2. `Equation 1` (Test of Serial Correlation $\longrightarrow$ Lagrange Multiplier Test)

`Equation 2` is:

$$\texttt{@RES}_t = \rho\texttt{@RES}_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma_\epsilon^2),$$

where $\texttt{@RES}_t = \texttt{RCONS}_t - \hat{\beta}_1 - \hat{\beta}_2\texttt{RYD}_t$, and $\hat{\beta}_1$ and $\hat{\beta}_2$ are OLSEs.

The null hypothesis is $H_0 : \rho = 0$

```
 @RES(-1)   .950693        .053301        17.8362        [.000]
```

Therefore, the Lagrange multiplier test statistic is $17.8362^2 = 318.13 > 6.635$.

$H_0: \rho = 0$ is rejected.

3. `Equation 3` (Test of Serial Correlation $\longrightarrow$ Wald Test)

`Equation 3` is:

$$\mathtt{RCONS}_t = \beta_1 + \beta_2 \mathtt{RYD}_t + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

The null hypothesis is $H_0: \rho = 0$

```
 RHO          .945025       .045843       20.6143       [.000]
```

The Wald test statistics is $20.6143^2 = 424.95$, which is compared with $\chi^2(1)$.

4. `Equation 1` vs. `NONLINEAR LEAST SQUARES` (Choice of Functional Form – linear):

`NONLINEAR LEAST SQUARES` estimates:

$$\mathtt{RCONS}_t = a1 + a2\frac{\mathtt{RYD}_t^{a3} - 1}{a3} + u_t.$$

When $a3 = 1$, we have:

$$\mathtt{RCONS}_t = (a1 - a2) + a2\mathtt{RYD}_t + u_t,$$

which is equivalent to `Equation 1`.

The null hypothesis is $H_0: \ a3 = 1$, where $G = 1$.

- MLE with $a3 = 1$ MLE (`Equation 1`)

  `Log of likelihood function = -431.289`

- MLE without $a3 = 1$ (`NONLINEAR LEAST SQUARES`)

  `Log of likelihood function = -414.362`

The likelihood ratio test statistic is given by:

$$-2\log(\lambda) = -2\Big(-431.289 - (-414.362)\Big) = 33.854.$$

The 1% upper probability point of $\chi^2(1)$ is 6.635.

$$33.854 > 6.635$$

$H_0: a3 = 1$ is rejected.

Therefore, the functional form of the regression model is not linear.

5. `Equation 4` vs. `NONLINEAR LEAST SQUARES` (Choice of Functional Form – log-linear):

   In `NONLINEAR LEAST SQUARES`, i.e.,

$$\mathtt{RCONS}_t = a1 + a2\frac{\mathtt{RYD}_t^{a3} - 1}{a3} + u_t,$$

   if $a3 = 0$, we have:

$$\mathtt{RCONS}_t = a1 + a2\log(\mathtt{RYD}_t) + u_t,$$

which is equivalent to `Equation 3`.

The null hypothesis is $H_0: \ a3 = 0$, where $G = 1$.

- MLE with $a3 = 0$ (`Equation 3`)

  `Log of likelihood function = -495.418`

- MLE without $a3 = 0$ (`NONLINEAR LEAST SQUARES`)

  `Log of likelihood function = -414.362`

The likelihood ratio test statistic is:

$$-2\log(\lambda) = -2\Big(-495.418 - (-414.362)\Big) = 162.112 > 6.635.$$

Therefore, $H_0: \ a3 = 0$ is rejected.

As a result, the functional form of the regression model is not log-linear, either.

6. `Equation 1` vs. `Equation 5` (Simultaneous Test of Serial Correlation and Linear Function):

   `Equation 5` is:

$$\mathtt{RCONS}_t = a1 + a2\frac{\mathtt{RYD}_t^{a3} - 1}{a3} + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2)$$

   The null hypothesis is $H_0: \ a3 = 1, \ \rho = 0$

Restricted MLE $\Longrightarrow$ `Equation 1`

Unrestricted MLE $\Longrightarrow$ `Equation 4`

Remark: In Lines 14–16 of `PROGRAM`, we have estimated `Equation 4`, given $a3 = 0.00, 0.01, 0.02, \cdots$.

As a result, $a3 = 1.15$ gives us the maximum log-likelihood.

The likelihood ratio test statistic is:

$$-2\log(\lambda) = -2\Big(-431.289 - (-383.807)\Big) = 94.964.$$

$-2\log(\lambda) \sim \chi^2(2)$ in this case.

The 1% upper probability point of $\chi^2(2)$ is 9.210.

$$94.964 > 9.210$$

$H_0:\ a3 = 1, \rho = 0$ is rejected.

