Econometrics II

(Tue., 8:50-10:20)

TA Session (by Mr. Kinoshita):

Thu., 14:40 - 16:10

Room # 605 (法経大学院総合研究棟)

Econometrics (Undergraduate Course)

Wed., 10:30-12:00

Fri., 8:50-10:20

- If you have not taken Econometrics in undergraduate level, attend the class.
- Textbook: 『計量経済学』(山本 拓 著，新世社)
- The prerequisite of this class is to have knowledge of **Econometrics I** (last semester) and **Econometrics** (undergraduate level).

1 Regression Analysis (回帰分析)

1.1 Setup of the Model

When (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) are available, suppose that there is a linear relationship between y and x, i.e.,

$$y_i = \beta_1 + \beta_2 x_i + u_i, \tag{1}$$

for $i = 1, 2, \cdots, n$. x_i and y_i denote the ith observations.

$\longrightarrow$ **Single (or simple) regression model (単回帰モデル)**

y_i is called the **dependent variable (従属変数)** or the **explained variable (被説明変数)**, while x_i is known as the **independent variable (独立変数)** or the **explanatory (or explaining) variable (説明変数)**.

β_1 = **Intercept (切片)** β_2 = **Slope (傾き)**

β_1 and β_2 are unknown **parameters (パラメータ，母数)** to be estimated.

β_1 and β_2 are called the **regression coefficients** (回帰係数).

u_i is the unobserved **error term (誤差項)** assumed to be a random variable with mean zero and variance σ^2.
σ^2 is also a parameter to be estimated.

x_i is assumed to be **nonstochastic (非確率的)**, but y_i is **stochastic (確率的)** because y_i depends on the error u_i.

The error terms u_1, u_2, $\cdots$, u_n are assumed to be mutually independently and identically distributed, which is called ***iid***.

It is assumed that u_i has a distribution with mean zero, i.e., $\mathrm{E}(u_i) = 0$ is assumed.

Taking the expectation on both sides of (1), the expectation of y_i is represented as:

$$\begin{aligned}\mathrm{E}(y_i) &= \mathrm{E}(\beta_1 + \beta_2 x_i + u_i) = \beta_1 + \beta_2 x_i + \mathrm{E}(u_i) \\ &= \beta_1 + \beta_2 x_i,\end{aligned} \tag{2}$$

for $i = 1, 2, \cdots, n$.

Using $\mathrm{E}(y_i)$ we can rewrite (1) as $y_i = \mathrm{E}(y_i) + u_i$.

(2) represents the true regression line.

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be estimates of β_1 and β_2.

Replacing β_1 and β_2 by $\hat{\beta}_1$ and $\hat{\beta}_2$, (1) turns out to be:

$$y_i = \hat{\beta}_1 + \hat{\beta}_2 x_i + e_i, \qquad (3)$$

for $i = 1, 2, \cdots, n$, where e_i is called the **residual (残差)**.

The residual e_i is taken as the experimental value (or realization) of u_i.

We define $\hat{y}_i$ as follows:

$$\hat{y}_i = \hat{\beta}_1 + \hat{\beta}_2 x_i, \qquad (4)$$

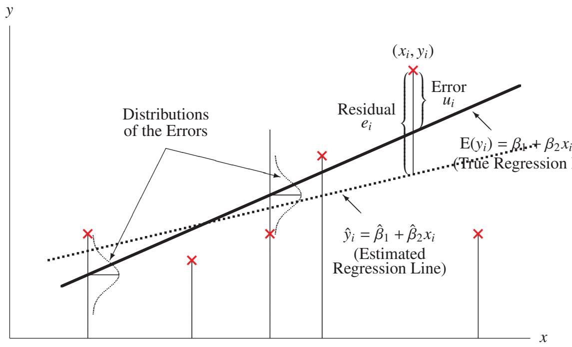
for $i = 1, 2, \cdots, n$, which is interpreted as the **predicted value (予測値)** of y_i.

(4) indicates the estimated regression line, which is different from (2).

Moreover, using $\hat{y}_i$ we can rewrite (3) as $y_i = \hat{y}_i + e_i$.

(2) and (4) are displayed in Figure 1.

Figure 1. True and Estimated Regression Lines (回帰直線)



Consider the case of $n = 6$ for simplicity.

$\times$ indicates the observed data series.

The true regression line (2) is represented by the solid line, while the estimated regression line (4) is drawn with the dotted line.

Based on the observed data, β_1 and β_2 are estimated as: $\hat{\beta}_1$ and $\hat{\beta}_2$.

In the next section, we consider how to obtain the estimates of β_1 and β_2, i.e., $\hat{\beta}_1$ and $\hat{\beta}_2$.

1.2 Ordinary Least Squares Estimation

Suppose that (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) are available.
For the regression model (1), we consider estimating β_1 and β_2.
Replacing β_1 and β_2 by their estimates $\hat{\beta}_1$ and $\hat{\beta}_2$, remember that the residual e_i is given by:

$$e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i.$$

The sum of squared residuals is defined as follows:

$$S(\hat{\beta}_1, \hat{\beta}_2) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2.$$

It might be plausible to choose the $\hat{\beta}_1$ and $\hat{\beta}_2$ which minimize the sum of squared residuals, i.e., $S(\hat{\beta}_1, \hat{\beta}_2)$.
This method is called the **ordinary least squares estimation (最小二乗法，OLS)**.
To minimize $S(\hat{\beta}_1, \hat{\beta}_2)$ with respect to $\hat{\beta}_1$ and $\hat{\beta}_2$, we set the partial derivatives equal to zero:

$$\frac{\partial S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i) = 0,$$

$$\frac{\partial S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_2} = -2 \sum_{i=1}^{n} x_i (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i) = 0,$$

which yields the following two equations:

$$\overline{y} = \hat{\beta}_1 + \hat{\beta}_2 \overline{x}, \tag{5}$$

$$\sum_{i=1}^{n} x_i y_i = n\overline{x}\hat{\beta}_1 + \hat{\beta}_2 \sum_{i=1}^{n} x_i^2, \tag{6}$$

where $\overline{y} = \dfrac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$.

Multiplying (5) by $n\overline{x}$ and subtracting (6), we can derive $\hat{\beta}_2$ as follows:

$$\hat{\beta}_2 = \frac{\sum_{i=1}^n x_i y_i - n\overline{x}\overline{y}}{\sum_{i=1}^n x_i^2 - n\overline{x}^2} = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2}. \tag{7}$$

From (5), $\hat{\beta}_1$ is directly obtained as follows:

$$\hat{\beta}_1 = \overline{y} - \hat{\beta}_2 \overline{x}. \tag{8}$$

When the observed values are taken for y_i and x_i for $i = 1, 2, \cdots, n$, we say that $\hat{\beta}_1$ and $\hat{\beta}_2$ are called the **ordinary least squares estimate**s (or simply the **least squares estimate**s, **最小二乗推定値**) of β_1 and β_2.

When y_i for $i = 1, 2, \cdots, n$ are regarded as the random sample, we say that $\hat{\beta}_1$ and $\hat{\beta}_2$ are called the **ordinary least squares estimator**s (or the **least squares estimator**s, **最小二乗推定量**) of β_1 and β_2.

1.3 Properties of Least Squares Estimator

Equation (7) is rewritten as:

$$\begin{aligned}\hat{\beta}_2 &= \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n (x_i - \overline{x})y_i}{\sum_{i=1}^n (x_i - \overline{x})^2} - \frac{\overline{y}\sum_{i=1}^n (x_i - \overline{x})}{\sum_{i=1}^n (x_i - \overline{x})^2} \\ &= \sum_{i=1}^{n} \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2} Y_i = \sum_{i=1}^{n} \omega_i y_i. \end{aligned} \tag{9}$$

In the third equality, $\sum_{i=1}^{n}(x_i - \overline{x}) = 0$ is utilized because of $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$.

In the fourth equality, ω_i is defined as: $\omega_i = \dfrac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2}$.

ω_i is nonstochastic because x_i is assumed to be nonstochastic.

ω_i has the following properties:

$$\sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n (x_i - \overline{x})}{\sum_{i=1}^n (x_i - \overline{x})^2} = 0, \tag{10}$$

$$\sum_{i=1}^{n} \omega_i x_i = \sum_{i=1}^{n} \omega_i (x_i - \overline{x}) = \frac{\sum_{i=1}^n (x_i - \overline{x})^2}{\sum_{i=1}^n (x_i - \overline{x})^2} = 1, \tag{11}$$

$$\sum_{i=1}^{n} \omega_i^2 = \sum_{i=1}^{n} \left(\frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2} \right)^2 = \frac{\sum_{i=1}^n (x_i - \overline{x})^2}{\left(\sum_{i=1}^n (x_i - \overline{x})^2\right)^2} = \frac{1}{\sum_{i=1}^n (x_i - \overline{x})^2}. \tag{12}$$

The first equality of (11) comes from (10).

From now on, we focus only on $\hat{\beta}_2$, because usually β_2 is more important than β_1 in the regression model (1).

In order to obtain the properties of the least squares estimator $\hat{\beta}_2$, we rewrite (9) as:

$$\begin{aligned}\hat{\beta}_2 &= \sum_{i=1}^{n} \omega_i y_i = \sum_{i=1}^{n} \omega_i (\beta_1 + \beta_2 x_i + u_i) \\ &= \beta_1 \sum_{i=1}^{n} \omega_i + \beta_2 \sum_{i=1}^{n} \omega_i x_i + \sum_{i=1}^{n} \omega_i u_i = \beta_2 + \sum_{i=1}^{n} \omega_i u_i. \end{aligned} \tag{13}$$

In the fourth equality of (13), (10) and (11) are utilized.

Mean and Variance of $\hat{\beta}_2$: $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed with mean zero and variance σ^2, but they are not necessarily normal.

Remember that we do not need normality assumption to obtain mean and variance but the normality assumption is required to test a hypothesis.

From (13), the expectation of $\hat{\beta}_2$ is derived as follows:

$$\begin{aligned}\mathrm{E}(\hat{\beta}_2) &= \mathrm{E}(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = \beta_2 + \mathrm{E}(\sum_{i=1}^{n} \omega_i u_i) \\ &= \beta_2 + \sum_{i=1}^{n} \omega_i \mathrm{E}(u_i) = \beta_2. \end{aligned} \tag{14}$$

It is shown from (14) that the ordinary least squares estimator $\hat{\beta}_2$ is an unbiased estimator of β_2.

From (13), the variance of $\hat{\beta}_2$ is computed as:

$$\begin{aligned} V(\hat{\beta}_2) &= V(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = V(\sum_{i=1}^{n} \omega_i u_i) = \sum_{i=1}^{n} V(\omega_i u_i) = \sum_{i=1}^{n} \omega_i^2 V(u_i) \\ &= \sigma^2 \sum_{i=1}^{n} \omega_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \end{aligned} \tag{15}$$

The third equality holds because $u_1, u_2, \cdots, u_n$ are mutually independent.

The last equality comes from (12).

Thus, $E(\hat{\beta}_2)$ and $V(\hat{\beta}_2)$ are given by (14) and (15).

Gauss-Markov Theorem (ガウス・マルコフ定理): It has been discussed above that $\hat{\beta}_2$ is represented as (9), which implies that $\hat{\beta}_2$ is a linear estimator, i.e., linear in y_i.

In addition, (14) indicates that $\hat{\beta}_2$ is an unbiased estimator.

Therefore, summarizing these two facts, it is shown that $\hat{\beta}_2$ is a **linear unbiased estimator (線形不偏推定量)**.

Furthermore, here we show that $\hat{\beta}_2$ has minimum variance within a class of the linear unbiased estimators.

Consider the alternative linear unbiased estimator $\tilde{\beta}_2$ as follows:

$$\tilde{\beta}_2 = \sum_{i=1}^{n} c_i y_i = \sum_{i=1}^{n} (\omega_i + d_i) y_i,$$

where $c_i = \omega_i + d_i$ is defined and d_i is nonstochastic.

Then, $\tilde{\beta}_2$ is transformed into:

$$\begin{aligned} \tilde{\beta}_2 &= \sum_{i=1}^{n} c_i y_i = \sum_{i=1}^{n} (\omega_i + d_i)(\beta_1 + \beta_2 x_i + u_i) \\ &= \beta_1 \sum_{i=1}^{n} \omega_i + \beta_2 \sum_{i=1}^{n} \omega_i x_i + \sum_{i=1}^{n} \omega_i u_i + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} d_i u_i \\ &= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} \omega_i u_i + \sum_{i=1}^{n} d_i u_i. \end{aligned}$$

Equations (10) and (11) are used in the forth equality.

Taking the expectation on both sides of the above equation, we obtain:

$$\begin{aligned} E(\tilde{\beta}_2) &= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} \omega_i E(u_i) + \sum_{i=1}^{n} d_i E(u_i) \\ &= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i. \end{aligned}$$

Note that d_i is not a random variable and that $E(u_i) = 0$.

Since $\tilde{\beta}_2$ is assumed to be unbiased, we need the following conditions:

$$\sum_{i=1}^{n} d_i = 0, \qquad \sum_{i=1}^{n} d_i x_i = 0.$$

When these conditions hold, we can rewrite $\tilde{\beta}_2$ as:

$$\tilde{\beta}_2 = \beta_2 + \sum_{i=1}^{n} (\omega_i + d_i) u_i.$$

The variance of $\tilde{\beta}_2$ is derived as:

$$\begin{aligned} V(\tilde{\beta}_2) &= V\Big(\beta_2 + \sum_{i=1}^{n} (\omega_i + d_i) u_i\Big) = V\Big(\sum_{i=1}^{n} (\omega_i + d_i) u_i\Big) = \sum_{i=1}^{n} V\Big((\omega_i + d_i) u_i\Big) \\ &= \sum_{i=1}^{n} (\omega_i + d_i)^2 V(u_i) = \sigma^2 (\sum_{i=1}^{n} \omega_i^2 + 2 \sum_{i=1}^{n} \omega_i d_i + \sum_{i=1}^{n} d_i^2) \\ &= \sigma^2 (\sum_{i=1}^{n} \omega_i^2 + \sum_{i=1}^{n} d_i^2). \end{aligned}$$

From unbiasedness of $\tilde{\beta}_2$, using $\sum_{i=1}^n d_i = 0$ and $\sum_{i=1}^n d_i x_i = 0$, we obtain:

$$\sum_{i=1}^n \omega_i d_i = \frac{\sum_{i=1}^n (x_i - \overline{x}) d_i}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n x_i d_i - \overline{X} \sum_{i=1}^n d_i}{\sum_{i=1}^n (x_i - \overline{x})^2} = 0,$$

which is utilized to obtain the variance of $\tilde{\beta}_2$ in the third line of the above equation.

From (15), the variance of $\hat{\beta}_2$ is given by: $\mathrm{V}(\hat{\beta}_2) = \sigma^2 \sum_{i=1}^n \omega_i^2$.

Therefore, we have:

$$\mathrm{V}(\tilde{\beta}_2) \geq \mathrm{V}(\hat{\beta}_2),$$

because of $\sum_{i=1}^n d_i^2 \geq 0$.

When $\sum_{i=1}^n d_i^2 = 0$, i.e., when $d_1 = d_2 = \cdots = d_n = 0$, we have the equality: $\mathrm{V}(\tilde{\beta}_2) = \mathrm{V}(\hat{\beta}_2)$.

Thus, in the case of $d_1 = d_2 = \cdots = d_n = 0$, $\hat{\beta}_2$ is equivalent to $\tilde{\beta}_2$.

As shown above, the least squares estimator $\hat{\beta}_2$ gives us the **minimum variance linear unbiased estimator (最小分散線形不偏推定量)**, or equivalently the **best linear unbiased estimator (最良線形不偏推定量，BLUE)**, which is called the **Gauss-Markov theorem (ガウス・マルコフ定理)**.

Asymptotic Properties of $\hat{\beta}_2$: We assume that as n goes to infinity we have the following:

$$\frac{1}{n} \sum_{i=1}^n (x_i - \overline{x})^2 \longrightarrow m < \infty,$$

where m is a constant value. From (12), we obtain:

$$n \sum_{i=1}^n \omega_i^2 = \frac{1}{(1/n) \sum_{i=1}^n (x_i - \overline{x})} \longrightarrow \frac{1}{m}.$$

Note that $f(x_n) \longrightarrow f(m)$ when $x_n \longrightarrow m$, called **Slutsky's theorem (スルツキー定理)**, where m is a constant value and $f(\cdot)$ is a function.

We show both consistency of $\hat{\beta}_2$ and asymptotic normality of $\sqrt{n}(\hat{\beta}_2 - \beta_2)$.

● First, we prove that $\hat{\beta}_2$ is a consistent estimator of β_2.

Chebyshev's inequality is given by:

$$P(|X - \mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2},$$

where $\mu = \mathrm{E}(X)$ and $\sigma^2 = \mathrm{V}(X)$.

Replace X, $\mathrm{E}(X)$ and $\mathrm{V}(X)$ by:

$$\hat{\beta}_2, \qquad \mathrm{E}(\hat{\beta}_2) = \beta_2, \quad \text{and} \quad \mathrm{V}(\hat{\beta}_2) = \sigma^2 \sum_{i=1}^n \omega_i^2 = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \overline{x})},$$

respectively.

Then, when $n \longrightarrow \infty$, we obtain the following result:

$$P(|\hat{\beta}_2 - \beta_2| > \epsilon) \leq \frac{\sigma^2 \sum_{i=1}^n \omega_i^2}{\epsilon^2} = \frac{\sigma^2 n \sum_{i=1}^n \omega_i^2}{n\epsilon^2} \longrightarrow 0,$$

where $\sum_{i=1}^n \omega_i^2 \longrightarrow 0$ because $n \sum_{i=1}^n \omega_i^2 \longrightarrow \dfrac{1}{m}$ from the assumption.

Thus, we obtain the result that $\hat{\beta}_2 \longrightarrow \beta_2$ as $n \longrightarrow \infty$.

Therefore, we can conclude that $\hat{\beta}_2$ is a consistent estimator of β_2.

● Next, we want to show that $\sqrt{n}(\hat{\beta}_2 - \beta_2)$ is asymptotically normal.

Note that $\hat{\beta}_2 = \beta_2 + \sum_{i=1}^n \omega_i u_i$ as in (13).

From the central limit theorem, asymptotic normality is shown as follows:

$$\frac{\sum_{i=1}^n \omega_i u_i - \mathrm{E}(\sum_{i=1}^n \omega_i u_i)}{\sqrt{\mathrm{V}(\sum_{i=1}^n \omega_i u_i)}} = \frac{\sum_{i=1}^n \omega_i u_i}{\sigma \sqrt{\sum_{i=1}^n \omega_i^2}} = \frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \longrightarrow N(0,1),$$

where $\mathrm{E}(\sum_{i=1}^n \omega_i u_i) = 0$, $\mathrm{V}(\sum_{i=1}^n \omega_i u_i) = \sigma^2 \sum_{i=1}^n \omega_i^2$, and $\sum_{i=1}^n \omega_i u_i = \hat{\beta}_2 - \beta_2$ are substituted in the first and second equalities.

Moreover, we can rewrite as follows:

$$\frac{\hat\beta_2-\beta_2}{\sigma/\sqrt{\sum_{i=1}^n(x_i-\overline{x})^2}} = \frac{\sqrt{n}(\hat\beta_2-\beta_2)}{\sigma/\sqrt{(1/n)\sum_{i=1}^n(x_i-\overline{x})^2}} \longrightarrow \frac{\sqrt{n}(\hat\beta_2-\beta_2)}{\sigma/\sqrt{m}} \longrightarrow N(0,1),$$

or equivalently,

$$\sqrt{n}(\hat\beta_2-\beta_2) \longrightarrow N(0,\frac{\sigma^2}{m}).$$

Thus, the asymptotic normality of $\sqrt{n}(\hat\beta_2-\beta_2)$ is shown.

