(c) Assume $\epsilon_t \sim$ iid $(0, \sigma_\epsilon^2)$. Define $X_T(r)$ for $r \in [0, 1]$ as follows:

$$X_T(r) = \begin{cases} 0, & 0 \le r < \dfrac{1}{T} \\ \dfrac{\epsilon_1}{T}, & \dfrac{1}{T} \le r < \dfrac{2}{T} \\ \dfrac{\epsilon_1 + \epsilon_2}{T}, & \dfrac{2}{T} \le r < \dfrac{3}{T} \\ \quad\vdots & \quad\vdots \\ \dfrac{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_T}{T}, & r = 1 \end{cases}$$

Let $[Tr]$ be the largest integer which is less than or equal to $T \times r$.

$$X_T(r) \equiv \frac{1}{T}\sum_{t=1}^{[Tr]} \epsilon_t, \qquad \sqrt{T} X_T(r) \longrightarrow N(0, r\sigma_\epsilon^2).$$

Note that

$$\frac{1}{T}\sum_{t=1}^{[Tr]} \epsilon_t = \frac{[Tr]}{T}\frac{1}{[Tr]}\sum_{t=1}^{[Tr]} \epsilon_t,$$

$$\frac{[Tr]}{T} \longrightarrow r, \qquad \frac{1}{\sqrt{[Tr]}}\sum_{t=1}^{[Tr]} \epsilon_t \longrightarrow N(0, \sigma_\epsilon^2),$$

$$\sqrt{T} X_T(r) = \frac{[Tr]}{T}\sqrt{\frac{T}{[Tr]}}\frac{1}{\sqrt{[Tr]}}\sum_{t=1}^{[Tr]} \epsilon_t, \qquad \sqrt{\frac{T}{[Tr]}} \longrightarrow \frac{1}{\sqrt{r}}.$$

Therefore, we obtain:

$$\sqrt{T} X_T(r) \longrightarrow N(0, r\sigma_\epsilon^2).$$

Moreover, we have the following results:

$$\frac{\sqrt{T} X_T(r)}{\sigma_\epsilon} \longrightarrow N(0, r) = W(r),$$

$$\frac{\sqrt{T}\,(X_T(r_2) - X_T(r_1))}{\sigma_\epsilon} \longrightarrow W(r_2) - W(r_1) = N(0, r_2 - r_1).$$

For example, consider:

$$X_T(1) = \frac{1}{T} \sum_{t=1}^{T} \epsilon_t.$$

Then,

$$\frac{\sqrt{T} X_T(1)}{\sigma_\epsilon} = \frac{1}{\sigma_\epsilon \sqrt{T}} \sum_{t=1}^{T} \epsilon_t \longrightarrow W(1) = N(0,1).$$

(d) Consider $y_t = y_{t-1} + \epsilon_t,\ y_0 = 0$ and $\epsilon_t \sim N(0, \sigma_\epsilon^2)$.

$X_T(r)$ is defined as follows:

$$X_T(r) = \begin{cases} 0, & 0 \le r < \dfrac{1}{T}, \\ \dfrac{y_1}{T}, & \dfrac{1}{T} \le r < \dfrac{2}{T}, \\ \dfrac{y_2}{T}, & \dfrac{2}{T} \le r < \dfrac{3}{T}, \\ \quad\vdots & \quad\vdots \\ \dfrac{y_{T-1}}{T}, & \dfrac{T-1}{T} \le r < 1, \\ \dfrac{y_T}{T}, & r = 1. \end{cases}$$

Define $S_T(r)$ as follows:

$$S_T(r) = \begin{cases} 0, & 0 \le r < \dfrac{1}{T}, \\ \dfrac{y_1^2}{T}, & \dfrac{1}{T} \le r < \dfrac{2}{T}, \\ \dfrac{y_2^2}{T}, & \dfrac{2}{T} \le r < \dfrac{3}{T}, \\ \vdots & \vdots \\ \dfrac{y_{T-1}^2}{T}, & \dfrac{T-1}{T} \le r < 1, \\ \dfrac{y_T^2}{T}, & r = 1. \end{cases}$$