`Equation 3` vs. `Equation 5` vs. (Taking into account serially correlated errors, Choice of Functional Form – linear):

The null hypothesis is $H_0: \ a3 = 1, \ \rho = 0$

From `Equation 3`,

```
 Log likelihood = -385.419
```

From `Equation 5`,

```
 Log likelihood = -383.807
```

$$2(-383.807 - (-385.419)) = 3.224 < 6.635.$$

$H_0: \ a3 = 1$ is not rejected, given $\rho \neq 0$.

Thus, if serial correlation is taken into account, the regression model is linear.

14 Time Series Analysis (時系列分析)

14.1 Introduction

Textbooks

・J.D. Hamilton (1994) *Econometric Analysis*

沖本・井上訳 (2006)『時系列解析 (上・下)』

・A.C. Harvey (1981) *Time Series Models*

国友・山本訳 (1985)『時系列モデル入門』

・沖本竜義 (2010)『経済・ファイナンスデータの計量時系列分析』

1. **Stationarity (定常性) :**

Let $y_1, y_2, \cdots, y_T$ be time series data.

(a) **Weak Stationarity (弱定常性) :**

$$\mathrm{E}(y_t) = \mu,$$

$$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

The first and second moments depend on time difference, not time itself.

(b) **Strong Stationarity (強定常性) :**

Let $f(y_{t_1}, y_{t_2}, \cdots, y_{t_r})$ be the joint distribution of $y_{t_1}, y_{t_2}, \cdots, y_{t_r}$.

$$f(y_{t_1}, y_{t_2}, \cdots, y_{t_r}) = f(y_{t_1+\tau}, y_{t_2+\tau}, \cdots, y_{t_r+\tau})$$

All the moments are same for all τ.

2. **Auto-covariance Function (自己共分散関数) :**

$$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

$$\gamma(\tau) = \gamma(-\tau)$$

3. **Auto-correlation Function (自己相関関数) :**

$$\rho(\tau) = \frac{\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu))}{\sqrt{\mathrm{Var}(y_t)}\sqrt{\mathrm{Var}(y_{t-\tau})}} = \frac{\gamma(\tau)}{\gamma(0)}$$

Note that $\mathrm{Var}(y_t) = \mathrm{Var}(y_{t-\tau}) = \gamma(0)$.

4. **Sample Mean (標本平均) :**

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

5. **Sample Auto-covariance (標本自己共分散) :**

$$\hat{\gamma}(\tau) = \frac{1}{T}\sum_{t=\tau+1}^{T}(y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu})$$

6. **Correlogram (コレログラム, or 標本自己相関関数) :**

$$\hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}$$

7. **Lag Operator (ラグ作要素) :**

$$L^\tau y_t = y_{t-\tau}, \qquad \tau = 1, 2, \cdots$$

8. **Likelihood Function (尤度関数)** — Innovation Form :

The joint distribution of $y_1, y_2, \cdots, y_T$ is written as:

$$
\begin{aligned}
f(y_1, , y_2, \cdots, y_T) &= f(y_T|y_{T-1}, \cdots, y_1) f(y_{T-1}, \cdots, y_1) \\
&= f(y_T|y_{T-1}, \cdots, y_1) f(y_{T-1}|y_{T-2}, \cdots, y_1) f(y_{T-2}, \cdots, y_1) \\
&\quad \vdots \\
&= f(y_T|y_{T-1}, \cdots, y_1) f(y_{T-1}|y_{T-2}, \cdots, y_1) \ \cdots \ f(y_2|y_1) f(y_1) \\
&= f(y_1) \prod_{t=2}^{T} f(y_t|y_{t-1}, \cdots, y_1).
\end{aligned}
$$

Therefore, the log-likelihood function is given by:

$$\log f(y_1, y_2, \cdots, y_T) = \log f(y_1) + \sum_{t=2}^{T} \log f(y_t | y_{t-1}, \cdots, y_1).$$

Under the normality assumption, $f(y_t|y_{t-1}, \cdots, y_1)$ is given by the normal distribution with conditional mean $\mathrm{E}(y_t|y_{t-1}, \cdots, y_1)$ and conditional variance $\mathrm{Var}(y_t|y_{t-1}, \cdots, y_1)$.

14.2 Autoregressive Model (自己回帰モデル or AR モデル)

1. **AR(p) Model :**

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

which is rewritten as:

$$\phi(L) y_t = \epsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p.$$

2. **Stationarity (定常性) :**

   Suppose that all the p solutions of x from $\phi(x) = 0$ are real numbers

   When the p solutions are greater than one, y_t is stationary.