Finally, replacing σ^2 by its consistent estimator s^2, it is known as follows:

$$\frac{\hat\beta_2-\beta_2}{s/\sqrt{\sum_{i=1}^n(x_i-\overline{x})^2}} \longrightarrow N(0,1), \tag{16}$$

where s^2 is defined as:

$$s^2=\frac{1}{n-2}\sum_{i=1}^n e_i^2=\frac{1}{n-2}\sum_{i=1}^n(y_i-\hat\beta_1-\hat\beta_2x_i)^2, \tag{17}$$

which is a consistent and unbiased estimator of σ^2. $\longrightarrow$ Proved later.

Thus, using (16), in large sample we can construct the confidence interval and test the hypothesis.

Exact Distribution of $\hat\beta_2$: We have shown asymptotic normality of $\sqrt{n}(\hat\beta_2-\beta_2)$, which is one of the large sample properties.

Now, we discuss the small sample properties of $\hat\beta_2$.

In order to obtain the distribution of $\hat\beta_2$ in small sample, the distribution of the error term has to be assumed.

Therefore, the extra assumption is that $u_i\sim N(0,\sigma^2)$.

Writing (13), again, $\hat\beta_2$ is represented as:

$$\hat\beta_2=\beta_2+\sum_{i=1}^n\omega_iu_i.$$

First, we obtain the distribution of the second term in the above equation.

Using the moment-generating function, $\sum_{i=1}^n\omega_iu_i$ is distributed as:

$$\sum_{i=1}^n\omega_iu_i\sim N(0,\ \sigma^2\sum_{i=1}^n\omega_i^2).$$

Therefore, $\hat\beta_2$ is distributed as:

$$\hat\beta_2=\beta_2+\sum_{i=1}^n\omega_iu_i\sim N(\beta_2,\ \sigma^2\sum_{i=1}^n\omega_i^2),$$

or equivalently,

$$\frac{\hat\beta_2-\beta_2}{\sigma\sqrt{\sum_{i=1}^n\omega_i^2}}=\frac{\hat\beta_2-\beta_2}{\sigma/\sqrt{\sum_{i=1}^n(x_i-\overline{x})^2}}\sim N(0,1),$$

for any n.

Moreover, replacing σ^2 by its estimator s^2 defined in (17), it is known that we have:

$$\frac{\hat\beta_2-\beta_2}{s/\sqrt{\sum_{i=1}^n(x_i-\overline{x})^2}}\sim t(n-2),$$

where $t(n-2)$ denotes t distribution with $n-2$ degrees of freedom.

Thus, under normality assumption on the error term u_i, the $t(n-2)$ distribution is used for the confidence interval and the testing hypothesis in small sample.

Or, taking the square on both sides,

$$\Big(\frac{\hat\beta_2-\beta_2}{s/\sqrt{\sum_{i=1}^n(x_i-\overline{x})^2}}\Big)^2\sim F(1,n-2),$$

which will be proved later.

Before going to **multiple regression model (重回帰モデル)**,

2 Some Formulas of Matrix Algebra

1. Let $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lk}\end{pmatrix}=[a_{ij}]$,

   which is a $l\times k$ matrix, where a_{ij} denotes ith row and jth column of A.

The **transposed matrix (転置行列)** of A, denoted by A', is defined as:

$$A' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{l1} \\ a_{12} & a_{22} & \cdots & a_{l2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{lk} \end{pmatrix} = [a_{ji}],$$

where the ith row of A' is the ith column of A.

2. $(Ax)' = x'A'$,

   where A and x are a $l \times k$ matrix and a $k \times 1$ vector, respectively.

3. $a' = a$,

   where a denotes a scalar.

4. $\dfrac{\partial a'x}{\partial x} = a$,

   where a and x are $k \times 1$ vectors.

5. $\dfrac{\partial x'Ax}{\partial x} = (A + A')x$,

   where A and x are a $k \times k$ matrix and a $k \times 1$ vector, respectively.

Especially, when A is symmetric,

$\dfrac{\partial x'Ax}{\partial x} = 2Ax.$

6. Let A and B be $k \times k$ matrices, and I_k be a $k \times k$ **identity matrix (単位行列)** (one in the diagonal elements and zero in the other elements).

   When $AB = I_k$, B is called the **inverse matrix (逆行列)** of A, denoted by $B = A^{-1}$.

   That is, $AA^{-1} = A^{-1}A = I_k$.

7. Let A be a $k \times k$ matrix and x be a $k \times 1$ vector.

   If A is a **positive definite matrix (正値定符号行列)**, for any x except for $x = 0$ we have:

   $$x'Ax > 0.$$

   If A is a **positive semidefinite matrix (非負値定符号行列)**, for any x except for $x = 0$ we have:

   $$x'Ax \geq 0.$$

If A is a **negative definite matrix (負値定符号行列)**, for any x except for $x = 0$ we have:

$$x'Ax < 0.$$

If A is a **negative semidefinite matrix (非正値定符号行列)**, for any x except for $x = 0$ we have:

$$x'Ax \leq 0.$$

Trace, Rank and etc.: $A : k \times k$, $B : n \times k$, $C : k \times n$.

1. The **trace (トレース)** of A is: $\mathrm{tr}(A) = \sum_{i=1}^{k} a_{ii}$, where $A = [a_{ij}]$.

2. The **rank (ランク，階数)** of A is the maximum number of linearly independent column (or row) vectors of A, which is denoted by $\mathrm{rank}(A)$.

3. If A is an **idempotent matrix (べき等行列)**, $A = A^2$.

4. If A is an idempotent and symmetric matrix, $A = A^2 = A'A$.

5. A is idempotent if and only if the eigen values of A consist of 1 and 0.

6. If A is idempotent, $\mathrm{rank}(A) = \mathrm{tr}(A)$.

7. $\mathrm{tr}(BC) = \mathrm{tr}(CB)$

Distributions in Matrix Form:

1. Let X, μ and Σ be $k \times 1$, $k \times 1$ and $k \times k$ matrices.

   When $X \sim N(\mu, \Sigma)$, the density function of X is given by:

$$f(x) = \frac{1}{(2\pi)^{k/2}|\Sigma|} \exp\Big(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big).$$

   $\mathrm{E}(X) = \mu$ and $\mathrm{V}(X) = \mathrm{E}\big((X-\mu)(X-\mu)'\big) = \Sigma$

   The moment-generating function: $\phi(\theta) = \mathrm{E}\big(\exp(\theta' X)\big) = \exp(\theta'\mu + \frac{1}{2}\theta'\Sigma\theta)$

2. If $X \sim N(\mu, \Sigma)$, then $(X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2(k)$.

   Note that $X'X \sim \chi^2(k)$ when $X \sim N(0, I_k)$.

3. X: $n \times 1$, Y: $m \times 1$, $X \sim N(\mu_x, \Sigma_x)$, $Y \sim N(\mu_y, \Sigma_y)$

   X is independent of Y, i.e., $\mathrm{E}\big((X-\mu_x)(Y-\mu_y)'\big) = 0$ in the case of normal random variables.

$$\frac{(X-\mu_x)'\Sigma_x^{-1}(X-\mu_x)/n}{(Y-\mu_y)'\Sigma_y^{-1}(Y-\mu_y)/m} \sim F(n, m)$$

4. If $X \sim N(0, \sigma^2 I_n)$ and A is a symmetric idempotent $n \times n$ matrix of rank G, then $X'AX/\sigma^2 \sim \chi^2(G)$.

   Note that $X'AX = (AX)'(AX)$ and $\mathrm{rank}(A) = \mathrm{tr}(A)$ because A is idempotent.

5. If $X \sim N(0, \sigma^2 I_n)$, A and B are symmetric idempotent $n \times n$ matrices of rank G and K, and $AB = 0$, then

$$\frac{X'AX}{G\sigma^2}\Big/\frac{X'BX}{K\sigma^2} = \frac{X'AX/G}{X'BX/K} \sim F(G, K).$$

3 Multiple Regression Model (重回帰モデル)

Up to now, only one independent variable, i.e., x_i, is taken into the regression model.

In this section, we extend it to more independent variables, which is called the **multiple regression (重回帰)**.

We consider the following regression model:

$$\begin{aligned} y_i &= \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_k x_{i,k} + u_i \\ &= (x_{i,1}, x_{i,2}, \cdots, x_{i,k}) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + u_i \\ &= x_i\beta + u_i, \end{aligned}$$

for $i = 1, 2, \cdots, n$,

where x_i and β denote a $1 \times k$ vector of the independent variables and a $k \times 1$ vector of the unknown parameters to be estimated, which are represented as:

$$x_i = (x_{i,1}, x_{i,2}, \cdots, x_{i,k}), \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}.$$

$x_{i,j}$ denotes the ith observation of the jth independent variable.

The case of $k = 2$ and $x_{i,1} = 1$ for all i is exactly equivalent to (1).

Therefore, the matrix form above is a generalization of (1).

Writing all the equations for $i = 1, 2, \cdots, n$, we have:

$$\begin{aligned} y_1 &= \beta_1 x_{1,1} + \beta_2 x_{1,2} + \cdots + \beta_k x_{1,k} + u_1 = x_1\beta + u_1, \\ y_2 &= \beta_1 x_{2,1} + \beta_2 x_{2,2} + \cdots + \beta_k x_{2,k} + u_2 = x_2\beta + u_2, \\ &\vdots \\ y_n &= \beta_1 x_{n,1} + \beta_2 x_{n,2} + \cdots + \beta_k x_{n,k} + u_n = x_n\beta + u_n, \end{aligned}$$

which is rewritten as:

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} &= \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \beta + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}. \end{aligned}$$

Again, the above equation is compactly rewritten as:

$$y = X\beta + u, \tag{18}$$

where y, X and u are denoted by:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \qquad X = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}.$$

Utilizing the matrix form (18), we derive the ordinary least squares estimator of β, denoted by $\hat{\beta}$.

In (18), replacing β by $\hat{\beta}$, we have the following equation:

$$y = X\hat{\beta} + e,$$

where e denotes a $n \times 1$ vector of the residuals.

The ith element of e is given by e_i.

The sum of squared residuals is written as follows:

$$\begin{aligned} S(\hat{\beta}) &= \sum_{i=1}^{n} e_i^2 = e'e = (y - X\hat{\beta})'(y - X\hat{\beta}) = (y' - \hat{\beta}'X')(y - X\hat{\beta}) \\ &= y'y - y'X\hat{\beta} - \hat{\beta}'X'y + \hat{\beta}'X'X\hat{\beta} = y'y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}. \end{aligned}$$

In the last equality, note that $\hat{\beta}'X'y = y'X\hat{\beta}$ because both are scalars.

To minimize $S(\hat{\beta})$ with respect to $\hat{\beta}$, we set the first derivative of $S(\hat{\beta})$ equal to zero, i.e.,

$$\frac{\partial S(\hat{\beta})}{\partial \hat{\beta}} = -2X'y + 2X'X\hat{\beta} = 0.$$

Solving the equation above with respect to $\hat{\beta}$, the **ordinary least squares estimator (OLS, 最小自乗推定量)** of β is given by:

$$\hat{\beta} = (X'X)^{-1}X'y. \tag{19}$$

Thus, the ordinary least squares estimator is derived in the matrix form.

(*) Remark

The second order condition for minimization:

$$\frac{\partial^2 S(\hat\beta)}{\partial\hat\beta\partial\hat\beta'} = 2X'X$$

is a positive definite matrix.

Set $c = Xd$.

For any $d \neq 0$, we have $c'c = d'X'Xd > 0$.

Now, in order to obtain the properties of $\hat\beta$ such as mean, variance, distribution and so on, (19) is rewritten as follows:

$$\begin{aligned}\hat\beta &= (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta + u) = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u \\ &= \beta + (X'X)^{-1}X'u. \qquad (20)\end{aligned}$$

Taking the expectation on both sides of (20), we have the following:

$$\mathrm{E}(\hat\beta) = \mathrm{E}(\beta + (X'X)^{-1}X'u) = \beta + (X'X)^{-1}X'\mathrm{E}(u) = \beta,$$

because of $\mathrm{E}(u) = 0$ by the assumption of the error term u_i.
Thus, unbiasedness of $\hat\beta$ is shown.

The variance of $\hat\beta$ is obtained as:

$$\begin{aligned}\mathrm{V}(\hat\beta) &= \mathrm{E}((\hat\beta-\beta)(\hat\beta-\beta)') = \mathrm{E}\Big((X'X)^{-1}X'u((X'X)^{-1}X'u)'\Big) \\ &= \mathrm{E}((X'X)^{-1}X'uu'X(X'X)^{-1}) = (X'X)^{-1}X'\mathrm{E}(uu')X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} = \sigma^2(X'X)^{-1}.\end{aligned}$$

The first equality is the definition of variance in the case of vector.
In the fifth equality, $\mathrm{E}(uu') = \sigma^2 I_n$ is used, which implies that $\mathrm{E}(u_i^2) = \sigma^2$ for all i and $\mathrm{E}(u_i u_j) = 0$ for $i \neq j$.
Remember that $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed with mean zero and variance σ^2.

Under normality assumption on the error term u, it is known that the distribution of $\hat\beta$ is given by:

$$\hat\beta \sim N(\beta, \sigma^2(X'X)^{-1}).$$

Proof:

θ_u: $n\times 1$, $\quad u$: $n \times 1$, $\quad \theta_\beta$: $k\times 1$, $\quad \hat\beta$: $k \times 1$

The moment-generating function of u, i.e., $\phi_u(\theta_u)$, is:

$$\phi_u(\theta_u) = \mathrm{E}\Big(\exp(\theta_u' u)\Big) = \exp\Big(\frac{\sigma^2}{2}\theta_u'\theta_u\Big),$$

which is $N(0, \sigma^2 I_n)$.

The moment-generating function of $\hat\beta$, i.e., $\phi_\beta(\theta_\beta)$, is:

$$\begin{aligned}\phi_\beta(\theta_\beta) &= \mathrm{E}\Big(\exp(\theta_\beta'\hat\beta)\Big) = \mathrm{E}\Big(\exp(\theta_\beta'\beta + \theta_\beta'(X'X)^{-1}X'u)\Big) \\ &= \exp(\theta_\beta'\beta)\mathrm{E}\Big(\theta_\beta'(X'X)^{-1}X'u\Big) = \exp(\theta_\beta'\beta)\phi_u\Big(\theta_\beta'(X'X)^{-1}X'\Big) \\ &= \exp(\theta_\beta'\beta)\exp\Big(\frac{\sigma^2}{2}\theta_\beta'(X'X)^{-1}\theta_\beta\Big) = \exp\Big(\theta_\beta'\beta + \frac{\sigma^2}{2}\theta_\beta'(X'X)^{-1}\theta_\beta\Big),\end{aligned}$$

which is equivalent to the normal distribution with mean β and variance $\sigma^2(X'X)^{-1}$.
Note that $\quad \theta_u = X(X'X)^{-1}\theta_\beta$. QED

Taking the jth element of $\hat\beta$, its distribution is given by:

$$\hat\beta_j \sim N(\beta_j, \sigma^2 a_{jj}), \qquad \text{i.e.,} \qquad \frac{\hat\beta_j - \beta_j}{\sigma\sqrt{a_{jj}}} \sim N(0,1),$$

where a_{jj} denotes the jth diagonal element of $(X'X)^{-1}$.

Replacing σ^2 by its estimator s^2, we have the following t distribution:

$$\frac{\hat\beta_j - \beta_j}{s\sqrt{a_{jj}}} \sim t(n-k),$$

where $t(n-k)$ denotes the t distribution with $n-k$ degrees of freedom.

s^2 is taken as follows:

$$s^2 = \frac{1}{n-k}\sum_{i=1}^{n} e_i^2 = \frac{1}{n-k}e'e = \frac{1}{n-k}(y - X\hat{\beta})'(y - X\hat{\beta}),$$

which leads to an unbiased estimator of σ^2.

Proof:

Substitute $y = X\beta + u$ and $\hat{\beta} = \beta + (X'X)^{-1}X'u$ into $e = y - X\hat{\beta}$.

$$\begin{aligned} e &= y - X\hat{\beta} = X\beta + u - X(\beta + (X'X)^{-1}X'u) \\ &= u - X(X'X)^{-1}X'u = (I_n - X(X'X)^{-1}X')u \end{aligned}$$

$I_n - X(X'X)^{-1}X'$ is idempotent and symmetric, because we have:

$$\begin{aligned} &(I_n - X(X'X)^{-1}X')(I_n - X(X'X)^{-1}X') = I_n - X(X'X)^{-1}X,' \\ &(I_n - X(X'X)^{-1}X')' = I_n - X(X'X)^{-1}X'. \end{aligned}$$

s^2 is rewritten as follows:

$$\begin{aligned} s^2 &= \frac{1}{n-k}e'e = \frac{1}{n-k}((I_n - X(X'X)^{-1}X')u)'(I_n - X(X'X)^{-1}X')u \\ &= \frac{1}{n-k}u'(I_n - X(X'X)^{-1}X')'(I_n - X(X'X)^{-1}X')u \\ &= \frac{1}{n-k}u'(I_n - X(X'X)^{-1}X')u \end{aligned}$$

Take the expectation of $u'(I_n - X(X'X)^{-1}X')u$ and note that $\mathrm{tr}(a) = a$ for a scalar a.

$$\begin{aligned} \mathrm{E}(s^2) &= \frac{1}{n-k}\mathrm{E}\Big(\mathrm{tr}\big(u'(I_n - X(X'X)^{-1}X')u\big)\Big) = \frac{1}{n-k}\mathrm{E}\Big(\mathrm{tr}\big((I_n - X(X'X)^{-1}X')uu'\big)\Big) \\ &= \frac{1}{n-k}\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')\mathrm{E}(uu')\Big) = \frac{1}{n-k}\sigma^2\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')I_n\Big) \\ &= \frac{1}{n-k}\sigma^2\mathrm{tr}(I_n - X(X'X)^{-1}X') = \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}(X(X'X)^{-1}X')) \\ &= \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}((X'X)^{-1}X'X)) = \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}(I_k)) \\ &= \frac{1}{n-k}\sigma^2(n-k) = \sigma^2 \end{aligned}$$

$\longrightarrow$ s^2 is an unbiased estimator of σ^2.

Note that we do not need normality assumption for unbiasedness of s^2.

Under normality assumption for u, the distribution of s^2 is:

$$\frac{(n-k)s^2}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2(\mathrm{tr}(I_n - X(X'X)^{-1}X'))$$

Note that $\mathrm{tr}(I_n - X(X'X)^{-1}X') = n - k$, because

$$\begin{aligned} &\mathrm{tr}(I_n) = n \\ &\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k \end{aligned}$$

Trace (トレース):

1. A: $n \times n$, $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$, where a_{ij} denotes an element in the ith row and the jth column of a matrix A.

2. a: scalar (1×1), $\mathrm{tr}(a) = a$

3. A: $n \times k$, B: $k \times n$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

4. $\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$

5. When X is a vector of random variables, $\mathrm{E}(\mathrm{tr}(X)) = \mathrm{tr}(\mathrm{E}(X))$

Asymptotic Normality (without normality assumption on u): Using the central limit theorem, without normality assumption we can show that as $n \longrightarrow \infty$, under the condition of $\frac{1}{n}X'X \longrightarrow M$ we have the following result:

$$\frac{\hat{\beta}_j - \beta_j}{s\sqrt{a_{jj}}} \longrightarrow N(0,1),$$

where M denotes a $k \times k$ constant matrix.