To obtain $\int_0^1 X_T(r)\mathrm{d}r$ and $\int_0^1 S_T(r)\mathrm{d}r$, we compute a sum of rectangulars as follows:

$$\int_0^1 X_T(r)\mathrm{d}r \approx \frac{y_1}{T}\left(\frac{2}{T}-\frac{1}{T}\right)+\frac{y_2}{T}\left(\frac{3}{T}-\frac{2}{T}\right)+\ \cdots\ +\frac{y_{T-1}}{T}\left(1-\frac{T-1}{T}\right)$$
$$=\frac{y_1}{T^2}+\frac{y_2}{T^2}+\ \cdots\ +\frac{y_{T-1}}{T^2}=\frac{1}{T^2}\sum_{t=1}^{T}y_t,$$

$$\int_0^1 S_T(r)\mathrm{d}r \approx \frac{y_1^2}{T}\left(\frac{2}{T}-\frac{1}{T}\right)+\frac{y_2^2}{T}\left(\frac{3}{T}-\frac{2}{T}\right)+\ \cdots\ +\frac{y_{T-1}^2}{T}\left(1-\frac{T-1}{T}\right)$$
$$=\frac{y_1^2}{T^2}+\frac{y_2^2}{T^2}+\ \cdots\ +\frac{y_{T-1}^2}{T^2}=\frac{1}{T^2}\sum_{t=1}^{T}y_t^2.$$

We have already known that $\sqrt{T} X_T(r) \longrightarrow \sigma_\epsilon W(r)$.

Therefore,

$$\int_0^1 \sqrt{T} X_T(r) \mathrm{d}r \longrightarrow \sigma_\epsilon \int_0^1 W(r) \mathrm{d}r.$$

That is,

$$\frac{1}{T^{3/2}} \sum_{t=1}^{T} y_t \longrightarrow \sigma_\epsilon \int_0^1 W(r) \mathrm{d}r.$$

From $S_T(r) \equiv \left(\sqrt{T}X_T(r)\right)^2$,

$$S_T(r) \equiv \left(\sqrt{T}X_T(r)\right)^2 \longrightarrow \sigma_\epsilon^2 (W(r))^2,$$

which is called the continuous mapping theorem.

(*) **Continuous Mapping Theorem (連続写像定理):**

if $x_T \longrightarrow x$ (convergence in distribution) and $g(\cdot)$ is a continuous function, then $g(x_T) \longrightarrow g(x)$ (convergence in distribution).

Threfore, we have the follwoing result:

$$\int_0^1 S_T(r)\mathrm{d}r \longrightarrow \sigma_\epsilon^2 \int_0^1 (W(r))^2 \,\mathrm{d}r.$$

That is,

$$\frac{1}{T^2}\sum_{t=1}^{T} y_t^2 \longrightarrow \sigma_\epsilon^2 \int_0^1 (W(r))^2 \,\mathrm{d}r.$$

8. Asymptotic Distribution of AR(1) Model:

(a) $\boldsymbol{H_0 : y_t = y_{t-1} + \epsilon_t}$ **and** $\boldsymbol{H_1 : y_t = \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{|\phi_1| < 1}$

OLSE of ϕ_1, denoted by $\hat{\phi}_1$, is given by:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^T y_{t-1} y_t}{\sum_{t=1}^T y_{t-1}^2} = \phi_1 + \frac{\sum_{t=1}^T y_{t-1}\epsilon_t}{\sum_{t=1}^T y_{t-1}^2}$$

Using $\phi_1 = 1$ and some formulas shown above, we obtain:

$$T(\hat{\phi}_1 - 1) = \frac{T^{-1}\sum_{t=1}^T y_{t-1}\epsilon_t}{T^{-2}\sum_{t=1}^T y_{t-1}^2} \longrightarrow \frac{\frac{1}{2}\Big((W(1))^2 - 1\Big)}{\int_0^1 (W(r))^2\,\mathrm{d}r}$$

Remember that

$$T^{-1}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow \frac{1}{2}\sigma_\epsilon^2\left((W(1))^2-1\right)$$

and

$$T^{-2}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \sigma_\epsilon^2\int_0^1 (W(r))^2\,\mathrm{d}r,$$

where $(W(1))^2 = \chi^2(1)$.

