLSE is a consitent estimator.

Asymptotic Normality:

1. Asymptotic Normality of OLSE

$$\sqrt{n}(\hat{\beta}_n - \beta) \longrightarrow N(0.\sigma^2 M_{xx}^{-1}) \text{ when } n \longrightarrow \infty.$$

2. **Central Limit Theorem:** Greenberg and Webster (1983)

$Z_1, Z_2, \cdots, Z_n$ are mutually indelendently distributed with mean μ and variance Σ_i.

Then, we have the following result:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_i - \mu) \longrightarrow N(0, \Sigma),$$

where

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\Sigma_i\right).$$

The distribution of Z_i is not assumed.

3. Define $Z_i = x_i u_i$. Then, $\Sigma_i = \mathrm{Var}(Z_i) = \sigma^2 x_i' x_i$.

4. Σ is defined as:

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma^2 x_i' x_i\right) = \sigma^2 \lim_{n\to\infty}\left(\frac{1}{n}X'X\right) = \sigma^2 M_{xx},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

5. Applying Central Limit Theorem (Greenberg and Webster (1983), we obtain the following:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i' u_i = \frac{1}{\sqrt{n}}X'u \longrightarrow N(0, \sigma^2 M_{xx}).$$

On the other hand, from $\hat{\beta}_n = \beta + (X'X)^{-1}X'u$, we can rewrite as:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u.$$

$$\mathrm{Var}\left(\left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u\right) = \mathrm{E}\left(\left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u\left(\left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u\right)'\right)$$

$$= \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'\mathrm{E}(uu')X\right)\left(\frac{1}{n}X'X\right)^{-1}$$
$$= \sigma^2\left(\frac{1}{n}X'X\right)^{-1} \longrightarrow \sigma^2 M_{xx}^{-1}.$$

Therefore,

$$\sqrt{n}(\hat{\beta} - \beta) \longrightarrow N(0, \sigma^2 M_{xx}^{-1})$$

$\Longrightarrow$ Asymptotic normality (漸近的正規性) of OLSE

The distribution of u_i is not assumed.

12 Instrumental Variable (操作変数法)

12.1 Measurement Error (測定誤差)

Errors in Variables

1. True regression model:

$$y = \tilde{X}\beta + u$$

2. Observed variable:

$$X = \tilde{X} + V$$

V: is called the **measurement error (測定誤差 or 観測誤差)**.

3. For the elements which do not include measurement errors in X, the corresponding elements in V are zeros.

4. Regression using observed variable:

$$y = X\beta + (u - V\beta)$$

OLS of β is:

$$\hat\beta = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'(u - V\beta)$$

5. Assumptions:

(a) The measurement error in X is uncorrelated with $\tilde X$ in the limit. i.e.,

$$\text{plim}\Big(\frac{1}{n}\tilde X'V\Big) = 0.$$

Therefore, we obtain the following:

$$\text{plim}\Big(\frac{1}{n}X'X\Big) = \text{plim}\Big(\frac{1}{n}\tilde X'\tilde X\Big) + \text{plim}\Big(\frac{1}{n}V'V\Big) = \Sigma + \Omega$$

(b) u is not correlated with V.

u is not correlated with $\tilde X$.

That is,

$$\text{plim}\Big(\frac{1}{n}V'u\Big) = 0, \qquad \text{plim}\Big(\frac{1}{n}\tilde X'u\Big) = 0.$$

6. OLSE of β is:

$$\hat\beta = \beta + (X'X)^{-1}X'(u - V\beta) = \beta + (X'X)^{-1}(\tilde X + V)'(u - V\beta).$$

Therefore, we obtain the following:

$$\text{plim}\,\hat\beta = \beta - (\Sigma + \Omega)^{-1}\Omega\beta$$

7. **Example: The Case of Two Variables:**

The regression model is given by:

$$y_t = \alpha + \beta\tilde x_t + u_t, \qquad x_t = \tilde x_t + v_t.$$

Under the above model,

$$\Sigma = \text{plim}\Big(\frac{1}{n}\tilde X'\tilde X\Big) = \text{plim}\begin{pmatrix} 1 & \frac{1}{n}\sum \tilde x_i \\ \frac{1}{n}\sum \tilde x_i & \frac{1}{n}\sum \tilde x_i^2 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix},$$

where μ and σ^2 represent the mean and variance of $\tilde x_i$.

$$\Omega = \text{plim}\Big(\frac{1}{n}V'V\Big) = \text{plim}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{n}\sum v_i^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \text{plim}\begin{pmatrix} \hat\alpha \\ \hat\beta \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \left(\begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \frac{1}{\sigma^2 + \sigma_v^2}\begin{pmatrix} -\mu\sigma_v^2\beta \\ \sigma_v^2\beta \end{pmatrix} \end{aligned}$$

Now we focus on β.

$\hat\beta$ is not consistent. because of:

$$\text{plim}(\hat\beta) = \beta - \frac{\sigma_v^2\beta}{\sigma^2 + \sigma_v^2} = \frac{\beta}{1 + \sigma_v^2/\sigma^2} < \beta$$

12.2 Instrumental Variable (IV) Method (操作変数法 or IV 法)

Instrumental Variable (IV)

1. Consider the regression model: $y = X\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

   In the case of $\text{E}(X'u) \neq 0$, OLSE of β is inconsistent.

2. **Proof:**

$$\hat\beta = \beta + (\frac{1}{n}X'X)^{-1}\frac{1}{n}X'u \longrightarrow \beta + M_{xx}^{-1}M_{xu},$$

where

$$\frac{1}{n}X'X \longrightarrow M_{xx}, \qquad \frac{1}{n}X'u \longrightarrow M_{xu} \neq 0$$

3. Find the Z which satisfies $\frac{1}{n}Z'u \longrightarrow M_{zu} = 0$.

   Multiplying Z' on both sides of the regression model: $y = X\beta + u$,

$$Z'y = Z'X\beta + Z'u$$

   Dividing n on both sides of the above equation, we take plim on both sides.

Then, we obtain the following:

$$\text{plim}\left(\frac{1}{n}Z'y\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta + \text{plim}\left(\frac{1}{n}Z'u\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta.$$

Accordingly, we obtain:

$$\beta = \left(\text{plim}\left(\frac{1}{n}Z'X\right)\right)^{-1} \text{plim}\left(\frac{1}{n}Z'y\right).$$

Therefore, we consider the following estimator:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

which is taken as an estimator of β.

$\Longrightarrow$ **Instrumental Variable Method (操作変数法 or IV 法)**

4. Assume the followings:

$$\frac{1}{n}Z'X \longrightarrow M_{zx}, \qquad \frac{1}{n}Z'Z \longrightarrow M_{zz}, \qquad \frac{1}{n}Z'u \longrightarrow 0$$

5. **Distribution of β_{IV}:**

$$\beta_{IV} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u,$$

   which is rewritten as:

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big)$$

Applying the Central Limit Theorem to $\Big(\frac{1}{\sqrt{n}}Z'u\Big)$, we have the following result:

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz}).$$

Therefore,

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}'^{-1})$$

$\Longrightarrow$ Consistency and Asymptotic Normality