Thus, we can construct the confidence interval and the testing procedure, using the t distribution under the normality assumption or the normal distribution without the normality assumption.

4 Properties of OLSE

1. Properties of $\hat{\beta}$: **BLUE (best linear unbiased estimator，最良線形不偏推定量)**, i.e., minimum variance within the class of linear unbiased estimators **(Gauss-Markov theorem，ガウス・マルコフの定理)**

   Proof:

   Consider another linear unbiased estimator, which is denoted by $\tilde{\beta} = Cy$.

   $$\tilde{\beta} = Cy = C(X\beta + u) = CX\beta + Cu,$$

   where C is a $k \times n$ matrix.

Taking the expectation of $\tilde{\beta}$, we obtain:

$$\mathrm{E}(\tilde{\beta}) = CX\beta + C\mathrm{E}(u) = CX\beta$$

Because we have assumed that $\tilde{\beta} = Cy$ is unbiased, $\mathrm{E}(\tilde{\beta}) = \beta$ holds.

That is, we need the condition: $CX = I_k$.

Next, we obtain the variance of $\tilde{\beta} = Cy$.

$$\tilde{\beta} = C(X\beta + u) = \beta + Cu.$$

Therefore, we have:

$$\mathrm{V}(\tilde{\beta}) = \mathrm{E}((\tilde{\beta} - \beta)(\tilde{\beta} - \beta)') = \mathrm{E}(Cuu'C') = \sigma^2 CC'$$

Defining $C = D + (X'X)^{-1}X'$, $\mathrm{V}(\tilde{\beta})$ is rewritten as:

$$\mathrm{V}(\tilde{\beta}) = \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')'.$$

Moreover, because $\hat{\beta}$ is unbiased, we have the following:

$$CX = I_k = (D + (X'X)^{-1}X')X = DX + I_k.$$

Therefore, we have the following condition:

$$DX = 0.$$

Accordingly, $\mathrm{V}(\tilde{\beta})$ is rewritten as:

$$\begin{aligned}\mathrm{V}(\tilde{\beta}) &= \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')' \\ &= \sigma^2 (X'X)^{-1} + \sigma^2 DD' = \mathrm{V}(\hat{\beta}) + \sigma^2 DD'\end{aligned}$$

Thus, $\mathrm{V}(\tilde{\beta}) - \mathrm{V}(\hat{\beta})$ is a positive definite matrix.

$\Longrightarrow \mathrm{V}(\tilde{\beta}_i) - \mathrm{V}(\hat{\beta}_i) > 0$

$\Longrightarrow \hat{\beta}$ is a minimum variance (i.e., best) linear unbiased estimator of β.

Note as follows:

$\Longrightarrow A$ is positive definite when $d'Ad > 0$ except $d = 0$.

$\Longrightarrow$ The ith diagonal element of A, i.e., a_{ii}, is positive (choose d such that the ith element of d is one and the other elements are zeros).

F Distribution ($H_0: \beta = 0$):

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$.

   Therefore, $\dfrac{(\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)}{\sigma^2} \sim \chi^2(k)$.

2. **Proof:**

   Using $\hat{\beta} - \beta = (X'X)^{-1}X'u$, we obtain:

   $$\begin{aligned}(\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) &= ((X'X)^{-1}X'u)'X'X(X'X)^{-1}X'u \\ &= u'X(X'X)^{-1}X'X(X'X)^{-1}X'u = u'X(X'X)^{-1}X'u\end{aligned}$$

Note that $X(X'X)^{-1}X'$ is symmetric and idempotent, i.e., $A'A = A$.

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(X(X'X)^{-1}X')\Big)$$

The degree of freedom is given by:

$$\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$$

Therefore, we obtain:

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k)$$

3. (*) Formula:

   Suppose that $X \sim N(0, I_k)$.

   If A is symmetric and idempotent, i.e., $A'A = A$, then $X'AX \sim \chi^2(\mathrm{tr}(A))$.

   Here, $X = \dfrac{1}{\sigma}u \sim N(0, I_n)$ from $u \sim N(0, \sigma^2 I_n)$, and $A = X(X'X)^{-1}X'$.

4. **Sum of Residuals:** e is rewritten as:

   $$e = (I_n - X(X'X)^{-1}X')u.$$

   Therefore, the sum of residuals is given by:

   $$e'e = u'(I_n - X(X'X)^{-1}X')u.$$

   Note that $I_n - X(X'X)^{-1}X'$ is symmetric and idempotent.

   We obtain the following result:

   $$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(I_n - X(X'X)^{-1}X')\Big),$$

where the trace is:

$$\mathrm{tr}(I_n - X(X'X)^{-1}X') = n - k.$$

Therefore, we have the following result:

$$\frac{e'e}{\sigma^2} = \frac{(n-k)s^2}{\sigma^2} \sim \chi^2(n-k),$$

where

$$s^2 = \frac{1}{n-k}e'e.$$

5. We show that $\hat\beta$ is independent of e.

   Proof:

   Because $u \sim N(0, \sigma^2 I_n)$, we show that $\mathrm{Cov}(e, \hat\beta) = 0$.

   $$\begin{aligned}
   \mathrm{Cov}(e,\hat\beta) &= \mathrm{E}(e(\hat\beta - \beta)') = \mathrm{E}\Big((I_n - X(X'X)^{-1}X')u((X'X)^{-1}X'u)'\Big) \\
   &= \mathrm{E}\Big((I_n - X(X'X)^{-1}X')uu'X(X'X)^{-1}\Big) = (I_n - X(X'X)^{-1}X')\mathrm{E}(uu')X(X'X)^{-1} \\
   &= (I_n - X(X'X)^{-1}X')(\sigma^2 I_n)X(X'X)^{-1} = \sigma^2(I_n - X(X'X)^{-1}X')X(X'X)^{-1} \\
   &= \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}X'X(X'X)^{-1}) = \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}) = 0.
   \end{aligned}$$

   Therefore, $\hat\beta$ is independent of e.

6. Therefore, we obtain the following distribution:

   $$\frac{(\hat\beta - \beta)'X'X(\hat\beta - \beta)}{\sigma^2} = \frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k),$$
   $$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2(n-k)$$

   $\hat\beta$ is independent of e.

   Accordingly, we can derive:

   $$\frac{\dfrac{(\hat\beta - \beta)'X'X(\hat\beta - \beta)}{\sigma^2}\Big/k}{\dfrac{e'e}{\sigma^2}\Big/(n-k)} = \frac{(\hat\beta - \beta)'X'X(\hat\beta - \beta)/k}{s^2} \sim F(k, n-k)$$

Note as follows:

$$\frac{\dfrac{(\hat\beta-\beta)'X'X(\hat\beta-\beta)}{\sigma^2}\Big/k}{\dfrac{e'e}{\sigma^2}\Big/(n-k)} = \frac{u'X(X'X)^{-1}X'u/k}{u'(I_n - X(X'X)^{-1}X')u/(n-k)} \sim F(k, n-k),$$

because $X(X'X)^{-1}X'(I_n - X(X'X)^{-1}X') = 0$.

(*) Formula:

When $X \sim N(0, I_n)$, A and B are $n \times n$ symmetric idempotent matrices, $\mathrm{Rank}(A) = \mathrm{tr}(A) = G$, $\mathrm{Rank}(B) = \mathrm{tr}(B) = K$ and $AB = 0$, then $\dfrac{X'AX/G}{X'BX/K} \sim F(G, K)$.

Coefficient of Determination (決定係数), R^2:

1. Definition of the Coefficient of Determination, R^2: $\quad R^2 = 1 - \dfrac{\sum_{i=1}^n e_i^2}{\sum_{i=1}^n (y_i - \overline{y})^2}$

2. Numerator: $\quad \displaystyle\sum_{i=1}^n e_i^2 = e'e$

3. Denominator: $\quad \displaystyle\sum_{i=1}^n (y_i - \overline{y})^2 = y'(I_n - \frac{1}{n}ii')'(I_n - \frac{1}{n}ii')y = y'(I_n - \frac{1}{n}ii')y$

(*) Remark

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_T - \overline{y} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} - \begin{pmatrix} \overline{y} \\ \overline{y} \\ \vdots \\ \overline{y} \end{pmatrix} = y - \frac{1}{n}ii'y = (I_n - \frac{1}{n}ii')y,$$

where $i = (1, 1, \cdots, 1)'$.

4. In a matrix form, we can rewrite as: $\quad R^2 = 1 - \dfrac{e'e}{y'(I_n - \frac{1}{n}ii')y}$

***F* Distribution and Coefficient of Determination:**

$\Longrightarrow$ This will be discussed later.

Testing Linear Restrictions (*F* Distribution):

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat\beta \sim N(\beta, \sigma^2 (X'X)^{-1})$.

   Consider testing the hypothesis $H_0 : R\beta = r$.

   $R:\ G \times k, \qquad \mathrm{rank}(R) = G \le k.$

   $R\hat\beta \sim N(R\beta, \sigma^2 R(X'X)^{-1}R')$.

Therefore, $\quad \dfrac{(R\hat\beta - r)'(R(X'X)^{-1}R')^{-1}(R\hat\beta - r)}{\sigma^2} \sim \chi^2(G)$.

Note that $R\beta = r$.

(a) When $\hat\beta \sim N(\beta, \sigma^2(X'X)^{-1})$, the mean is:

$\mathrm{E}(R\hat\beta) = R\mathrm{E}(\hat\beta) = R\beta$.

(b) When $\hat\beta \sim N(\beta, \sigma^2(X'X)^{-1})$, the variance is:

$$\begin{aligned} \mathrm{V}(R\hat\beta) &= \mathrm{E}((R\hat\beta - R\beta)(R\hat\beta - R\beta)') = \mathrm{E}(R(\hat\beta - \beta)(\hat\beta - \beta)'R') \\ &= R\mathrm{E}((\hat\beta - \beta)(\hat\beta - \beta)')R' = R\mathrm{V}(\hat\beta)R' = \sigma^2 R(X'X)^{-1}R'. \end{aligned}$$

2. We have the following:

$$\frac{\dfrac{(R\hat\beta - r)'(R(X'X)^{-1}R')^{-1}(R\hat\beta - r)}{G}}{\dfrac{(y - X\hat\beta)'(y - X\hat\beta)}{n-k}} \sim F(G, n-k)$$

3. **Some Examples:**

   (a) t Test:

   The case of $G = 1$, $r = 0$ and $R = (0, \cdots, 1, \cdots, 0)$ (the ith element of R is one and the other elements are zero):

That is, test of $\beta_i = 0$:

Define: $s^2 = \dfrac{(y - X\hat\beta)'(y - X\hat\beta)}{n-k}$.

Then,

$$\frac{\dfrac{(R\hat\beta - r)'(R(X'X)^{-1}R')^{-1}(R\hat\beta - r)}{G}}{s^2} = \frac{\hat\beta_i^2}{s^2 a_{ii}} \sim F(1, n-k),$$

where $R\hat\beta = \hat\beta_i$ and $a_{ii} =$ the i row and ith column of $(X'X)^{-1}$.

*) Recall that $Y \sim F(1, m)$ when $X \sim t(m)$ and $Y = X^2$.

Therefore, the test of $\beta_i = 0$ is given by:

$$\frac{\hat\beta_i}{s\sqrt{a_{ii}}} \sim t(n-k).$$

(b) Test of structural change (Part 1):

$$y_i = \begin{cases} x_i\beta_1 + u_i, & i = 1, 2, \cdots, m \\ x_i\beta_2 + u_i, & i = m+1, m+2, \cdots, n \end{cases}$$

Assume that $u_i \sim N(0, \sigma^2)$.

In a matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ \vdots & \vdots \\ x_m & 0 \\ 0 & x_{m+1} \\ 0 & x_{m+2} \\ \vdots & \vdots \\ 0 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ u_{m+1} \\ u_{m+2} \\ \vdots \\ u_n \end{pmatrix}$$

Moreover, rewriting,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u$$

Again, rewriting,

$$Y = X\beta + u$$

The null hypothesis is $H_0 : \beta_1 = \beta_2$.

Apply the F test, using $R = (I_k \quad -I_k)$ and $r = 0$.

In this case, $G = \mathrm{rank}(R) = k$ and β is a $2k \times 1$ vector.

The distribution is $F(k, n-2k)$.

(c) The hypothesis in which sum of the 1st and 2nd coefficients is equal to one:

$R = (1, 1, 0, \cdots, 0)$, $r = 1$

In this case, $G = \mathrm{rank}(R) = 1$

The distribution of the test statistic is $F(1, n-k)$.

(d) Testing seasonality:

The regression model: The case of **quarterly data (四半期データ)**

$$y = \alpha + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + X\beta_0 + u$$

$D_j =$ 1 in the jth quarter and 0 otherwise, i.e., D_j, $j = 1, 2, 3$, are seasonal dummy variables.

Testing seasonality $\Longrightarrow H_0: \ \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\beta = \begin{pmatrix} \alpha \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case, $G = \mathrm{rank}(R) = 3$, and β is a $k \times 1$ vector.

The distribution of the test statistic is $F(3, n-k)$.

(e) Cobb-Douglas Production Function:

Let Q_i, K_i and L_i be production, capital stock and labor.

We estimate the following production function:

$$\log(Q_i) = \beta_1 + \beta_2 \log(K_i) + \beta_3 \log(L_i) + u_i.$$

We want to test a linear homogeneous (一次同次) production function, i.e., $\beta_2 + \beta_3 = 1$.

The null and alternative hypotheses are:

$H_0:\ \beta_2 + \beta_3 = 1,$

$H_1:\ \beta_2 + \beta_3 \neq 1.$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \qquad r = 1.$$

(f) Test of structural change (Part 2):

Test the structural change between time periods m and $m+1$.

In the case where both the constant term and the slope are changed, the

regression model is as follows:

$$y_i = \alpha + \beta x_i + \gamma d_i + \delta d_i x_i + u_i,$$

where

$$d_i = \begin{cases} 0, & \text{for } i = 1, 2, \cdots, m, \\ 1, & \text{for } i = m+1, m+2, \cdots, n. \end{cases}$$

We consider testing the structural change at time $m+1$.

The null and alternative hypotheses are as follows:

$H_0:\ \gamma = \delta = 0,$

$H_1:\ \gamma \neq 0$, or, $\delta \neq 0$.

Then, set as follows:

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(g) Multiple regression model:

Consider the case of two explanatory variables:

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i.$$

We want to test the hypothesis that neither x_i nor z_i depends on y_i.

In this case, the null and alternative hypotheses are as follows:

$H_0:\ \beta = \gamma = 0,$

$H_1:\ \beta \neq 0$, or, $\gamma \neq 0$.

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Coefficient of Determination R^2 and F distribution:

● The regression model:

$$y_i = x_i\beta + u_i = \beta_1 + x_{2i}\beta_2 + u_i$$

where

$$x_i = \begin{pmatrix} 1 & x_{2i} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

$x_i:\ 1 \times k, \qquad x_{2i}:\ 1 \times (k-1), \qquad \beta:\ k \times 1, \qquad \beta_2:\ (k-1) \times 1$

$$y = X\beta + u = i\beta_1 + X_2\beta_2$$

where the first column of X corresponds to a constant term, i.e.,

$$X = (\, i \quad X_2 \,), \qquad i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

● The F distribution:

$$R = (\, 0 \quad I_{k-1} \,), \qquad r = 0$$

where R is a $(k-1) \times k$ matrix and r is a $(k-1) \times 1$ vector.

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k)$$

Note as follows:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2,$$

where $M = I_n - \dfrac{1}{n} ii'$.

Note that M is symmetric and idempotent, i.e., $M'M = M$.

$$\begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix} = My$$

$R(X'X)^{-1}R'$ is given by:

$$\begin{aligned} R(X'X)^{-1}R' &= (\, 0 \quad I_{k-1} \,) \left(\begin{pmatrix} i' \\ X_2' \end{pmatrix} (\, i \quad X_2 \,) \right)^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \\ &= (\, 0 \quad I_{k-1} \,) \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \end{aligned}$$

(*) The inverse of a partitioned matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square nonsingular matrices.

$$A^{-1} = \begin{pmatrix} B_{11} & -B_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1} \end{pmatrix},$$

where $B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}$, or alternatively,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -B_{22}A_{21}A_{11}^{-1} & B_{22} \end{pmatrix},$$

where $B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$

Go back to the F distribution.

$$\begin{aligned} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} &= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'X_2 - X_2'i(i'i)^{-1}i'X_2)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'(I_n - \frac{1}{n}ii')X_2)^{-1} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix} \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} &(\, 0 \quad I_{k-1} \,) \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \\ &= (\, 0 \quad I_{k-1} \,) \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = (X_2'MX_2)^{-1}. \end{aligned}$$

Thus, under $H_0 : \ \beta_2 = 0$, we obtain the following result:

$$\begin{aligned} &\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} \\ &= \frac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2/(k-1)}{e'e/(n-k)} \sim F(k-1, n-k) \end{aligned}$$

● Coefficient of Determination R^2:

Define e as $e = y - X\hat{\beta}$. The coefficient of determinant, R^2, is

$$R^2 = 1 - \frac{e'e}{y'My},$$

where $M = I_n - \frac{1}{n}ii'$, I_n is a $n \times n$ identity matrix and i is a $n \times 1$ vector consisting of 1, i.e., $i = (1, 1, \cdots, 1)'$.

$$Me = My - MX\hat{\beta}.$$

When $X = (\, i \quad X_2 \,)$ and $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$,

$$Me = e,$$

because $i'e = 0$, and

$$MX = M(\, i \quad X_2 \,) = (\, Mi \quad MX_2 \,) = (\, 0 \quad MX_2 \,)$$

because $Mi = 0$.

$$MX\hat{\beta} = (\, 0 \quad MX_2 \,)\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = MX_2\hat{\beta}_2$$

Thus,

$$My = MX\hat{\beta} + Me \qquad \Longrightarrow \qquad My = MX_2\hat{\beta}_2 + e$$

Therefore, $y'My$ is given by: $\quad y'My = \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 + e'e,$

because $X_2'e = 0$ and $Me = e$.