We say that $\hat{\phi}_1$ is **super-consistent (超一致性)** or ***T*-consistent**.

Remember that when $|\phi_1| < 1$ we have $\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1-\phi_1^2)$, and in this case we say that $\hat{\phi}_1$ is **$\sqrt{T}$-consistent**.

Conventional t test statistic is given by:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi},$$

where

$$s_\phi = \left(s^2 \Big/ \sum_{t=1}^{T} y_{t-1}^2 \right)^{1/2} \quad \text{and} \quad s^2 = \frac{1}{T-1} \sum_{t=1}^{T} (y_t - \hat{\phi}_1 y_{t-1})^2.$$

Next, consider t statistic.

The t test statistic, denoted by t, is represented as follows:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi} = \frac{T(\hat{\phi}_1 - 1)}{T s_\phi}$$

The denominator is:

$$\begin{aligned} T s_\phi &= \left(s^2 \Big/ \frac{1}{T^2} \sum_{t=1}^{T} y_{t-1}^2 \right)^{1/2} \\ &\longrightarrow \left(\sigma_\epsilon^2 \Big/ \left(\sigma_\epsilon^2 \int_0^1 (W(r))^2 \, \mathrm{d}r \right) \right)^{1/2} = \left(\int_0^1 (W(r))^2 \, \mathrm{d}r \right)^{-1/2}, \end{aligned}$$

where $s^2 \longrightarrow \sigma_\epsilon^2$ is utilized.

Therefore, we have the following asymptotic distribution:

$$
\begin{aligned}
t = \frac{\hat{\phi}_1 - 1}{s_\phi} \;\longrightarrow\; & \frac{\frac{1}{2}\Big((W(1))^2 - 1\Big)}{\displaystyle\int_0^1 (W(r))^2\,\mathrm{d}r} \Bigg/ \left(\int_0^1 (W(r))^2\,\mathrm{d}r\right)^{-1/2} \\
= & \frac{\frac{1}{2}\Big((W(1))^2 - 1\Big)}{\left(\displaystyle\int_0^1 (W(r))^2\,\mathrm{d}r\right)^{1/2}}.
\end{aligned}
$$

Therefore, the distribution of the t statistic shown above is different from the t distribution.

(b) $\boldsymbol{H_0 : y_t = y_{t-1} + \epsilon_t}$ **and** $\boldsymbol{H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{|\phi_1| < 1}$

$$\begin{pmatrix} \hat{\alpha}_0 \\ \hat{\phi}_1 \end{pmatrix} = \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_t \\ \sum y_{t-1} y_t \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1} \epsilon_t \end{pmatrix}$$

In the true model, $\alpha_0 = 0$ and $\phi_1 = 1$.

$$\begin{pmatrix} \hat{\alpha}_0 \\ \hat{\phi}_1 - 1 \end{pmatrix} = \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1} \epsilon_t \end{pmatrix}$$

$$= \begin{pmatrix} O_p(T) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) \end{pmatrix}^{-1} \begin{pmatrix} O_p(T^{1/2}) \\ O_p(T) \end{pmatrix}$$

(*) For random variable x and constant k, $x = O_p(k)$ implies that x/k converges in distribution.