   Suppose that the p solutions include imaginary numbers.

   When the p solutions are outside unit circle, y_t is stationary.

Example: AR(1) Model: $y_t = \phi_1 y_{t-1} + \epsilon_t$

1. The stationarity condition is: the solution of $\phi(x) = 1 - \phi_1 x = 0$, i.e., $x = 1/\phi_1$, is greater than one in absolute value, or equivalently, $-1 < \phi_1 < 1$.

2. Rewriting the AR(1) model,

$$\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\
&= \phi_1^3 y_{t-3} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} \\
&\ \vdots \\
&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1}.
\end{aligned}$$

As s is large, ϕ_1^s approaches zero. $\Longrightarrow$ Stationarity condition

3. For stationarity, $y_t = \phi_1 y_{t-1} + \epsilon_t$ is rewritten as:

$$y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots$$

4. Mean and Variance of AR(1) process, μ

$$\begin{aligned}
\mu = \mathrm{E}(y_t) &= \mathrm{E}(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots) \\
&= \mathrm{E}(\epsilon_t) + \phi_1 \mathrm{E}(\epsilon_{t-1}) + \phi_1^2 \mathrm{E}(\epsilon_{t-2}) + \cdots = 0 \\
\gamma(0) = \mathrm{V}(y_t) &= \mathrm{V}(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots) \\
&= \mathrm{V}(\epsilon_t) + \mathrm{V}(\phi_1 \epsilon_{t-1}) + \mathrm{V}(\phi_1^2 \epsilon_{t-2}) + \cdots \\
&= \sigma^2 + \phi_1^2 \sigma^2 + \phi_1^4 \sigma^2 + \cdots = \frac{\sigma^2}{1 - \phi_1^2}
\end{aligned}$$

5. Autocovariance and autocorrelation functions of the AR(1) process:

Rewriting the AR(1) process, we have:

$$y_t = \phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}.$$

Therefore, the autocovariance function of AR(1) process is:

$$\begin{aligned}\gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\ &= \mathrm{E}\Big((\phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}) y_{t-\tau}\Big) \\ &= \phi_1^\tau \mathrm{E}(y_{t-\tau} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau}) + \phi_1 \mathrm{E}(\epsilon_{t-1} y_{t-\tau}) + \cdots + \phi_1^{\tau-1} \mathrm{E}(\epsilon_{t-\tau+1} y_{t-\tau}) \\ &= \phi_1^\tau \gamma(0).\end{aligned}$$

The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi_1^{\tau}.$$

6. Another solution:

   Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation:

$$E(y_t y_{t-\tau}) = \phi_1 E(y_{t-1} y_{t-\tau}) + E(\epsilon_t y_{t-\tau})$$

$$\gamma(\tau) = \begin{cases} \phi_1 \gamma(\tau - 1), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by:

$$\gamma(0) = \phi_1\gamma(1) + \sigma^2 = \phi_1^2\gamma(0) + \sigma^2.$$

Note that $\gamma(1) = \phi_1\gamma(0)$.

Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1 - \phi_1^2}$$

7. Estimation of AR(1) model:

(a) Likelihood function

$$\begin{aligned}\log f(y_T, \cdots, y_1) &= \log f(y_1) + \sum_{t=2}^{T} \log f(y_t | y_{t-1}, \cdots, y_1) \\ &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\frac{\sigma^2}{1-\phi_1^2}\right) - \frac{1}{\sigma^2/(1-\phi_1^2)} y_1^2 \\ &\quad -\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2\end{aligned}$$

$$= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\log\left(\frac{1}{1-\phi_1^2}\right)$$
$$-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

Note as follows:

$$f(y_1) = \frac{1}{\sqrt{2\pi\sigma^2/(1-\phi_1^2)}}\exp\left(-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2\right)$$
$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right)$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4/(1-\phi_1^2)}y_1^2 + \frac{1}{2\sigma^4}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2 = 0$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \phi_1} = -\frac{\phi_1}{1-\phi_1^2} + \frac{\phi_1}{\sigma^2}y_1^2 + \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})y_{t-1} = 0$$

The MLE of ϕ_1 and σ^2 satisfies the above two equation.

$$\tilde{\sigma}^2 = \frac{1}{T}\left((1-\tilde{\phi}_1^2)y_1^2 + \sum_{t=2}^{T}(y_t - \tilde{\phi}_1 y_{t-1})^2\right)$$

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} + \left(\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2\tilde{\phi}_1}{1-\tilde{\phi}_1^2}\right)\Big/\sum_{t=2}^{T} y_{t-1}^2$$