The coefficient of determinant, R^2, is rewritten as:

$$R^2 = 1 - \frac{e'e}{y'My} \qquad \Longrightarrow \qquad e'e = (1 - R^2)y'My$$

$$R^2 = \frac{y'My - e'e}{y'My} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2}{y'My} \qquad \Longrightarrow \qquad \hat{\beta}_2'X_2'MX_2\hat{\beta}_2 = R^2 y'My$$

Therefore,

$$\begin{aligned} \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} &= \frac{R^2 y'My/(k-1)}{(1-R^2)y'My/(n-k)} \\ &= \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F(k-1, n-k) \end{aligned}$$

5 Restricted OLS (制約付き最小二乗法)

1. Minimize $(y - X\beta)'(y - X\beta)$ subject to $R\beta = r$.

   Let L be the Lagrangian for the minimization problem.

   $$L = (y - X\beta)'(y - X\beta) - 2\lambda'(R\beta - r)$$

   Let the solutions of β and λ for minimization be $\tilde{\beta}$ and $\tilde{\lambda}$.

   $$\frac{\partial L}{\partial \beta} = -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0$$

   $$\frac{\partial L}{\partial \lambda} = -2(R\tilde{\beta} - r) = 0$$

From $\partial L/\partial\beta = 0$, we obtain:

$$\begin{aligned} \tilde{\beta} &= (X'X)^{-1}X'y + (X'X)^{-1}R'\tilde{\lambda} \\ &= \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}. \end{aligned}$$

Multiplying R from the left, we have:

$$R\tilde{\beta} = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Because $R\tilde{\beta} = r$ has to be satisfied, we have the following expression:

$$r = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}.$$

Therefore, solving the above equation with respect to $\tilde{\lambda}$, we obtain:

$$\tilde{\lambda} = \Big(R(X'X)^{-1}R'\Big)^{-1}(r - R\hat{\beta})$$

Substituting $\tilde{\lambda}$ into $\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}$, the restricted OLSE is given by:

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\Big(R(X'X)^{-1}R'\Big)^{-1}(r - R\hat{\beta}).$$

(a) The expectation of $\tilde{\beta}$ is:

$$\begin{aligned} \mathrm{E}(\tilde{\beta}) &= \mathrm{E}(\hat{\beta}) + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\mathrm{E}(\hat{\beta})) \\ &= \beta + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\beta)) = \beta, \end{aligned}$$

which shows that $\tilde\beta$ is unbiased.

(b) The variance of $\tilde\beta$ is as follows.

First, rewrite as follows:

$$\begin{aligned}(\tilde\beta-\beta) &= (\hat\beta-\beta) + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\beta - R\hat\beta)\\ &= (\hat\beta-\beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(R\hat\beta - R\beta)\\ &= (\hat\beta-\beta) - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(\hat\beta-\beta)\\ &= \left(I-(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\right)^{-1}(\hat\beta-\beta)\\ &= W(\hat\beta-\beta).\end{aligned}$$

Then, we obtain the following variance:

$$\begin{aligned}\mathrm{V}(\tilde\beta) &\equiv \mathrm{E}((\tilde\beta-\beta)(\tilde\beta-\beta)') = \mathrm{E}(W(\hat\beta-\beta)(\hat\beta-\beta)'W')\\ &= W\mathrm{E}((\hat\beta-\beta)(\hat\beta-\beta)')W' = W\mathrm{V}(\hat\beta)W' = \sigma^2 W(X'X)^{-1}W'\\ &= \sigma^2\Big(I-(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)(X'X)^{-1}\\ &\qquad\times\Big(I-(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R\Big)'\\ &= \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}\\ &= \mathrm{V}(\hat\beta) - \sigma^2(X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R(X'X)^{-1}\end{aligned}$$

Thus, $\mathrm{V}(\hat\beta)-\mathrm{V}(\tilde\beta)$ is positive definite.

2. Another solution:

   (a) Again, write the first-order condition for minimization:

$$\frac{\partial L}{\partial\beta} = -2X'(y-X\tilde\beta) - 2R'\tilde\lambda = 0,$$
$$\frac{\partial L}{\partial\lambda} = -2(R\tilde\beta - r) = 0,$$

   which can be written as:

$$X'X\tilde\beta - R'\tilde\lambda = X'y,$$
$$R\tilde\beta = r.$$

Using the matrix form:

$$\begin{pmatrix} X'X & R' \\ R & 0\end{pmatrix}\begin{pmatrix}\tilde\beta \\ -\tilde\lambda\end{pmatrix} = \begin{pmatrix} X'y \\ r\end{pmatrix}$$

The solutions of $\tilde\beta$ and $-\tilde\lambda$ are given by:

$$\begin{pmatrix}\tilde\beta \\ -\tilde\lambda\end{pmatrix} = \begin{pmatrix} X'X & R' \\ R & 0\end{pmatrix}^{-1}\begin{pmatrix} X'y \\ r\end{pmatrix}.$$

   (b) Formula to the inverse matrix:

$$\begin{pmatrix} A & B \\ B' & D\end{pmatrix}^{-1} = \begin{pmatrix} E & F \\ F' & G\end{pmatrix},$$

where E, F and G are given by:

$$\begin{aligned}E &= (A-BD^{-1}B')^{-1}\\ &= A^{-1} + A^{-1}B(D-B'A^{-1}B)^{-1}B'A^{-1}\end{aligned}$$

$$\begin{aligned}F &= -(A-BD^{-1}B')^{-1}BD^{-1}\\ &= -A^{-1}B(D-B'A^{-1}B)^{-1}\end{aligned}$$

$$\begin{aligned}G &= (D-B'A^{-1}B)^{-1}\\ &= D^{-1} + D^{-1}B'(A-BD^{-1}B')^{-1}BD^{-1}\end{aligned}$$

   (c) In this case, E and F correspond to:

$$E = (X'X)^{-1} - (X'X)^{-1}R'(R(X'X)^{-1}R')R(X'X)^{-1}$$
$$F = (X'X)^{-1}R'(R(X'X)^{-1}R').$$

Therefore, $\tilde\beta$ is derived as follows:

$$\begin{aligned}\tilde\beta &= EX'y + Fr\\ &= \hat\beta + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat\beta).\end{aligned}$$

   (d) The variance is:

$$\mathrm{V}\begin{pmatrix}\tilde\beta \\ -\tilde\lambda\end{pmatrix} = \sigma^2\begin{pmatrix} X'X & R' \\ R & 0\end{pmatrix}^{-1}.$$

Therefore, $\mathrm{V}(\tilde{\beta})$ is:

$$\begin{aligned}\mathrm{V}(\tilde{\beta}) &= \sigma^2 E \\ &= \sigma^2\Big((X'X)^{-1} - (X'X)^{-1}R'(R(X'X)^{-1}R')R(X'X)^{-1}\Big)\end{aligned}$$

(e) Under the restriction: $R\beta = r$,

$$\mathrm{V}(\hat{\beta}) - \mathrm{V}(\tilde{\beta}) = \sigma^2(X'X)^{-1}R'(R(X'X)^{-1}R')R(X'X)^{-1}$$

is positive definite.

6 F Distribution (Restricted OLS and Unrestricted OLS)

1. As mentioned above, under the null hypothesis $H_0 : R\beta = r$,

$$\frac{\dfrac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{G}}{\dfrac{(y - X\hat{\beta})'(y - X\hat{\beta})}{n-k}} \sim F(G, n-k),$$

where $G = \mathrm{Rank}(R)$.

The numerator is written as follows:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = (\hat{\beta} - \tilde{\beta})'(X'X)(\hat{\beta} - \tilde{\beta}).$$

Remember that

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}(r - R\hat{\beta}).$$

Moreover, rewrite as follows:

$$\begin{aligned}(y - X\tilde{\beta})'(y - X\tilde{\beta}) =& (y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta}))'(y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta})) \\ =& (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) \\ & - (y - X\hat{\beta})'X(\tilde{\beta} - \hat{\beta}) - (\tilde{\beta} - \hat{\beta})'X'(y - X\hat{\beta}) \\ =& (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}).\end{aligned}$$

$X'(y - X\hat{\beta}) = X'e = 0$ is utilized.

Summarizing, we have following representation:

$$\begin{aligned}(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) =& (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) \\ =& (y - X\tilde{\beta})'(y - X\tilde{\beta}) - (y - X\hat{\beta})'(y - X\hat{\beta}) \\ =& \tilde{u}'\tilde{u} - e'e,\end{aligned}$$

where e and $\tilde{u}$ are the restricted residual and the unrestricted residual.

That is,

$$e = y - X\hat{\beta}, \qquad \text{and} \qquad \tilde{u} = y - X\tilde{\beta}.$$

Therefore, we obtain the following result:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} = \frac{(\tilde{u}'\tilde{u} - e'e)/G}{e'e/(n-k)}$$

7 Example: *F* Distribution (Restricted OLS and Unrestricted OLS)

Date file $\Longrightarrow$ cons99.txt (Next slide)

Each column denotes year, nominal household expenditures (家計消費，10 billion yen), household disposable income (家計可処分所得，10 billion yen) and household expenditure deflator (家計消費デフレータ，1990=100) from the left.

1955	5430.1	6135.0	18.1	1970	37784.1	45913.2	35.2	1985	185335.1	220655.6	93.9
1956	5974.2	6828.4	18.3	1971	42571.6	51944.3	37.5	1986	193069.6	229938.8	94.8
1957	6686.3	7619.5	19.0	1972	49124.1	60245.4	39.7	1987	202072.8	235924.0	95.3
1958	7169.7	8153.3	19.1	1973	59366.1	74924.8	44.1	1988	212939.9	247159.7	95.8
1959	8019.3	9274.3	19.7	1974	71782.1	93833.2	53.3	1989	227122.2	263940.5	97.7
1960	9234.9	10776.5	20.5	1975	83591.1	108712.8	59.4	1990	243035.7	280133.0	100.0
1961	10836.2	12869.4	21.8	1976	94443.7	123540.9	65.2	1991	255531.8	297512.9	102.5
1962	12430.8	14701.4	23.2	1977	105397.8	135318.4	70.1	1992	265701.6	309256.6	104.5
1963	14506.6	17042.7	24.9	1978	115960.3	147244.2	73.5	1993	272075.3	317021.6	105.9
1964	16674.9	19709.9	26.0	1979	127600.9	157071.1	76.0	1994	279538.7	325655.7	106.7
1965	18820.5	22337.4	27.8	1980	138585.0	169931.5	81.6	1995	283245.4	331967.5	106.2
1966	21680.6	25514.5	29.0	1981	147103.4	181349.2	85.4	1996	291458.5	340619.1	106.0
1967	24914.0	29012.6	30.1	1982	157994.0	190611.5	87.7	1997	298475.2	345522.7	107.3
1968	28452.7	34233.6	31.6	1983	166631.6	199587.8	89.5				
1969	32705.2	39486.3	32.9	1984	175383.4	209451.9	91.8				

Estimate using TSP 5.0.

```
LINE  ****************************************************
 |      1  freq a;
 |      2  smpl 1955 1997;
 |      3  read(file='cons99.txt') year cons yd price;
 |      4  rcons=cons/(price/100);
 |      5  ryd=yd/(price/100);
 |      6  d1=0.0;
 |      7  smpl 1974 1997;
 |      8  d1=1.0;
 |      9  smpl 1956 1997;
 |     10  d1ryd=d1*ryd;
 |     11  olsq rcons c ryd;
 |     12  olsq rcons c d1 ryd d1ryd;
 |     13  end;
*****************************************************
```



```
                 Equation   1
                 ============

   Method of estimation = Ordinary Least Squares


Dependent variable: RCONS
Current sample:  1956 to 1997
Number of observations:  42

 Mean of dependent variable = 149038.
Std. dev. of dependent var. = 78147.9
   Sum of squared residuals = .127951E+10
      Variance of residuals = .319878E+08
   Std. error of regression = 5655.77
                  R-squared = .994890
```



```
         Adjusted R-squared = .994762
    Durbin-Watson statistic = .116873
  F-statistic (zero slopes) = 7787.70
 Schwarz Bayes. Info. Crit. = 17.4101
 Log of likelihood function = -421.469

           Estimated     Standard
Variable  Coefficient     Error        t-statistic
C         -3317.80       1934.49       -1.71508
RYD       .854577        .968382E-02   88.2480
```



```
                 Equation   2
                 ============

   Method of estimation = Ordinary Least Squares


Dependent variable: RCONS
Current sample:  1956 to 1997
Number of observations:  42

 Mean of dependent variable = 149038.
Std. dev. of dependent var. = 78147.9
   Sum of squared residuals = .244501E+09
      Variance of residuals = .643423E+07
   Std. error of regression = 2536.58
                  R-squared = .999024
```

```
        Adjusted R-squared = .998946
    Durbin-Watson statistic = .420979
  F-statistic (zero slopes) = 12959.1
 Schwarz Bayes. Info. Crit. = 15.9330
 Log of likelihood function = -386.714

           Estimated    Standard
Variable  Coefficient    Error        t-statistic
C         4204.11       1440.45       2.91861
D1        -39915.3      3154.24       -12.6545
RYD       .786609       .015024       52.3561
D1RYD     .194495       .018731       10.3839
```


1. `Equation 1`

   Significance test:

   `Equation 1` is:

$$\mathtt{RCONS} = \beta_1 + \beta_2 \mathtt{RYD}$$

   $H_0: \ \beta_2 = 0$

   (No.1) t Test $\Longrightarrow$ Compare 10.3839 and $t(42-2)$.

   (No.2) F Test $\Longrightarrow$ Compare $\dfrac{R^2/G}{(1-R^2)/(n-k)} = \dfrac{.994890/1}{(1-.994890)/(42-2)} =$ 7787.8 and $F(1,40)$.

   1% point of $F(1,40) = 7.31$

   $H_0: \ \beta_2 = 0$ is rejected.

2. `Equation 1` vs. `Equation 2`

   Test the structural change between 1973 and 1974.

   `Equation 2` is:

$$\mathtt{RCONS} = \beta_1 + \beta_2 \mathtt{D1} + \beta_3 \mathtt{RYD} + \beta_4 \mathtt{RYD} \times \mathtt{D1}$$

   $H_0: \ \beta_2 = \beta_4 = 0$

Restricted OLS $\Longrightarrow$ `Equation 1`

Unrestricted OLS $\Longrightarrow$ `Equation 2`

$$\frac{(\tilde{u}'\tilde{u} - e'e)/G}{e'e/(n-k)} = \frac{(.127951\mathrm{E}+10 - .244501\mathrm{E}+09)/2}{.244501\mathrm{E}+09/(42-4)} = 80.43$$

which should be compared with $F(2,38)$.

1% point of $F(2,38) = 5.211 < 80.43$

$H_0: \ \beta_2 = \beta_4 = 0$ is rejected.

$\Longrightarrow$ The structure was changed in 1974.

8 Generalized Least Squares Method (GLS, 一般化最小自乗法)

1. Regression model: $\quad y = X\beta + u, \qquad u \sim (0, \sigma^2\Omega)$

2. **Heteroscedasticity (不等分散，不均一分散)**

$$\sigma^2\Omega = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n^2 \end{pmatrix}$$

First-Order Autocorrelation (一階の自己相関，系列相関)

In the case of time series data, the subscript is conventionally given by t, not i.

$u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \ \text{iid } N(0, \sigma_\epsilon^2)$

$$\sigma^2\Omega = \frac{\sigma_\epsilon^2}{1-\rho^2} \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}$$

$$\mathrm{V}(u_t) = \sigma^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}$$

3. The generalized least squares (GLS) estimator of β, denoted by b, solves the following minimization problem:

$$\min_\beta \ (y - X\beta)'\Omega^{-1}(y - X\beta)$$

The GLSE of β is:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

4. In general, when Ω is symmetric, Ω is decomposed as follows.

$$\Omega = A'\Lambda A$$

Λ is a diagonal matrix, where the diagonal elements of Λ are given by the eigen values.

A is a matrix consisting of eigen vectors.

When Ω is a positive definite matrix, all the diagonal elements of Λ are positive.

5. There exists P such that $\Omega = PP'$ (i.e., take $P = A'\Lambda^{1/2}$).

Multiply P^{-1} on both sides of $y = X\beta + u$.

We have:

$$y^\star = X^\star\beta + u^\star,$$

where $y^\star = P^{-1}y$, $X^\star = P^{-1}X$, and $u^\star = P^{-1}u$.

Note that

$$\mathrm{V}(u^\star) = \mathrm{V}(P^{-1}u) = P^{-1}\mathrm{V}(u)P'^{-1} = \sigma^2 P^{-1}\Omega P'^{-1} = \sigma^2 I_n,$$

because $\Omega = PP'$, i.e., $P^{-1}\Omega P'^{-1} = I_n$.

Accordingly, the regression model is rewritten as:

$$y^\star = X^\star\beta + u^\star, \qquad u^\star \sim (0, \sigma^2 I_n)$$

Apply OLS to the above model.

That is,

$$\min_\beta \ (y^\star - X^\star\beta)'(y^\star - X^\star\beta)$$

is equivalent to:

$$\min_\beta \ (y - X\beta)'\Omega^{-1}(y - X\beta)$$

$$b = (X^{\star\prime}X^\star)^{-1}X^{\star\prime}y^\star = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

$$b = \beta + (X^{\star\prime}X^\star)^{-1}X^{\star\prime}u^\star = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}u$$

$$\mathrm{E}(b) = \beta$$

$$\mathrm{V}(b) = \sigma^2(X^{\star\prime}X^\star)^{-1} = \sigma^2(X'\Omega^{-1}X)^{-1}$$

6. Suppose that the regression model is given by:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega).$$

In this case, when we use OLS, what happens?

$$\hat\beta = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u$$

$$\mathrm{V}(\hat\beta) = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}$$

Compare GLS and OLS.

(a) Expectation:

$$\mathrm{E}(\hat{\beta}) = \beta, \quad \text{and} \quad \mathrm{E}(b) = \beta$$

Thus, both $\hat{\beta}$ and b are unbiased estimator.

(b) Variance:

$$\mathrm{V}(\hat{\beta}) = \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1}$$
$$\mathrm{V}(b) = \sigma^2 (X'\Omega^{-1}X)^{-1}$$

Which is more efficient, OLS or GLS?.

$$\begin{aligned}
\mathrm{V}(\hat{\beta}) - \mathrm{V}(b) &= \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1} - \sigma^2 (X'\Omega^{-1}X)^{-1} \\
&= \sigma^2 \Big((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1} X'\Omega^{-1} \Big) \Omega \\
&\qquad \times \Big((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1} X'\Omega^{-1} \Big)' \\
&= \sigma^2 A\Omega A'
\end{aligned}$$

Ω is the variance-covariance matrix of u, which is a positive definite matrix.

Therefore, except for $\Omega = I_n$, $A\Omega A'$ is also a positive definite matrix.

This implies that $\mathrm{V}(\hat{\beta}_i) - \mathrm{V}(b_i) > 0$ for the ith element of β.

Accordingly, b is more efficient than $\hat{\beta}$.

7. If $u \sim N(0, \sigma^2\Omega)$, then $b \sim N(\beta, \sigma^2 (X'\Omega^{-1}X)^{-1})$.

   Consider testing the hypothesis $H_0 : R\beta = r$.

   $R : G \times k$, $\quad \mathrm{rank}(R) = G \leq k$.

   $Rb \sim N(R\beta, \sigma^2 R(X'\Omega^{-1}X)^{-1}R')$.

   Therefore, the following quadratic form is distributed as:

$$\frac{(Rb - r)'(R(X'\Omega^{-1}X)^{-1}R')^{-1}(Rb - r)}{\sigma^2} \sim \chi^2(G)$$

8. Because $(y^\star - X^\star b)'(y^\star - X^\star b)/\sigma^2 \sim \chi^2(n-k)$, we obtain:

$$\frac{(y - Xb)'\Omega^{-1}(y - Xb)}{\sigma^2} \sim \chi^2(n-k)$$

9. Furthermore, from the fact that b is independent of $y - Xb$, the following F distribution can be derived:

$$\frac{(Rb - r)'(R(X'\Omega^{-1}X)^{-1}R')^{-1}(Rb - r)/G}{(y - Xb)'\Omega^{-1}(y - Xb)/n - k} \sim F(G, n-k)$$

10. Let b be the unrestricted GLSE and $\tilde{b}$ be the restricted GLSE.

    Their residuals are given by e and $\tilde{e}$, respectively.

$$e = y - Xb, \qquad \tilde{e} = y - X\tilde{b}$$

   Then, the F test statistic is written as follows:

$$\frac{(\tilde{e}'\Omega^{-1}\tilde{e} - e'\Omega^{-1}e)/G}{e'\Omega^{-1}e/(n-k)} \sim F(G, n-k)$$

8.1 Example: Mixed Estimation (Theil and Goldberger Model)

A generalization of the restricted OLS $\Longrightarrow$ Stochastic linear restriction:

$$\begin{aligned}
r &= R\beta + v, \qquad && \mathrm{E}(v) = 0 \ \text{ and } \ \mathrm{V}(v) = \sigma^2\Psi \\
y &= X\beta + u, \qquad && \mathrm{E}(u) = 0 \ \text{ and } \ \mathrm{V}(u) = \sigma^2 I_n
\end{aligned}$$

Using a matrix form,

$$\begin{pmatrix} y \\ r \end{pmatrix} = \begin{pmatrix} X \\ R \end{pmatrix}\beta + \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathrm{E}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \text{ and } \ \mathrm{V}\begin{pmatrix} u \\ v \end{pmatrix} = \sigma^2 \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}$$

For estimation, we do not need normality assumption.