To change each element of the matrices to $O_p(1)$, we use the following matrix:

$$\Gamma = \begin{pmatrix} T^{1/2} & 0 \\ 0 & T \end{pmatrix}.$$

Multiplying the above matrix from the left, we obtain the following:

$$\Gamma \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\phi}_1 - 1 \end{pmatrix} = \begin{pmatrix} T^{1/2}\hat{\alpha}_0 \\ T(\hat{\phi}_1 - 1) \end{pmatrix} = \Gamma \begin{pmatrix} O_p(T) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) \end{pmatrix}^{-1} \Gamma\Gamma^{-1} \begin{pmatrix} O_p(T^{1/2}) \\ O_p(T) \end{pmatrix}$$

$$
= \left(\Gamma^{-1} \begin{pmatrix} O_p(T) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) \end{pmatrix} \Gamma^{-1} \right)^{-1} \Gamma^{-1} \begin{pmatrix} O_p(T^{1/2}) \\ O_p(T) \end{pmatrix}
$$

$$
= \left(\Gamma^{-1} \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix} \Gamma^{-1} \right)^{-1} \Gamma^{-1} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1}\epsilon_t \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & T^{-3/2} \sum y_{t-1} \\ T^{-3/2} \sum y_{t-1} & T^{-2} \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} T^{-1/2} \sum \epsilon_t \\ T^{-1} \sum y_{t-1}\epsilon_t \end{pmatrix}.
$$

Each matrix converges in distribution as follows:

$$\begin{pmatrix} 1 & T^{-3/2}\sum y_{t-1} \\ T^{-3/2}\sum y_{t-1} & T^{-2}\sum y_{t-1}^2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r \\ \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r & \sigma_\epsilon^2 \int_0^1 (W(r))^2\,\mathrm{d}r \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W(r)\mathrm{d}r \\ \int_0^1 W(r)\mathrm{d}r & \int_0^1 (W(r))^2\,\mathrm{d}r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix},$$

$$\begin{pmatrix} T^{-1/2}\sum \epsilon_t \\ T^{-1}\sum y_{t-1}\epsilon_t \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_\epsilon W(1) \\ \frac{1}{2}\sigma_\epsilon^2 \left((W(1))^2 - 1\right) \end{pmatrix} = \sigma_\epsilon \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} W(1) \\ \frac{1}{2}\left((W(1))^2 - 1\right) \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} T^{1/2}\hat{\alpha}_0 \\ T(\hat{\phi}_1 - 1) \end{pmatrix} \longrightarrow \left[\begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W(r)\mathrm{d}r \\ \int_0^1 W(r)\mathrm{d}r & \int_0^1 (W(r))^2\,\mathrm{d}r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix}\right]^{-1}$$

$$\times \sigma_\epsilon \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} W(1) \\ \frac{1}{2}\left((W(1))^2 - 1\right) \end{pmatrix}.$$

Finally, $T(\hat{\phi}_1 - 1)$ converges to the following distribution:

$$T(\hat{\phi}_1 - 1) \longrightarrow \frac{\frac{1}{2}\left((W(1))^2 - 1\right) - W(1) \displaystyle\int_0^1 W(r)\mathrm{d}r}{\displaystyle\int_0^1 (W(r))^2\,\mathrm{d}r - \left(\int_0^1 W(r)\mathrm{d}r\right)^2}.$$

The t test statistic is:

$$t = \frac{\hat{\phi}_1 - 1}{\left(s_\phi^2\right)^{1/2}} = \frac{T(\hat{\phi}_1 - 1)}{\left(T^2 s_\phi^2\right)^{1/2}},$$

where

$$s_\phi^2 = s^2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$s^2 = \frac{1}{T-2} \sum_{t=1}^{T} (y_t - \hat{\alpha}_0 - \hat{\phi}_1 y_{t-1})^2.$$

The denominator $T^2 s_\phi^2$ converges in distribution as follows:

$$T^2 s_\phi^2 \longrightarrow \sigma_\epsilon^2 \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W(r) \mathrm{d}r \\ \int_0^1 W(r) \mathrm{d}r & \int_0^1 (W(r))^2 \, \mathrm{d}r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\int_0^1 (W(r))^2 \, \mathrm{d}r - \left(\int_0^1 W(r) \mathrm{d}r \right)^2}$$