Applying GLS, we obtain:

$$\begin{aligned} b &= \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} y \\ r \end{pmatrix} \right) \\ &= \left(X'X + R'\Psi^{-1}R \right)^{-1} \left(X'y + R'\Psi^{-1}r \right). \end{aligned}$$

Mean and Variance of b: b is rewritten as follows:

$$\begin{aligned} b &= \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} y \\ r \end{pmatrix} \right) \\ &= \beta + \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

Therefore, the mean and variance are given by:

$$\mathrm{E}(b) = \beta \qquad \Longrightarrow \qquad b \text{ is unbiased.}$$

$$\begin{aligned} \mathrm{V}(b) &= \sigma^2 \left(\begin{pmatrix} X' & R' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \Psi \end{pmatrix}^{-1} \begin{pmatrix} X \\ R \end{pmatrix} \right)^{-1} \\ &= \sigma^2 \left(X'X + R'\Psi^{-1}R \right)^{-1} \end{aligned}$$

9 Maximum Likelihood Estimation (MLE, 最尤法)

$\Longrightarrow$ Review of Last Semester

1. The distribution function of $\{X_i\}_{i=1}^n$ is $f(x;\theta)$, where $x = (x_1, x_2, \cdots, x_n)$ and $\theta = (\mu, \Sigma)$.

   Note that X is a vector of random variables and x is a vector of their realizations (i.e., observed data).

   Likelihood function $L(\cdot)$ is defined as $L(\theta; x) = f(x;\theta)$.

Note that $f(x;\theta) = \prod_{i=1}^n f(x_i;\theta)$ when X_1, X_2, $\cdots$, X_n are mutually independently and identically distributed.

The maximum likelihood estimator (MLE) of θ is θ such that:

$$\max_{\theta} \; L(\theta; X). \qquad \Longleftrightarrow \qquad \max_{\theta} \; \log L(\theta; X).$$

MLE satisfies the following two conditions:

(a) $\dfrac{\partial \log L(\theta; X)}{\partial \theta} = 0.$

(b) $\dfrac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'}$ is a negative definite matrix.

2. **Fisher's information matrix (フィッシャーの情報行列)** is defined as:

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'} \Big),$$

   where we have the following equality:

$$-\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta \partial \theta'} \Big) = \mathrm{E}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta} \frac{\partial \log L(\theta; X)}{\partial \theta'} \Big) = \mathrm{V}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta} \Big)$$

   Proof of the above equality:

$$\int L(\theta; x) \mathrm{d}x = 1$$

Take a derivative with respect to θ.

$$\int \frac{\partial L(\theta; x)}{\partial \theta} \mathrm{d}x = 0$$

(We assume that (i) the domain of x does not depend on θ and (ii) the derivative $\dfrac{\partial L(\theta; x)}{\partial \theta}$ exists.)

Rewriting the above equation, we obtain:

$$\int \frac{\partial \log L(\theta; x)}{\partial \theta} L(\theta; x) \mathrm{d}x = 0,$$

i.e.,

$$\mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial \theta} \right) = 0.$$

Again, differentiating the above with respect to θ, we obtain:

$$\int \frac{\partial^2 \log L(\theta; x)}{\partial\theta\partial\theta'} L(\theta; x)\mathrm{d}x + \int \frac{\partial \log L(\theta; x)}{\partial\theta} \frac{\partial L(\theta; x)}{\partial'\theta}\mathrm{d}x$$
$$= \int \frac{\partial^2 \log L(\theta; x)}{\partial\theta\partial\theta'} L(\theta; x)\mathrm{d}x + \int \frac{\partial \log L(\theta; x)}{\partial\theta} \frac{\partial \log L(\theta; x)}{\partial\theta'} L(\theta; x)\mathrm{d}x$$
$$= \mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\Big) + \mathrm{E}\Big(\frac{\partial \log L(\theta; X)}{\partial\theta} \frac{\partial \log L(\theta; X)}{\partial\theta'}\Big) = 0.$$

Therefore, we can derive the following equality:

$$-\mathrm{E}\left(\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\right) = \mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial\theta} \frac{\partial \log L(\theta; X)}{\partial\theta'}\right) = \mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right),$$

where the second equality utilizes $\mathrm{E}\left(\dfrac{\partial \log L(\theta; X)}{\partial\theta}\right) = 0$.

3. **Cramer-Rao Lower Bound (クラメール・ラオの下限)**: $I(\theta)$

Suppose that an estimator of θ is given by $s(X)$.

The expectation of $s(X)$ is:

$$\mathrm{E}(s(X)) = \int s(x)L(\theta; x)\mathrm{d}x.$$

Differentiating the above with respect to θ,

$$\frac{\partial \mathrm{E}(s(X))}{\partial\theta} = \int s(x)\frac{\partial L(\theta; x)}{\partial\theta}\mathrm{d}x = \int s(x)\frac{\partial \log L(\theta; x)}{\partial\theta} L(\theta; x)\mathrm{d}x$$
$$= \mathrm{Cov}\left(s(X), \frac{\partial \log L(\theta; X)}{\partial\theta}\right)$$

For simplicity, let $s(X)$ and θ be scalars.

Then,

$$\left(\frac{\partial \mathrm{E}(s(X))}{\partial\theta}\right)^2 = \left(\mathrm{Cov}\left(s(X), \frac{\partial \log L(\theta; X)}{\partial\theta}\right)\right)^2 = \rho^2 \mathrm{V}\,(s(X))\,\mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right)$$
$$\leq \mathrm{V}\,(s(X))\,\mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right),$$

where ρ denotes the correlation coefficient between $s(X)$ and $\dfrac{\partial \log L(\theta; X)}{\partial\theta}$,

i.e.,

$$\rho = \frac{\mathrm{Cov}\left(s(X), \dfrac{\partial \log L(\theta; X)}{\partial\theta}\right)}{\sqrt{\mathrm{V}\,(s(X))}\sqrt{\mathrm{V}\left(\dfrac{\partial \log L(\theta; X)}{\partial\theta}\right)}}.$$

Note that $|\rho| \leq 1$.

Therefore, we have the following inequality:

$$\left(\frac{\partial \mathrm{E}(s(X))}{\partial\theta}\right)^2 \leq \mathrm{V}(s(X))\,\mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right),$$

i.e.,

$$\mathrm{V}(s(X)) \geq \frac{\left(\dfrac{\partial \mathrm{E}(s(X))}{\partial\theta}\right)^2}{\mathrm{V}\left(\dfrac{\partial \log L(\theta; X)}{\partial\theta}\right)}$$

Especially, when $\mathrm{E}(s(X)) = \theta$,

$$\mathrm{V}(s(X)) \geq \frac{1}{-\mathrm{E}\left(\dfrac{\partial^2 \log L(\theta; X)}{\partial\theta^2}\right)} = (I(\theta))^{-1}.$$

Even in the case where $s(X)$ is a vector, the following inequality holds.

$$\mathrm{V}(s(X)) \geq (I(\theta))^{-1},$$

where $I(\theta)$ is defined as:

$$I(\theta) = -\mathrm{E}\left(\frac{\partial^2 \log L(\theta; X)}{\partial\theta\partial\theta'}\right)$$
$$= \mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial\theta} \frac{\partial \log L(\theta; X)}{\partial\theta'}\right)$$
$$= \mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial\theta}\right).$$

4. Asymptotic Normality of MLE:

   Let $\tilde{\theta}$ be MLE of θ.

   As n goes to infinity, we have the following result:

$$\sqrt{n}(\tilde{\theta} - \theta) \longrightarrow N\left(0, \lim_{n\to\infty}\left(\frac{I(\theta)}{n}\right)^{-1}\right),$$

   where it is assumed that $\lim_{n\to\infty}\left(\dfrac{I(\theta)}{n}\right)$ converges.

   That is, when n is large, $\tilde{\theta}$ is approximately distributed as follows:

$$\tilde{\theta} \sim N\left(\theta, (I(\theta))^{-1}\right).$$

Suppose that $s(X) = \tilde{\theta}$.

When n is large, $\mathrm{V}(s(X))$ is approximately equal to $(I(\theta))^{-1}$.

5. Optimization (最適化):

$$0 = \frac{\partial \log L(\theta; x)}{\partial \theta} = \frac{\partial \log L(\theta^*; x)}{\partial \theta} + \frac{\partial^2 \log L(\theta^*; x)}{\partial \theta \partial \theta'}(\theta - \theta^*).$$

Solving the above equation with respect to θ, we obtain the following:

$$\theta = \theta^* - \left(\frac{\partial^2 \log L(\theta^*; x)}{\partial \theta \partial \theta'}\right)^{-1} \frac{\partial \log L(\theta^*; x)}{\partial \theta}.$$

Replace the variables as follows:

$$\theta \longrightarrow \theta^{(i+1)}$$
$$\theta^* \longrightarrow \theta^{(i)}$$

Then, we have:

$$\theta^{(i+1)} = \theta^{(i)} - \left(\frac{\partial^2 \log L(\theta^{(i)}; x)}{\partial \theta \partial \theta'}\right)^{-1} \frac{\partial \log L(\theta^{(i)}; x)}{\partial \theta}.$$

$\Longrightarrow$ **Newton-Raphson method (ニュートン・ラプソン法)**

Replacing $\dfrac{\partial^2 \log L(\theta^{(i)}; x)}{\partial \theta \partial \theta'}$ by $\mathrm{E}\left(\dfrac{\partial^2 \log L(\theta^{(i)}; x)}{\partial \theta \partial \theta'}\right)$, we obtain the following optimization algorithm:

$$\begin{aligned}\theta^{(i+1)} &= \theta^{(i)} - \left(\mathrm{E}\left(\frac{\partial^2 \log L(\theta^{(i)}; x)}{\partial \theta \partial \theta'}\right)\right)^{-1} \frac{\partial \log L(\theta^{(i)}; x)}{\partial \theta} \\ &= \theta^{(i)} + \left(I(\theta^{(i)})\right)^{-1} \frac{\partial \log L(\theta^{(i)}; x)}{\partial \theta}\end{aligned}$$

$\Longrightarrow$ **Method of Scoring (スコア法)**

9.1 MLE: The Case of Single Regression Model

The regression model:

$$y_i = \beta_1 + \beta_2 x_i + u_i,$$

1. $u_i \sim N(0, \sigma^2)$ is assumed.

2. The density function of u_i is:

$$f(u_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} u_i^2\right).$$

   Because $u_1, u_2, \cdots, u_n$ are mutually independently distributed, the joint den-

sity function of $u_1, u_2, \cdots, u_n$ is written as:

$$\begin{aligned}f(u_1, u_2, \cdots, u_n) &= f(u_1) f(u_2) \cdots f(u_n) \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n u_i^2\right)\end{aligned}$$

3. Using the transformation of variable ($u_i = y_i - \beta_1 - \beta_2 x_i$), the joint density function of $y_1, y_2, \cdots, y_n$ is given by:

$$\begin{aligned}f(y_1, y_2, \cdots, y_n) &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_1 - \beta_2 x_i)^2\right) \\ &\equiv L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n).\end{aligned}$$

$L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)$ is called the likelihood function.

$\log L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)$ is called the log-likelihood function.

$$\begin{aligned}&\log L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n) \\ &\qquad = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_t - \beta_1 - \beta_2 x_i)^2\end{aligned}$$

4. **Transformation of Variable (変数変換)**:

   Suppose that the density function of a random variable X is $f_x(x)$.

   Defining $X = g(Y)$, the density function of Y, $f_y(y)$, is given by:

   $$f_y(y) = f_x(g(y))\left|\frac{\mathrm{d}g(y)}{\mathrm{d}y}\right|.$$

   In the case where X and $g(Y)$ are $n \times 1$ vectors, $\left|\frac{\mathrm{d}g(y)}{\mathrm{d}y}\right|$ should be replaced by $\left|\frac{\partial g(y)}{\partial y'}\right|$, which is an absolute value of a determinant of the matrix $\frac{\partial g(y)}{\partial y'}$.

Example: When $X \sim U(0, 1)$, derive the density function of $Y = -\log(X)$.

$$f_x(x) = 1$$

$X = \exp(-Y)$ is obtained.

Therefore, the density function of Y, $f_y(y)$, is given by:

$$f_y(y) = \left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| f_x(g(y)) = |-\exp(-y)| = \exp(-y)$$

5. Given the observed data $y_1, y_2, \cdots, y_n$, the likelihood function $L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)$, or the log-likelihood function $\log L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)$ is maximized with respect to $(\alpha, \beta, \sigma^2)$.

   Solve the following three simultaneous equations:

   $$\frac{\partial \log L(\alpha, \beta, \sigma^2 | y_1, y_2, \cdots, y_n)}{\partial \alpha} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i) = 0,$$

   $$\frac{\partial \log L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)}{\partial \beta} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)x_i = 0,$$

$$\frac{\partial \log L(\beta_1, \beta_2, \sigma^2 | y_1, y_2, \cdots, y_n)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2 = 0.$$

The solutions of $(\beta_1, \beta_2, \sigma^2)$ are called the maximum likelihood estimates, denoted by $(\tilde\beta_1, \tilde\beta_2, \tilde\sigma^2)$.

The maximum likelihood estimates are:

$$\tilde\beta_2 = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2}, \qquad \tilde\beta_1 = \overline{y} - \tilde\beta_2\overline{x}, \qquad \tilde\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde\alpha - \tilde\beta x_i)^2.$$

The MLE of σ^2 is divided by n, not $n - 2$.

9.2 MLE: The Case of Multiple Regression Model I

1. Multivariate Normal Distribution: $X:\ n \times 1$ and $X \sim N(\mu, \Sigma)$

   The density function of X is:

   $$f(x) = (2\pi)^{n/2}|\Sigma|^{-1/2}\exp\Big(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big).$$

2. Regression model: $y = X\beta + u, \quad u \sim N(0, \sigma^2 I_n)$

Transformation of Variables from u to y:

$$f_u(u) = (2\pi\sigma^2)^{-n/2} \exp\Big(-\frac{1}{2\sigma^2}u'u\Big)$$

$$\begin{aligned} f_y(y) &= f_u(y - X\beta)\left|\frac{\partial u}{\partial y'}\right| \\ &= (2\pi\sigma^2)^{-n/2} \exp\Big(-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\Big) \\ &= L(\theta; y, X), \end{aligned}$$

where $\theta = (\beta, \sigma^2)$, because of $\dfrac{\partial u}{\partial y'} = I_n$.

Therefore, the log-likelihood function is:

$$\log L(\theta; y, X) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta),$$

Note that $|\Sigma|^{-1/2} = |\sigma^2 I_n|^{-1/2} = \sigma^{-n/2}$.

3. $\max\limits_{\theta} \quad \log L(\theta; y, X)$

(FOC) $\dfrac{\partial \log L(\theta; y, X)}{\partial\theta} = 0$

(SOC) $\dfrac{\partial^2 \log L(\theta; y, X)}{\partial\theta\partial\theta'}$ is a negative definite matrix.

We obtain MLE of β and σ^2:

$$\tilde\beta = (X'X)^{-1}X'y, \qquad \tilde\sigma^2 = \frac{(y - X\tilde\beta)'(y - X\tilde\beta)}{n},$$

where $\tilde\sigma^2$ is divided by n, not $n - k$.

4. Fisher's information matrix is:

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; y, X)}{\partial\theta\partial\theta'}\Big)$$

The inverse of the information matrix, $I(\theta)^{-1}$, provides a lower bound of the

variance - covariance matrix for unbiased estimators of θ .

$$I(\theta)^{-1} = \begin{pmatrix} \sigma^2(X'X)^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{pmatrix}$$

For large n, we approximately obtain: $\begin{pmatrix} \tilde\beta \\ \tilde\sigma^2 \end{pmatrix} \sim N\left(\begin{pmatrix} \beta \\ \sigma^2 \end{pmatrix}, \begin{pmatrix} \sigma^2(X'X)^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{pmatrix}\right)$.

9.3 MLE: The Case of Multiple Regression Model II

1. Regression model: $y = X\beta + u, \quad u \sim N(0, \sigma^2\Omega)$

Transformation of Variables from u to y:

$$f_u(u) = (2\pi\sigma^2)^{-n/2}|\Omega|^{-1/2} \exp\Big(-\frac{1}{2\sigma^2}u'\Omega^{-1}u\Big)$$

$$\begin{aligned} f_y(y) &= f_u(y - X\beta)\left|\frac{\partial u}{\partial y'}\right| \\ &= (2\pi\sigma^2)^{-n/2}|\Omega|^{-1/2} \exp\Big(-\frac{1}{2\sigma^2}(y - X\beta)'\Omega^{-1}(y - X\beta)\Big) \\ &= L(\theta; y, X), \end{aligned}$$

where $\theta = (\beta, \sigma^2)$, because of $\dfrac{\partial u}{\partial y'} = I_n$.

The log-likelihood function is:

$$\log L(\theta; y, X) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\log|\Omega| - \frac{1}{2\sigma^2}(y - X\beta)'\Omega^{-1}(y - X\beta),$$

where $\theta = (\beta, \sigma^2)$.

2. $\max\limits_{\theta} \quad \log L(\theta; y, X)$

(FOC) $\dfrac{\partial \log L(\theta; y, X)}{\partial\theta} = 0$

(SOC) $\dfrac{\partial^2 \log L(\theta; y, X)}{\partial\theta\partial\theta'}$ is a negative definite matrix.

Then, we obtain MLE of β and σ^2:

$$\tilde{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y, \qquad \tilde{\sigma}^2 = \frac{(y - X\tilde{\beta})'\Omega^{-1}(y - X\tilde{\beta})}{n}$$

3. Fisher's information matrix is defined as:

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; y, X)}{\partial\theta\partial\theta'}\Big)$$

The inverse of the information matrix, $I(\theta)^{-1}$, provides a lower bound of the variance - covariance matrix for unbiased estimators of θ, which is given by:

$$I(\theta)^{-1} = \begin{pmatrix} \sigma^2(X'\Omega^{-1}X)^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{pmatrix}$$

9.4 MLE: AR(1) Model

The pth-order Autoregressive Model, i.e., AR(p) Model (p 次の自己回帰モデル):

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + u_t$$

AR(1) Model: $\quad t = 2, 3, \cdots, n,$

$$y_t = \phi_1 y_{t-1} + u_t, \quad u_t \sim N(0, \sigma^2)$$

where $|\phi_1| < 1$ is assumed for now.

To obtain the joint density function of $y_1, y_2, \cdots, y_n$, $f(y_n, y_{n-1}, \cdots, y_1)$ is decomposed as follows:

$$f(y_n, y_{n-1}, \cdots, y_1) = f(y_1) \prod_{t=2}^{n} f(y_t | y_{t-1}, \cdots, y_1).$$

From $y_t = \phi_1 y_{t-1} + u_t$, we can obtain:

$$\mathrm{E}(y_t|y_{t-1}, \cdots, y_1) = \phi_1 y_{t-1}, \quad \text{and} \quad \mathrm{V}(y_t|y_{t-1}, \cdots, y_1) = \sigma^2.$$

Therefore, the conditional distribution $f(y_t|y_{t-1}, \cdots, y_1)$ is:

$$f(y_t|y_{t-1}, \cdots, y_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right).$$

To obtain the unconditional distribution $f(y_t)$, y_t is rewritten as follows:

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + u_t \\ &= \phi_1^2 y_{t-2} + u_t + \phi_1 u_{t-1} \\ &\ \ \vdots \\ &= \phi_1^j y_{t-j} + u_t + \phi_1 u_{t-1} + \cdots + \phi_1^j u_{t-j} \\ &\ \ \vdots \\ &= u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \cdots, \qquad \text{when } j \text{ goes to infinity.} \end{aligned}$$

The unconditional expectation and variance of y_t is:

$$\mathrm{E}(y_t) = 0, \quad \text{and} \quad \mathrm{V}(y_t) = \sigma^2(1 + \phi_1^2 + \phi_1^4 + \cdots) = \frac{\sigma^2}{1 - \phi_1^2}.$$

Therefore, the unconditional distribution of y_t is given by:

$$f(y_t) = \frac{1}{\sqrt{2\pi\sigma^2/(1 - \phi_1^2)}} \exp\left(-\frac{1}{2\sigma^2/(1 - \phi_1^2)} y_t^2\right).$$

Finally, the joint distribution of $y_1, y_2, \cdots, y_n$ is given by:

$$\begin{aligned} f(y_n, y_{n-1}, \cdots, y_1) &= f(y_1) \prod_{t=2}^{n} f(y_t|y_{t-1}, \cdots, y_1) \\ &= \frac{1}{\sqrt{2\pi\sigma^2/(1 - \phi_1^2)}} \exp\left(-\frac{1}{2\sigma^2/(1 - \phi_1^2)} y_1^2\right) \\ &\quad \times \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right) \end{aligned}$$

The log-likelihood function is:

$$\log L(\phi_1, \sigma^2; y_n, y_{n-1}, \cdots, y_1) = -\frac{1}{2}\log(2\pi\sigma^2/(1-\phi_1^2)) - \frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 \\ -\frac{n-1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=2}^{n}(y_t - \phi_1 y_{t-1})^2.$$

Maximize $\log L$ with respect to ϕ_1 and σ^2.

Maximization Procedure:

- Newton-Raphson Method, or Method of Scoring
- Simple Grid Search (search maximization within the range $-1 < \rho < 1$, changing the value of ϕ_1 by 0.01)

181

9.5 MLE: Regression Model with AR(1) Error

When the error term is autocorrelated, the regression model is written as:

$$y_t = x_t\beta + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2).$$

The joint distribution of $u_n, u_{n-1}, \cdots, u_1$ is:

$$f_u(u_n, u_{n-1}, \cdots, u_1; \rho, \sigma_\epsilon^2) = f_u(u_1; \rho, \sigma_\epsilon^2)\prod_{t=2}^{n} f_u(u_t | u_{t-1}, \cdots, u_1; \rho, \sigma_\epsilon^2)$$

182

$$= (2\pi\sigma_\epsilon^2/(1-\rho^2))^{-1/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2/(1-\rho^2)}u_1^2\right) \\ \times(2\pi\sigma_\epsilon^2)^{-(n-1)/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=2}^{n}(u_t - \rho u_{t-1})^2\right).$$

By transformation of variables from $u_n, u_{n-1}, \cdots, u_1$ to $y_n, y_{n-1}, \cdots, y_1$, the joint distribution of $y_n, y_{n-1}, \cdots, y_1$ is:

$$f_y(y_n, y_{n-1}, \cdots, y_1; \rho, \sigma_\epsilon^2, \beta) \\ = f_u(y_n - x_n\beta, y_{n-1} - x_{n-1}\beta, \cdots, y_1 - x_1\beta; \rho, \sigma_\epsilon^2)\left|\frac{\partial u}{\partial y'}\right|$$

183

$$= (2\pi\sigma_\epsilon^2/(1-\rho^2))^{-1/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2/(1-\rho^2)}(y_1 - x_1\beta)^2\right) \\ \times(2\pi\sigma_\epsilon^2)^{-(n-1)/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=2}^{n}\big((y_t - \rho y_{t-1}) - (x_t - \rho x_{t-1})\beta\big)^2\right) \\ = (2\pi\sigma_\epsilon^2)^{-1/2}(1-\rho^2)^{1/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2}(\sqrt{1-\rho^2}y_1 - \sqrt{1-\rho^2}x_1\beta)^2\right) \\ \times(2\pi\sigma_\epsilon^2)^{-(n-1)/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=2}^{n}\big((y_t - \rho y_{t-1}) - (x_t - \rho x_{t-1})\beta\big)^2\right) \\ = (2\pi\sigma_\epsilon^2)^{-n/2}(1-\rho^2)^{1/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2}(y_1^* - x_1^*\beta)^2\right)\times\exp\left(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=2}^{n}(y_t^* - x_t^*\beta)^2\right)$$

184

$$= (2\pi)^{-n/2}(\sigma_\epsilon^2)^{-n/2}(1-\rho^2)^{1/2}\exp\left(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=1}^{n}(y_t^* - x_t^*\beta)^2\right) \\ = L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1),$$

where y_t^* and x_t^* are given by:

$$y_t^* = \begin{cases} \sqrt{1-\rho^2}y_t, & \text{for } t = 1, \\ y_t - \rho y_{t-1}, & \text{for } t = 2, 3, \cdots, n, \end{cases}$$

$$x_t^* = \begin{cases} \sqrt{1-\rho^2}x_t, & \text{for } t = 1, \\ x_t - \rho x_{t-1}, & \text{for } t = 2, 3, \cdots, n, \end{cases}$$

185

◎ For maximization, the first derivative of $L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1)$ with respect to β should be zero.

$$\tilde\beta = (\sum_{t=1}^{T} x_t^{*\prime}x_t^*)^{-1}(\sum_{t=1}^{T} x_t^{*\prime}y_t^*) \\ = (X^{*\prime}X^*)^{-1}X^{*\prime}y^*$$

$\Longrightarrow$ This is equivalent to OLS from the regression model: $y^* = X^*\beta + \epsilon$ and $\epsilon \sim N(0, \sigma^2 I_n)$, where $\sigma^2 = \sigma_\epsilon^2/(1-\rho^2)$.

186

◎ For maximization, the first derivative of $L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1)$ with respect to σ_ϵ^2 should be zero.

$$\tilde{\sigma}_\epsilon^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t^* - x_t^*\beta)^2 = \frac{1}{n}(y^* - X^*\beta)'(y^* - X^*\beta),$$

where

$$y^* = \begin{pmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_n - \rho y_{n-1} \end{pmatrix}, \qquad X^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}x_1 \\ x_2 - \rho x_1 \\ \vdots \\ x_n - \rho x_{n-1} \end{pmatrix}.$$

◎ For maximization, the first derivative of $L(\rho, \sigma_\epsilon^2, \beta; y_n, y_{n-1}, \cdots, y_1)$ with respect to ρ should be zero.

$\max_{\beta,\sigma_\epsilon^2,\rho} L(\rho, \sigma_\epsilon^2, \beta; y)$ is equivalent to $\max_{\rho} L(\rho, \tilde{\sigma}_\epsilon^2, \tilde{\beta}; y)$.

$L(\rho, \tilde{\sigma}_\epsilon^2, \tilde{\beta}; y)$ is called the **concentrated log-likelihood function (集約対数尤度関数)**, which is a function of ρ, i.e., both $\tilde{\sigma}_\epsilon^2$ and $\tilde{\beta}$ depend only on ρ.

The log-likelihood function is written as:

$$\begin{aligned}\log L(\rho, \tilde{\sigma}_\epsilon^2, \tilde{\beta}; y) &= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\tilde{\sigma}_\epsilon^2) + \frac{1}{2}\log(1-\rho^2) - \frac{n}{2} \\ &= -\frac{n}{2}\log(2\pi) - \frac{n}{2} - \frac{n}{2}\log\big(\tilde{\sigma}_\epsilon^2(\rho)\big) + \frac{1}{2}\log(1-\rho^2)\end{aligned}$$

For maximization of $\log L$, use Newton-Raphson method, method of scoring or simple grid search

Note that $\tilde{\sigma}_\epsilon^2 = \tilde{\sigma}_\epsilon^2(\rho) = \frac{1}{n}(y^* - X^*\tilde{\beta})'(y^* - X^*\tilde{\beta})$ for $\tilde{\beta} = (X^{*\prime}X^*)^{-1}X^{*\prime}y^*$.

Remark: The regression model with AR(1) error is:

$$y_t = x_t\beta + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2).$$

$$\mathrm{V}(u) = \sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 & & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{n-3} \\ \rho^3 & \rho^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \rho \\ \rho^{n-1} & \rho^{n-2} & \cdots & \rho^2 & \rho & 1 \end{pmatrix} = \sigma^2\Omega, \qquad \text{where } \sigma^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}.$$

where $\mathrm{Cov}(u_i, u_j) = \mathrm{E}(u_i u_j) = \sigma^2\rho^{|i-j|}$, i.e., the ith row and jth column of Ω is $\rho^{|i-j|}$.

The regression model with AR(1) error is: $y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega)$.

There exists P which satisfies that $\Omega = PP'$, because ω is a positive definite matrix.

Multiply P^{-1} on both sides from the left.

$$\begin{aligned} P^{-1}y = P^{-1}X\beta + P^{-1}u \qquad &\Longrightarrow \qquad y^* = X^*\beta + u^* \text{ and } u^* \sim N(0, \sigma^2 I_n) \\ &\Longrightarrow \qquad \text{Apply OLS.}\end{aligned}$$

$$y^* = \begin{pmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_n - \rho y_{n-1} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & \cdots & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = P^{-1}y$$

$$X^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}x_1 \\ x_2 - \rho x_1 \\ \vdots \\ x_n - \rho x_{n-1} \end{pmatrix} = P^{-1}X \qquad \Longrightarrow \qquad \text{Check } P^{-1}\Omega P^{-1\prime} = aI_n, \text{ where } a \text{ is constant.}$$

9.6 MLE: Regression Model with Heteroscedastic Errors

In the case where the error term depends on the other exogenous variables, the regression model is written as follows:

$$y_i = x_i\beta + u_i, \qquad u_i \sim \text{ iid } N(0, \sigma_i^2), \qquad \sigma_i^2 = (z_i\alpha)^2.$$

The joint distribution of $u_n, u_{n-1}, \cdots, u_1$, denoted by $f_u(\cdot;\cdot)$, is given by:

$$\log f_u(u_n, u_{n-1}, \cdots, u_1; \sigma_1^2, \cdots, \sigma_n^2) = \sum_{i=1}^n \log f_u(u_i; \sigma_i^2)$$
$$= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^n \log(\sigma_i^2) - \frac{1}{2}\sum_{i=1}^n \Big(\frac{u_i}{\sigma_i}\Big)^2$$

$$= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^n \log(z_i\alpha)^2 - \frac{1}{2}\sum_{i=1}^n \Big(\frac{u_i}{z_i\alpha}\Big)^2$$

By the transformation of variables from $u_n, u_{n-1}, \cdots, u_1$ to $y_n, y_{n-1}, \cdots, y_1$, the log-likelihood function is:

$$L(\alpha, \beta; y_n, y_{n-1}, \cdots, y_1) = \log f_y(y_n, y_{n-1}, \cdots, y_1; \alpha, \beta)$$
$$= \log f_u(y_n - x_n\beta, y_{n-1} - x_{n-1}\beta, \cdots, y_1 - x_1\beta; \sigma_i^2) \Big|\frac{\partial u}{\partial y}\Big|$$
$$= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^n \log(z_i\alpha)^2 - \frac{1}{2}\sum_{i=1}^n \Big(\frac{y_i - x_i\beta}{z_i\alpha}\Big)^2$$

$\Longrightarrow$ Maximize the above log-likelihood function with respect to β and α.

10 Asymptotic Theory

1. **Definition: Convergence in Distribution (分布収束)**

   A series of random variables X_1, X_2, $\cdots$ have distribution functions F_1, F_2, $\cdots$, respectively.

   If

   $$\lim_{i\to\infty} F_i = F,$$

   then we say that a series of random variables X_1, X_2, $\cdots$ converges to F in distribution.

2. **Consistency (一致性):**

   (a) **Definition: Convergence in Probability (確率収束)**

   Let $\{Z_i :\ i = 1, 2, \cdots\}$ be a series of random variables.

   If the following holds,

   $$\lim_{i\to\infty} P(|Z_i - \theta| < \epsilon) = 1,$$

   for any positive ϵ, then we say that Z_i converges to θ in probability.

   θ is called a **probability limit (確率極限)** of Z_i.

   $$\text{plim } Z_i = \theta.$$

   (b) Let $\hat\theta_i$ be an estimator of parameter θ.

   If $\hat\theta_i$ converges to θ in probability, we say that $\hat\theta_T$ is a consistent estimator of θ.

3. **Chebyshev's inequality:**

   For $g(X) \geq 0$,

   $$P(g(X) \geq k) \leq \frac{\mathrm{E}(g(X))}{k},$$

   where k is a positive constant.

4. **Example:** For a random variable X, set $g(X) = (X-\mu)'(X-\mu)$, $\mathrm{E}(X) = \mu$

   and $\mathrm{Var}(X) = \Sigma$.

   Then, we have the following inequality:

   $$P((X-\mu)'(X-\mu) \geq k) \leq \frac{\mathrm{tr}(\Sigma)}{k}.$$

   Note as follows:

   $$\mathrm{E}((X-\mu)'(X-\mu)) = \mathrm{E}\Big(\mathrm{tr}((X-\mu)'(X-\mu))\Big) = \mathrm{E}\Big(\mathrm{tr}((x-\mu)(x-\mu)')\Big)$$
   $$= \mathrm{tr}\Big(\mathrm{E}((x-\mu)(x-\mu)')\Big) = \mathrm{tr}(\Sigma).$$

5. **Example 1:**

Suppose that $X_i \sim (\mu, \sigma^2)$, $i = 1, 2, \cdots, n$.

Then, the sample average $\overline{X}$ is a consistent estimator of μ.

Proof:

Note that $g(\overline{X}) = (\overline{X} - \mu)^2$, $\epsilon^2 = k$, $\mathrm{E}(g(\overline{X})) = \mathrm{V}(\overline{X}) = \dfrac{\sigma^2}{n}$.

Use Chebyshev's inequality.

If $n \longrightarrow \infty$,

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \longrightarrow 0, \qquad \text{for any } \epsilon.$$

That is. for any ϵ,

$$\lim_{n\to\infty} P(|\overline{X} - \mu| < \epsilon) = 1$$

6. **Example 2 (Multivariate Case):**

Suppose that $X_i \sim (\mu, \Sigma)$, $i = 1, 2, \cdots, n$.

Then, the sample average $\overline{X}$ is a consistent estimator of μ.

Proof:

Note that $g(\overline{X}) = (\overline{X} - \mu)'(\overline{X} - \mu)$, $\epsilon^2 = k$, $\mathrm{E}(g(\overline{X})) = \mathrm{V}(\overline{X}) = \dfrac{1}{n}\Sigma$.

Use Chebyshev's inequality.

If $n \longrightarrow \infty$,

$$P((\overline{X} - \mu)'(\overline{X} - \mu) \geq k) \leq \frac{\mathrm{tr}(\Sigma)}{nk} \longrightarrow 0, \qquad \text{for any positive } k.$$

That is. for any positive k,

$$\lim_{n\to\infty} P((\overline{X} - \mu)'(\overline{X} - \mu) < k) = 1$$

7. **Some Formulas:**

Let X_n and Y_n be the random variables which satisfy $\operatorname{plim} X_n = c$ and $\operatorname{plim} Y_n = d$.

Then,

(a) $\operatorname{plim}\,(X_n + Y_n) = c + d$

(b) $\operatorname{plim}\, X_n Y_n = cd$

(c) $\operatorname{plim}\, X_n/Y_n = c/d$ for $d \neq 0$

(d) $\operatorname{plim}\, g(X_n) = g(c)$ for a function $g(\cdot)$

$\Longrightarrow$ **Slutsky's Theorem (スルツキー定理)**

8. **Central Limit Theorem (中心極限定理)**

$X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \Sigma)$.

Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \ \longrightarrow\ N(0, \Sigma)$$

9. **Central Limit Theorem (Generalization)**

$X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \Sigma_i)$.

Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \ \longrightarrow\ N(0, \Sigma),$$

where

$$\Sigma = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \Sigma_i \right).$$

10. **Definition:** Let $\hat{\theta}_n$ be a consistent estimator of θ.

Suppose that $\sqrt{n}(\hat{\theta}_n - \theta)$ converges to $N(0, \Sigma)$ in distribution.

Then, we say that $\hat{\theta}_n$ has an **asymptotic distribution (漸近分布)**: $N(\theta, \Sigma/n)$.

11. **Definition:** We say that $\hat{\theta}_n$ is consistent uniformly asymptotically normal, when the following three conditions are satisfied:

    (a) $\hat{\theta}_n$ is consistent,

    (b) $\sqrt{n}(\hat{\theta}_n - \theta)$ converges to $N(0, \Sigma)$ in distribution,

    (c) Uniform convergence.

12. **Definition:** Suppose that $\hat{\theta}_n$ and $\tilde{\theta}_n$ are consistent, uniformly, asymptotically normal, and that the asymptotic variances are given by Σ/n and Ω/n.

    If $\Omega - \Sigma$ is positive semidefinite, $\hat{\theta}_n$ is **asymptotically more efficient (漸近的**

に有効) than $\tilde{\theta}_n$.

13. **Definition:** If a consistent, uniformly, asymptotically normal estimator is asymptotically more efficient than any other consistent, uniformly, asymptotically normal estimators, we say that the consistent, uniformly, asymptotically normal estimator is asymptotically efficient (漸近的有効).

14. The sufficient condition for an asymptotically efficient and consistent, uniformly, asymptotically normal estimator is that the asymptotic variance is equivalent to Cramer-Rao lower bound.

15. $X_1, X_2, \cdots, X_n$ are random variables with density function $f(x;\theta)$.

    Let $\hat{\theta}_n$ be a maximum likelihood estimator of θ.

    Then, under some regularity conditions. $\hat{\theta}_n$ is a consistent estimator of θ and the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is given by: $N\left(0, \lim\left(\dfrac{I(\theta)}{n}\right)^{-1}\right)$.

16. Regularity Conditions:

    (a) The domain of X_i does not depend on θ.

    (b) There exists at least third-order derivative of $f(x;\theta)$ with respect to θ, and their derivatives are finite.

17. Thus, MLE is

    (i) consistent,

    (ii) asymptotically normal, and

    (iii) asymptotically efficient.

18. **Slutsky's Theorem**

    Let $\hat{\theta}$ be a consistent estimator of θ.

    Then, $g(\hat{\theta})$ is also a consistent estimator of $g(\theta)$, where $g(\cdot)$ is a well-defined continuous function.

19. **Invariance of Maximum Likelihood Estimation (最尤法の不変性)**

    Let $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k$ be maximum likelihood estimators of $\theta_1, \theta_2, \cdots, \theta_k$.

    Consider the following one-to-one transformation:

    $$\alpha_1 = \alpha_1(\theta_1, \theta_2, \cdots, \theta_k),\ \alpha_2 = \alpha_2(\theta_1, \theta_2, \cdots, \theta_k),\ \cdots,\ \alpha_k = \alpha_k(\theta_1, \theta_2, \cdots, \theta_k)$$

    Then, MLEs of $\alpha_1, \alpha_2, \cdots, \alpha_k$ are given by:

    $$\hat{\alpha}_1 = \alpha_1(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k),\ \hat{\alpha}_2 = \alpha_2(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k),\ \cdots,\ \hat{\alpha}_k = \alpha_k(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k).$$

11 Consistency and Asymptotic Normality of OLSE

Regression model:

$$y = X\beta + u, \qquad u \sim (0, \sigma^2 I_n)$$

Consistency:

1. Let $\hat{\beta}_n = (X'X)^{-1}X'y$ be the OLS with sample size n.

   Consistency: As n is large, $\hat{\beta}_n$ converges to β.

2. Assume the stationarity assumption for X, i.e.,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

Then, we have the following result:

$$\frac{1}{n}X'u \longrightarrow 0.$$

Proof:

According to Chebyshev's inequality, for $g(x) \geq 0$,

$$P(g(X) \geq k) \leq \frac{\mathrm{E}(g(X))}{k},$$

where k is a positive constant.

Set $g(X) = X'X$, and X is replaced by $\frac{1}{n}X'u$.

Apply Chebyshev's inequality.

$$\begin{aligned}\mathrm{E}\Big((\frac{1}{n}X'u)'\frac{1}{T}X'u\Big) &= \frac{1}{n^2}\mathrm{E}\Big(u'XX'u\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(u'XX'u)\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(XX'uu')\Big)\\ &= \frac{1}{n^2}\mathrm{tr}\Big(XX'\mathrm{E}(uu')\Big) = \frac{\sigma^2}{n^2}\mathrm{tr}(XX') = \frac{\sigma^2}{n^2}\mathrm{tr}(X'X) = \frac{\sigma^2}{n}\mathrm{tr}(\frac{1}{n}X'X).\end{aligned}$$

Therefore,

$$P\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u \geq k\Big) \leq \frac{\sigma^2}{nk}\mathrm{tr}(\frac{1}{n}X'X) \longrightarrow 0 \times \mathrm{tr}(M_{xx}) = 0$$

Note that from the assumption,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

Therefore, we have:

$$(\frac{1}{n}X'u)'\frac{1}{n}X'u \longrightarrow 0,$$

which implies:

$$\frac{1}{n}X'u \longrightarrow 0,$$

because $(\frac{1}{n}X'u)'\frac{1}{n}X'u$ indicates a quadratic form.

3. Note that

$$\frac{1}{n}X'X \longrightarrow M_{xx}$$

results in

$$(\frac{1}{n}X'X)^{-1} \longrightarrow M_{xx}^{-1}$$

$\Longrightarrow$ Slutsky's Theorem

(*) **Slutsky's Theorem** $\quad g(\hat{\theta}) \longrightarrow g(\theta)$, when $\hat{\theta} \longrightarrow \theta$.

4. OLS is given by:

$$\hat{\beta}_n = \beta + (X'X)^{-1}X'u$$

$$= \beta + (\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'u).$$

Therefore,

$$\hat{\beta}_n \longrightarrow \beta + M_{xx}^{-1} \times 0 = \beta$$

Thus, OLSE is a consitent estimator.

Asymptotic Normality:

1. Asymptotic Normality of OLSE

$\sqrt{n}(\hat{\beta}_n - \beta) \longrightarrow N(0.\sigma^2 M_{xx}^{-1})$ when $n \longrightarrow \infty$.

2. **Central Limit Theorem:** Greenberg and Webster (1983)

$Z_1, Z_2, \cdots, Z_n$ are mutually indelendently distributed with mean μ and variance Σ_i.

Then, we have the following result:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_i-\mu) \longrightarrow N(0,\Sigma),$$

where

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\Sigma_i\right).$$

The distribution of Z_i is not assumed.

3. Define $Z_i = x_i u_i$. Then, $\Sigma_i = \mathrm{Var}(Z_i) = \sigma^2 x_i' x_i$.

4. Σ is defined as:

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma^2 x_i' x_i\right) = \sigma^2 \lim_{n\to\infty}\left(\frac{1}{n}X'X\right) = \sigma^2 M_{xx},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

5. Applying Central Limit Theorem (Greenberg and Webster (1983), we obtain the following:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}x_i' u_i = \frac{1}{\sqrt{n}}X'u \longrightarrow N(0,\sigma^2 M_{xx}).$$

On the other hand, from $\hat\beta_n = \beta + (X'X)^{-1}X'u$, we can rewrite as:

$$\sqrt{n}(\hat\beta-\beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u.$$

$$\mathrm{Var}\left(\left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u\right) = \mathrm{E}\left(\left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u\left(\left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u\right)'\right)$$

$$\begin{aligned} &= \left(\frac{1}{n}X'X\right)^{-1}\left(\frac{1}{n}X'\mathrm{E}(uu')X\right)\left(\frac{1}{n}X'X\right)^{-1} \\ &= \sigma^2\left(\frac{1}{n}X'X\right)^{-1} \longrightarrow \sigma^2 M_{xx}^{-1}. \end{aligned}$$

Therefore,

$$\sqrt{n}(\hat\beta-\beta) \longrightarrow N(0,\sigma^2 M_{xx}^{-1})$$

$\Longrightarrow$ Asymptotic normality (漸近的正規性) of OLSE

The distribution of u_i is not assumed.

12 Instrumental Variable (操作変数法)

12.1 Measurement Error (測定誤差)

Errors in Variables

1. True regression model:

$$y = \tilde{X}\beta + u$$

2. Observed variable:

$$X = \tilde{X} + V$$

V: is called the **measurement error (測定誤差 or 観測誤差)**.

3. For the elements which do not include measurement errors in X, the corresponding elements in V are zeros.

4. Regression using observed variable:

$$y = X\beta + (u - V\beta)$$

OLS of β is:

$$\hat\beta = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'(u - V\beta)$$

5. Assumptions:

   (a) The measurement error in X is uncorrelated with $\tilde{X}$ in the limit. i.e.,

   $$\text{plim}\Big(\frac{1}{n}\tilde{X}'V\Big) = 0.$$

   Therefore, we obtain the following:

   $$\text{plim}\Big(\frac{1}{n}X'X\Big) = \text{plim}\Big(\frac{1}{n}\tilde{X}'\tilde{X}\Big) + \text{plim}\Big(\frac{1}{n}V'V\Big) = \Sigma + \Omega$$

   (b) u is not correlated with V.

   u is not correlated with $\tilde{X}$.

That is,

$$\text{plim}\Big(\frac{1}{n}V'u\Big) = 0, \qquad \text{plim}\Big(\frac{1}{n}\tilde{X}'u\Big) = 0.$$

6. OLSE of β is:

   $$\hat{\beta} = \beta + (X'X)^{-1}X'(u - V\beta) = \beta + (X'X)^{-1}(\tilde{X} + V)'(u - V\beta).$$

   Therefore, we obtain the following:

   $$\text{plim}\,\hat{\beta} = \beta - (\Sigma + \Omega)^{-1}\Omega\beta$$

7. **Example: The Case of Two Variables:**

   The regression model is given by:

   $$y_t = \alpha + \beta\tilde{x}_t + u_t, \qquad x_t = \tilde{x}_t + v_t.$$

   Under the above model,

   $$\Sigma = \text{plim}\Big(\frac{1}{n}\tilde{X}'\tilde{X}\Big) = \text{plim}\begin{pmatrix} 1 & \frac{1}{n}\sum\tilde{x}_i \\ \frac{1}{n}\sum\tilde{x}_i & \frac{1}{n}\sum\tilde{x}_i^2 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix},$$

   where μ and σ^2 represent the mean and variance of $\tilde{x}_i$.

   $$\Omega = \text{plim}\Big(\frac{1}{n}V'V\Big) = \text{plim}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{n}\sum v_i^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}\text{plim}\begin{pmatrix}\hat{\alpha} \\ \hat{\beta}\end{pmatrix} &= \begin{pmatrix}\alpha \\ \beta\end{pmatrix} - \left(\begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + \sigma^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \\ &= \begin{pmatrix}\alpha \\ \beta\end{pmatrix} - \frac{1}{\sigma^2 + \sigma_v^2}\begin{pmatrix} -\mu\sigma_v^2\beta \\ \sigma_v^2\beta \end{pmatrix}\end{aligned}$$

Now we focus on β.

$\hat{\beta}$ is not consistent. because of:

$$\text{plim}(\hat{\beta}) = \beta - \frac{\sigma_v^2\beta}{\sigma^2 + \sigma_v^2} = \frac{\beta}{1 + \sigma_v^2/\sigma^2} < \beta$$

12.2 Instrumental Variable (IV) Method (操作変数法 or IV 法)

Instrumental Variable (IV)

1. Consider the regression model: $y = X\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

   In the case of $\mathrm{E}(X'u) \neq 0$, OLSE of β is inconsistent.

2. **Proof:**

   $$\hat{\beta} = \beta + (\frac{1}{n}X'X)^{-1}\frac{1}{n}X'u \longrightarrow \beta + M_{xx}^{-1}M_{xu},$$

where

$$\frac{1}{n}X'X \longrightarrow M_{xx}, \qquad \frac{1}{n}X'u \longrightarrow M_{xu} \neq 0$$

3. Find the Z which satisfies $\frac{1}{n}Z'u \longrightarrow M_{zu} = 0$.

   Multiplying Z' on both sides of the regression model: $y = X\beta + u$,

   $$Z'y = Z'X\beta + Z'u$$

   Dividing n on both sides of the above equation, we take plim on both sides.

Then, we obtain the following:

$$\text{plim}\left(\frac{1}{n}Z'y\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta + \text{plim}\left(\frac{1}{n}Z'u\right) = \text{plim}\left(\frac{1}{n}Z'X\right)\beta.$$

Accordingly, we obtain:

$$\beta = \left(\text{plim}\left(\frac{1}{n}Z'X\right)\right)^{-1} \text{plim}\left(\frac{1}{n}Z'y\right).$$

Therefore, we consider the following estimator:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

which is taken as an estimator of β.

$\Longrightarrow$ **Instrumental Variable Method (操作変数法 or IV 法)**

4. Assume the followings:

$$\frac{1}{n}Z'X \longrightarrow M_{zx}, \qquad \frac{1}{n}Z'Z \longrightarrow M_{zz}, \qquad \frac{1}{n}Z'u \longrightarrow 0$$

5. **Distribution of β_{IV}:**

$$\beta_{IV} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u,$$

which is rewritten as:

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big)$$

Applying the Central Limit Theorem to $\Big(\frac{1}{\sqrt{n}}Z'u\Big)$, we have the following result:

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz}).$$

Therefore,

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\frac{1}{n}Z'X\Big)^{-1}\Big(\frac{1}{\sqrt{n}}Z'u\Big) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}'^{-1})$$

$\Longrightarrow$ Consistency and Asymptotic Normality

6. The variance of β_{IV} is given by:

$$\mathrm{V}(\beta_{IV}) = s^2 (Z'X)^{-1}Z'Z(X'Z)^{-1},$$

where

$$s^2 = \frac{(y - X\beta_{IV})'(y - X\beta_{IV})}{n - k}.$$

12.3 Two-Stage Least Squares Method (2 段階最小二乗法, 2SLS or TSLS)

1. Regression Model:

   $$y = X\beta + u, \quad u \sim N(0, \sigma^2 I),$$

   In the case of $\mathrm{E}(X'u) \neq 0$, OLSE is not consistent.

2. Find the variable Z which satisfies $\frac{1}{n}Z'u \longrightarrow M_{zu} = 0$.

3. Use $Z = \hat{X}$ for the instrumental variable.

$\hat{X}$ is the predicted value which regresses X on the other exogenous variables, say W.

That is, consider the following regression model:

$$X = WB + V.$$

Estimate B by OLS.

Then, we obtain the prediction:

$$\hat{X} = W\hat{B},$$

where $\hat{B} = (W'W)^{-1}W'X$.

Or, equivalently,

$$\hat{X} = W(W'W)^{-1}W'X.$$

$\hat{X}$ is used for the instrumental variable of X.

4. The IV method is rewritten as:

$$\beta_{IV} = (\hat{X}'X)^{-1}\hat{X}'y = (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'y.$$

Furthermore, β_{IV} is written as follows:

$$\beta_{IV} = \beta + (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'u.$$

Therefore, we obtain the following expression:

$$\sqrt{n}(\beta_{IV} - \beta) = \Big(\big(\frac{1}{n}X'W\big)\big(\frac{1}{n}W'W\big)^{-1}\big(\frac{1}{n}XW'\big)'\Big)^{-1}\big(\frac{1}{n}X'W\big)\big(\frac{1}{n}W'W\big)^{-1}\big(\frac{1}{\sqrt{n}}W'u\big)$$
$$\longrightarrow N\Big(0, (M_{xz}M_{zz}^{-1}M_{xz}')^{-1}\Big).$$

5. Clearly, there is no correlation between W and u at least in the limit, i.e.,

$$\text{plim}\Big(\frac{1}{n}W'u\Big) = 0.$$

6. **Remark:**

$$\hat{X}'X = X'W(W'W)^{-1}W'X = X'W(W'W)^{-1}W'W(W'W)^{-1}W'X = \hat{X}'\hat{X}.$$

Therefore,

$$\beta_{IV} = (\hat{X}'X)^{-1}\hat{X}'y = (\hat{X}'\hat{X})^{-1}\hat{X}'y,$$

which implies the OLS estimator of β in the regression model: $y = \hat{X}\beta + u$ and $u \sim N(0, \sigma^2 I_n)$.

13 Large Sample Tests

13.1 Wald, LM and LR Tests

$\theta : K \times 1$

$h(\theta) : G \times 1$ vector function, $G \leq K$

$\theta : K \times 1$

The null hypothesis $H_0 : h(\theta) = 0 \implies G$ restrictions

$\tilde{\theta} : k \times 1$, restricted maximum likelihood estimate

$\hat{\theta} : k \times 1$, unrestricted maximum likelihood estimate

$I(\theta) : k \times k$, information matrix, i.e.,

$$I(\theta) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta)}{\partial\theta\partial\theta'}\Big).$$

$\log L(\theta)$: log-likelihood function

$R_\theta = \dfrac{\partial h(\theta)}{\partial\theta'} : G \times k$

$F_\theta = \dfrac{\partial \log L(\theta)}{\partial\theta} : k \times 1$

1. **Wald Test (ワルド検定):** $W = h(\hat{\theta})'\Big(R_{\hat{\theta}}(I(\hat{\theta}))^{-1}R_{\hat{\theta}}'\Big)^{-1}h(\hat{\theta})$

   (a) $h(\theta) \approx h(\hat{\theta}) + \dfrac{\partial h(\hat{\theta})}{\partial\theta'}(\theta - \hat{\theta}) \quad \Longleftarrow \quad h(\theta)$ is linearized around $\theta = \hat{\theta}$.

Under the null hypothesis $h(\theta) = 0$,

$$h(\hat{\theta}) \approx \frac{\partial h(\hat{\theta})}{\partial\theta'}(\hat{\theta} - \theta) = R_{\hat{\theta}}(\hat{\theta} - \theta)$$

(b) $\hat{\theta}$ is MLE.

From the properties of MLE,

$$\sqrt{n}(\hat{\theta} - \theta) \longrightarrow N\Big(0, \lim_{n\to\infty}\Big(\frac{I(\theta)}{n}\Big)^{-1}\Big),$$

That is, approximately, we have the following result:

$$(\hat{\theta} - \theta) \sim N\Big(0, (I(\theta))^{-1}\Big).$$

(c) The distribution of $h(\hat{\theta})$ is approximately given by:

$$h(\hat{\theta}) \sim N\Big(0, R_{\hat{\theta}}(I(\theta))^{-1}R'_{\hat{\theta}}\Big)$$

(d) Therefore, the $\chi^2(G)$ distribution is derived as follows:

$$h(\hat{\theta})\Big(R_{\hat{\theta}}(I(\theta))^{-1}R'_{\hat{\theta}}\Big)^{-1}h(\hat{\theta})' \longrightarrow \chi^2(G).$$

Furthermore, from the fact that $I(\hat{\theta}) \longrightarrow I(\theta)$ as $n \longrightarrow \infty$ (i.e., convergence in probability, 確率収束), we can replace θ by $\hat{\theta}$ as follows:

$$h(\hat{\theta})\Big(R_{\hat{\theta}}(I(\hat{\theta}))^{-1}R'_{\hat{\theta}}\Big)^{-1}h(\hat{\theta})' \longrightarrow \chi^2(G).$$

2. **Lagrange Multiplier Test (ラグランジェ乗数検定):** $LM = F'_{\tilde{\theta}}(I(\tilde{\theta}))^{-1}F_{\tilde{\theta}}$

(a) MLE with the constraint $h(\theta) = 0$:

$$\max_{\theta} \ \log L(\theta), \quad \text{subject to} \quad h(\theta) = 0$$

The Lagrangian function:

$$L = \log L(\theta) + \lambda h(\theta)$$

(b) For maximization, we have the following two equations:

$$\frac{\partial L}{\partial \theta} = \frac{\partial \log L(\theta)}{\partial \theta} + \lambda \frac{\partial h(\theta)}{\partial \theta} = 0$$
$$\frac{\partial L}{\partial \lambda} = h(\theta) = 0$$

(c) Mean and variance of $\dfrac{\partial \log L(\theta)}{\partial \theta}$ are given by:

$$\mathrm{E}\Big(\frac{\partial \log L(\theta)}{\partial \theta}\Big) = 0, \qquad \mathrm{V}\Big(\frac{\partial \log L(\theta)}{\partial \theta}\Big) = -\mathrm{E}\Big(\frac{\partial^2 \log L(\theta)}{\partial \theta \partial \theta'}\Big) = I(\theta).$$

(d) Therefore, using the central limit theorem,

$$\frac{1}{\sqrt{n}}\frac{\partial \log L(\theta)}{\partial \theta} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial \log f(X_i;\theta)}{\partial \theta} \longrightarrow N\Big(0, \lim_{n\to\infty}\Big(\frac{1}{n}I(\theta)\Big)\Big)$$

(e) Therefore,

$$\frac{\partial \log L(\theta)}{\partial \theta}(I(\theta))^{-1}\frac{\partial \log L(\theta)}{\partial \theta'} \longrightarrow \chi^2(G)$$

Because MLE is consistent, i.e., $\tilde{\theta} \longrightarrow \theta$, we have the result:

$$F'_{\tilde{\theta}}(I(\tilde{\theta}))^{-1}F_{\tilde{\theta}} \longrightarrow \chi^2(G).$$

3. **Likelihood Ratio Test (尤度比検定):** $LR = -2\log\lambda \longrightarrow \chi^2(G)$

$$\lambda = \frac{L(\tilde{\theta})}{L(\hat{\theta})}$$

(a) By Taylor series expansion evaluated at $\theta = \hat{\theta}$, $\log L(\theta)$ is given by:

$$\begin{aligned}\log L(\theta) &= \log L(\hat{\theta}) + \frac{\partial \log L(\hat{\theta})}{\partial \theta}(\theta - \hat{\theta}) \\ &\quad + \frac{1}{2}(\theta-\hat{\theta})'\frac{\partial^2 \log L(\hat{\theta})}{\partial \theta \partial \theta'}(\theta - \hat{\theta}) + \cdots \\ &= \log L(\hat{\theta}) + \frac{1}{2}(\theta-\hat{\theta})'\frac{\partial^2 \log L(\hat{\theta})}{\partial \theta \partial \theta'}(\theta - \hat{\theta}) + \cdots\end{aligned}$$

Note that $\dfrac{\partial \log L(\hat{\theta})}{\partial \theta} = 0$ because $\hat{\theta}$ is MLE.

$$\begin{aligned}-2(\log L(\theta) - \log L(\hat{\theta})) &\approx -(\theta-\hat{\theta})'\Big(\frac{\partial^2 \log L(\hat{\theta})}{\partial \theta \partial \theta'}\Big)(\theta-\hat{\theta}) \\ &= \sqrt{n}(\hat{\theta}-\theta)'\Big(-\frac{1}{n}\frac{\partial^2 \log L(\hat{\theta})}{\partial \theta \partial \theta'}\Big)\sqrt{n}(\hat{\theta}-\theta) \\ &\longrightarrow \chi^2(G)\end{aligned}$$

Note:

(1) $\hat{\theta} \longrightarrow \theta$,

(2) $-\dfrac{1}{n}\dfrac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'} \longrightarrow -\lim_{n\to\infty}\Big(\dfrac{1}{n}\mathrm{E}\Big(\dfrac{\partial^2 \log L(\hat{\theta})}{\partial\theta\partial\theta'}\Big)\Big) = \lim_{n\to\infty}\Big(\dfrac{1}{n}I(\theta)\Big)$,

(3) $\sqrt{n}(\hat{\theta}-\theta) \longrightarrow N\Big(0, \lim_{n\to\infty}\Big(\dfrac{1}{n}I(\theta)\Big)\Big)$.

(b) Under H_0 : $h(\theta) = 0$,

$$-2(\log L(\tilde{\theta}) - \log L(\hat{\theta})) \longrightarrow \chi^2(G).$$

Remember that $h(\tilde{\theta}) = 0$ is always satisfied.

For proof, see Theil (1971, p.396).

4. All of W, LM and LR are asymptotically distributed as $\chi^2(G)$ random variables under the null hypothesis $H_0 : h(\theta) = 0$.

5. Under some comditions, we have $W \geq LR \geq LM$. See Engle (1981) "Wald, Likelihood and Lagrange Multiplier Tests in Econometrics," Chap. 13 in *Handbook of Econometrics*, Vol.2, Grilliches and Intriligator eds, North-Holland.

13.2 Example: W, LM and LR Tests

Date file $\Longrightarrow$ cons99.txt (same data as before)

Each column denotes year, nominal household expenditures (家計消費，10 billion yen), household disposable income (家計可処分所得，10 billion yen) and household expenditure deflator (家計消費デフレータ，1990=100) from the left.

1955	5430.1	6135.0	18.1	1970	37784.1	45913.2	35.2	1985	185335.1	220655.6	93.9
1956	5974.2	6828.4	18.3	1971	42571.6	51944.3	37.5	1986	193069.6	229938.8	94.8
1957	6686.3	7619.5	19.0	1972	49124.1	60245.4	39.7	1987	202072.8	235924.0	95.3
1958	7169.7	8153.3	19.1	1973	59366.1	74924.8	44.1	1988	212939.9	247159.7	95.8
1959	8019.3	9274.3	19.7	1974	71782.1	93833.2	53.3	1989	227122.2	263940.5	97.7
1960	9234.9	10776.5	20.5	1975	83591.1	108712.8	59.4	1990	243035.7	280133.0	100.0
1961	10836.2	12869.4	21.8	1976	94443.7	123540.9	65.2	1991	255531.8	297512.9	102.5
1962	12430.8	14701.4	23.2	1977	105397.8	135318.4	70.1	1992	265701.6	309256.6	104.5
1963	14506.6	17042.7	24.9	1978	115960.3	147244.2	73.5	1993	272075.3	317021.6	105.9
1964	16674.9	19709.9	26.0	1979	127600.9	157071.1	76.0	1994	279538.7	325655.7	106.7
1965	18820.5	22337.4	27.8	1980	138585.0	169931.5	81.6	1995	283245.4	331967.5	106.2
1966	21680.6	25514.5	29.0	1981	147103.4	181349.2	85.4	1996	291458.5	340619.1	106.0
1967	24914.0	29012.6	30.1	1982	157994.0	190611.5	87.7	1997	298475.2	345522.7	107.3
1968	28452.7	34233.6	31.6	1983	166631.6	199587.8	89.5				
1969	32705.2	39486.3	32.9	1984	175383.4	209451.9	91.8				


```
          PROGRAM
LINE  *************************************************
|       freq a;
|       smpl 1955 1997;
|       read(file='cons99.txt') year cons yd price;
|       rcons=cons/(price/100);
|       ryd=yd/(price/100);
|       lyd=log(ryd);
|       olsq rcons c ryd;
|       olsq @res @res(-1);
|       ar1  rcons c ryd;
|       olsq rcons c lyd;
|       param a1 0 a2 0 a3 1;
|       frml eq rcons=a1+a2*((ryd**a3)-1.)/a3;
|       lsq(tol=0.00001,maxit=100) eq;
|       a3=1.15;
|       rryd=((ryd**a3)-1.)/a3;
|       ar1  rcons c rryd;
|       end;
*******************************************************
```



```
                          Equation   1
                          ============
             Method of estimation = Ordinary Least Squares

Dependent variable: RCONS
Current sample:  1955 to 1997
Number of observations:  43

        Mean of dep. var. = 146270.          LM het. test = .207443 [.649]
   Std. dev. of dep. var. = 79317.2         Durbin-Watson = .115101 [.000,.000]
Sum of squared residuals = .129697E+10  Jarque-Bera test = 9.47539 [.009]
    Variance of residuals = .316335E+08   Ramsey's RESET2 = 53.6424 [.000]
Std. error of regression = 5624.36       F (zero slopes) = 8311.90 [.000]
                R-squared = .995092        Schwarz B.I.C. = 435.051
       Adjusted R-squared = .994972        Log likelihood = -431.289

           Estimated    Standard
Variable  Coefficient     Error       t-statistic   P-value
C         -2919.54      1847.55       -1.58022      [.122]
RYD       .852879       .935486E-02   91.1696       [.000]
```

Equation 2
============
Method of estimation = Ordinary Least Squares

Dependent variable: @RES
Current sample: 1956 to 1997
Number of observations: 42

Mean of dep. var. = -95.5174
Std. dev. of dep. var. = 5588.52
Sum of squared residuals = .146231E+09
Variance of residuals = .356662E+07
Std. error of regression = 1888.55
R-squared = .885884
Adjusted R-squared = .885884
LM het. test = .760256 [.383]
Durbin-Watson = 1.40409 [.023,.023]
Durbin's h = 1.97732 [.048]
Durbin's h alt. = 1.91077 [.056]
Jarque-Bera test = 6.49360 [.039]

Ramsey's RESET2 = .186107 [.668]
Schwarz B.I.C. = 377.788
Log likelihood = -375.919

Variable	Estimated Coefficient	Standard Error	t-statistic	P-value
@RES(-1)	.950693	.053301	17.8362	[.000]

Equation 3
============
FIRST-ORDER SERIAL CORRELATION OF THE ERROR
Objective function: Exact ML (keep first obs.)

Dependent variable: RCONS
Current sample: 1955 to 1997
Number of observations: 43

Mean of dep. var. = 146270.	R-squared = .999480
Std. dev. of dep. var. = 79317.2	Adjusted R-squared = .999454
Sum of squared residuals = .145826E+09	Durbin-Watson = 1.38714
Variance of residuals = .364564E+07	Schwarz B.I.C. = 391.061
Std. error of regression = 1909.36	Log likelihood = -385.419

Parameter	Estimate	Standard Error	t-statistic	P-value
C	1672.42	6587.40	.253881	[.800]
RYD	.840011	.027182	30.9032	[.000]
RHO	.945025	.045843	20.6143	[.000]

Equation 4
============
Method of estimation = Ordinary Least Squares

Dependent variable: RCONS
Current sample: 1955 to 1997
Number of observations: 43

Mean of dep. var. = 146270.	LM het. test = 2.21031 [.137]
Std. dev. of dep. var. = 79317.2	Durbin-Watson = .029725 [.000,.000]
Sum of squared residuals = .256040E+11	Jarque-Bera test = 3.72023 [.156]
Variance of residuals = .624487E+09	Ramsey's RESET2 = 344.855 [.000]
Std. error of regression = 24989.7	F (zero slopes) = 382.117 [.000]
R-squared = .903100	Schwarz B.I.C. = 499.179
Adjusted R-squared = .900737	Log likelihood = -495.418

Variable	Estimated Coefficient	Standard Error	t-statistic	P-value
C	-.115228E+07	66538.5	-17.3175	[.000]
LYD	109305.	5591.69	19.5478	[.000]

NONLINEAR LEAST SQUARES
======================
CONVERGENCE ACHIEVED AFTER 84 ITERATIONS

Number of observations = 43 Log likelihood = -414.362
Schwarz B.I.C. = 420.004

Parameter	Estimate	Standard Error	t-statistic	P-value
A1	16544.5	2615.60	6.32530	[.000]
A2	.063304	.024133	2.62307	[.009]
A3	1.21694	.031705	38.3839	[.000]

Standard Errors computed from quadratic form of analytic first derivatives (Gauss)

Equation: EQ
Dependent variable: RCONS

Mean of dep. var. = 146270.

Std. dev. of dep. var. = 79317.2
Sum of squared residuals = .590213E+09
Variance of residuals = .147553E+08
Std. error of regression = 3841.27
R-squared = .997766
Adjusted R-squared = .997655
LM het. test = .174943 [.676]
Durbin-Watson = .253234 [.000,.000]

```
                              Equation   5
                              ============
              FIRST-ORDER SERIAL CORRELATION OF THE ERROR
              Objective function:  Exact ML (keep first obs.)

Dependent variable: RCONS
Current sample:  1955 to 1997
Number of observations:  43

       Mean of dep. var. = 146270.             R-squared = .999470
  Std. dev. of dep. var. = 79317.2    Adjusted R-squared = .999443
Sum of squared residuals = .140391E+09     Durbin-Watson = 1.43657
   Variance of residuals = .350977E+07    Schwarz B.I.C. = 389.449
Std. error of regression = 1873.44        Log likelihood = -383.807

                          Standard
Parameter  Estimate        Error        t-statistic   P-value
C          12034.8        3346.47       3.59628       [.000]
RRYD       .140723        .282614E-02   49.7933       [.000]
RHO        .876924        .068199       12.8583       [.000]
```


1. `Equation 1` vs. `Equation 3` (Test of Serial Correlation)

`Equation 1` is:

$$\texttt{RCONS}_t = \beta_1 + \beta_2 \texttt{RYD}_t + u_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

`Equation 3` is:

$$\texttt{RCONS}_t = \beta_1 + \beta_2 \texttt{RYD}_t + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

The null hypothesis is $H_0: \ \rho = 0$

Restricted MLE $\Longrightarrow$ `Equation 1`

Unrestricted MLE $\Longrightarrow$ `Equation 3`

The log-likelihood function of `Equation 3` is:

$$\begin{aligned}\log L(\beta, \sigma_\epsilon^2, \rho) &= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_\epsilon^2) + \frac{1}{2}\log(1-\rho^2) \\ &\quad - \frac{1}{2\sigma_\epsilon^2}\sum_{t=1}^{n}(\texttt{RCONS}_t^* - \beta_1 \texttt{CONST}_t^* - \beta_2 \texttt{RYD}_t^*)^2,\end{aligned}$$

where

$$\texttt{RCONS}_t^* = \begin{cases} \sqrt{1-\rho^2}\,\texttt{RCONS}_t, & \text{for } t = 1, \\ \texttt{RCONS}_t - \rho \texttt{RCONS}_{t-1}, & \text{for } t = 2, 3, \cdots, n, \end{cases}$$

$$\texttt{CONST}_t^* = \begin{cases} \sqrt{1-\rho^2}, & \text{for } t = 1, \\ 1-\rho, & \text{for } t = 2, 3, \cdots, n, \end{cases}$$

$$\texttt{RYD}_t^* = \begin{cases} \sqrt{1-\rho^2}\,\texttt{RYD}_t, & \text{for } t = 1, \\ \texttt{RYD}_t - \rho \texttt{RYD}_{t-1}, & \text{for } t = 2, 3, \cdots, n. \end{cases}$$

- MLE with the restriction $\rho = 0$ (`Equation 1`) solves:

  $$\max_{\beta, \sigma_\epsilon^2} \log L(\beta, \sigma_\epsilon^2, 0)$$

  Restricted MLE $\Longrightarrow \tilde{\beta}, \tilde{\sigma}_\epsilon^2$

  `Log of likelihood function = -431.289`

- MLE without the restriction $\rho = 0$ (`Equation 3`) solves:

  $$\max_{\beta, \sigma_\epsilon^2, \rho} \log L(\beta, \sigma_\epsilon^2, \rho)$$

  Unrestricted MLE $\Longrightarrow \hat{\beta}, \hat{\sigma}_\epsilon^2, \hat{\rho}$

  `Log of likelihood function = -385.419`

The likelihood ratio test statistic is:

$$\begin{aligned}-2\log(\lambda) &= -2\log\Big(\frac{L(\tilde{\beta}, \tilde{\sigma}_\epsilon^2, 0)}{L(\hat{\beta}, \hat{\sigma}_\epsilon^2, \hat{\rho})}\Big) = -2\Big(\log L(\tilde{\beta}, \tilde{\sigma}_\epsilon^2, 0) - \log L(\hat{\beta}, \hat{\sigma}_\epsilon^2, \hat{\rho})\Big) \\ &= -2\Big(-431.289 - (-385.419)\Big) = 91.74.\end{aligned}$$

The asymptotic distribution is given by:

$$-2\log(\lambda) \sim \chi^2(G),$$

where G is the number of the restrictions, i.e., $G = 1$ in this case.

The 1% upper probability point of $\chi^2(1)$ is 6.635.

$$91.74 > 6.635$$

Therefore, $H_0: \ \rho = 0$ is rejected.

There is serial correlation in the error term.

2. `Equation 1` (Test of Serial Correlation $\longrightarrow$ Lagrange Multiplier Test)

`Equation 2` is:

$$\texttt{@RES}_t = \rho\texttt{@RES}_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma_\epsilon^2),$$

where $\texttt{@RES}_t = \texttt{RCONS}_t - \hat{\beta}_1 - \hat{\beta}_2\texttt{RYD}_t$, and $\hat{\beta}_1$ and $\hat{\beta}_2$ are OLSEs.

The null hypothesis is $H_0: \ \rho = 0$

```
@RES(-1)  .950693       .053301       17.8362       [.000]
```

Therefore, the Wald test statistic is $17.8362^2 = 318.13 > 6.635$.

$H_0: \ \rho = 0$ is rejected.

3. `Equation 3` (Test of Serial Correlation $\longrightarrow$ Wald Test)

`Equation 3` is:

$$\texttt{RCONS}_t = \beta_1 + \beta_2\texttt{RYD}_t + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

The null hypothesis is $H_0: \ \rho = 0$

```
RHO        .945025       .045843       20.6143       [.000]
```

The Wald teststatistics is $20.6143^2 = 424.95$, which is compared with $\chi^2(1)$.

$535.48 > 6.635 \implies H_0: \ \rho = 0$ is rejected by Wald test.

4. `Equation 1` vs. `NONLINEAR LEAST SQUARES` (Choice of Functional Form – linear):

`NONLINEAR LEAST SQUARES` estimates:

$$\texttt{RCONS}_t = a1 + a2\frac{\texttt{RYD}_t^{a3} - 1}{a3} + u_t.$$

When $a3 = 1$, we have:

$$\texttt{RCONS}_t = (a1 - a2) + a2\texttt{RYD}_t + u_t,$$

which is equivalent to `Equation 1`.

The null hypothesis is $H_0: \ a3 = 1$, where $G = 1$.

- MLE with $a3 = 1$ MLE (`Equation 1`)

  `Log of likelihood function = -431.289`

- MLE without $a3 = 1$ (`NONLINEAR LEAST SQUARES`)

  `Log of likelihood function = -414.362`

The likelihood ratio test statistic is given by:

$$-2\log(\lambda) = -2\Big(-431.289 - (-414.362)\Big) = 33.854.$$

The 1% upper probability point of $\chi^2(1)$ is 6.635.

$$33.854 > 6.635$$

$H_0: \ a3 = 1$ is rejected.

Therefore, the functional form of the regression model is not linear.

5. `Equation 4` vs. `NONLINEAR LEAST SQUARES` (Choice of Functional Form – log-linear):

In `NONLINEAR LEAST SQUARES`, i.e.,

$$\texttt{RCONS}_t = a1 + a2\frac{\texttt{RYD}_t^{a3} - 1}{a3} + u_t,$$

if $a3 = 0$, we have:

$$\texttt{RCONS}_t = a1 + a2\log(\texttt{RYD}_t) + u_t,$$

which is equivalent to `Equation 3`.

The null hypothesis is $H_0: \ a3 = 0$, where $G = 1$.

- MLE with $a3 = 0$ (`Equation 3`)

  `Log of likelihood function = -495.418`

- MLE without $a3 = 0$ (`NONLINEAR LEAST SQUARES`)

  `Log of likelihood function = -414.362`

The likelihood ratio test statistic is:

$$-2\log(\lambda) = -2\Big(-495.418 - (-414.362)\Big) = 162.112 > 6.635.$$

Therefore, $H_0:\ a3 = 0$ is rejected.

As a result, the functional form of the regression model is not log-linear, either.

6. `Equation 1` vs. `Equation 5` (Simultaneous Test of Serial Correlation and Linear Function):

`Equation 5` is:

$$\texttt{RCONS}_t = a1 + a2\frac{\texttt{RYD}_t^{a3} - 1}{a3} + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

The null hypothesis is $H_0:\ a3 = 1,\ \rho = 0$

Restricted MLE $\Longrightarrow$ `Equation 1`

Unrestricted MLE $\Longrightarrow$ `Equation 4`

Remark: In Lines 14–16 of `PROGRAM`, we have estimated `Equation 4`, given $a3 = 0.00, 0.01, 0.02, \cdots$.

As a result, $a3 = 1.15$ gives us the maximum log-likelihood.

The likelihood ratio test statistic is:

$$-2\log(\lambda) = -2\Big(-431.289 - (-383.807)\Big) = 94.964.$$

$-2\log(\lambda) \sim \chi^2(2)$ in this case.

The 1% upper probability point of $\chi^2(2)$ is 9.210.

$$94.964 > 9.210$$

$H_0:\ a3 = 1, \rho = 0$ is rejected.

Thus, even if serial correlation is taken into account, the regression model is not linear.

14 その他のトピック

1. Time Series Analysis (時系列分析)
   $\longrightarrow$ Econometrics III (Spring Semester, 2013)
2. Bayesian Estimation (ベイズ推定)
   $\longrightarrow$ Econometrics III (Spring Semester, 2013)
3. Panel Data (パネル・データ)
4. Discrete Dependent Variable (離散従属変数) and Truncated Regression Model (切断回帰モデル)

5. Nonparametric Estimation and Test (ノンパラメトリック推定・検定)
6. Generalized Method of Moment (GMM, 一般化積率法)
7. Etc.... (その他)