Thus, the t test statistic converges to the following distribution:

$$t \longrightarrow \frac{\frac{1}{2}\left((W(1))^2 - 1\right) - W(1)\displaystyle\int_0^1 W(r)\mathrm{d}r}{\left(\displaystyle\int_0^1 (W(r))^2\,\mathrm{d}r - \left(\int_0^1 W(r)\mathrm{d}r\right)^2\right)^{1/2}}.$$

(c) $\boldsymbol{H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t}$ **and** $\boldsymbol{H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t}$ **for** $|\boldsymbol{\phi_1}| < 1$

$$\begin{pmatrix} T^{1/2}(\hat{\alpha}_0 - \alpha_0) \\ T^{3/2}(\hat{\phi}_1 - 1) \end{pmatrix} \longrightarrow N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma_\epsilon^2 \begin{pmatrix} 1 & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \frac{\alpha_0^2}{3} \end{pmatrix}\right).$$

(abbr.)

(d) $\boldsymbol{H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t}$ **and**

$\boldsymbol{H_1 : y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t}$ **for** $|\boldsymbol{\phi_1}| \boldsymbol{< 1}$

(abbr.)

9. The distributions of the t statistic: $\dfrac{\hat{\phi}_1 - 1}{s_\phi}$

t Distribution

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	–2.49	–2.06	–1.71	–1.32	1.32	1.71	2.06	2.49
50	–2.40	–2.01	–1.68	–1.30	1.30	1.68	2.01	2.40
100	–2.36	–1.98	–1.66	–1.29	1.29	1.66	1.98	2.36
250	–2.34	–1.97	–1.65	–1.28	1.28	1.65	1.97	2.34
500	–2.33	–1.96	–1.65	–1.28	1.28	1.65	1.96	2.33
∞	–2.33	–1.96	–1.64	–1.28	1.28	1.64	1.96	2.33

(a) $\boldsymbol{H_0 : \ y_t = y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1 : \ y_t = \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$ **or** $\boldsymbol{-1 < \phi_1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00

(b) $H_0: \; y_t = y_{t-1} + \epsilon_t$

$H_1: \; y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t$ **for** $\phi_1 < 1$ **or** $-1 < \phi_1$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−3.75	−3.33	−3.00	−2.63	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60

(d) $\boldsymbol{H_0: \ y_t = \alpha_0 + y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1: \ y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$ **or** $\boldsymbol{-1 < \phi_1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

15.2 Serially Correlated Errors

Consider the case where the error term is serially correlated.

15.2.1 Augmented Dickey-Fuller (ADF) Test

Consider the following AR(p) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \qquad \epsilon_t \sim \text{iid}(0, \sigma_\epsilon^2),$$

which is rewritten as:

$$\phi(L) y_t = \epsilon_t.$$

When the above model has a unit root, we have $\phi(1) = 0$, i.e., $\phi_1 + \phi_2 + \cdots + \phi_p = 1$.

The above AR(p) model is written as:

$$y_t = \rho y_{t-1} + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + +\delta_{p-1} \Delta y_{t-p+1} + \epsilon_t,$$

where $\rho = \phi_1 + \phi_2 + \cdots + \phi_p$ and $\delta_j = -(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p)$.

The null and alternative hypotheses are:

$$H_0 : \ \rho = 1 \text{ (Unit root)},$$

$$H_1 : \ \rho < 1 \text{ (Stationary)}.$$

Use the t test, where we have the same asymptotic distributions.

We can utilize the same tables as before.

Choose p by AIC or SBIC.

Use $N(0,1)$ to test $H_0:\ \delta_j = 0$ against $H_1:\ \delta_j \neq 0$ for $j = 1, 2, \cdots, p-1$.

Reference

Kurozumi (2008) "Economic Time Series Analysis and Unit Root Tests: Development and Perspective," *Japan Statistical Society*, Vol.38, Series J, No.1, pp.39 – 57.

Download the above paper from:

`http://ci.nii.ac.jp/vol_issue/nels/AA11989749/ISS0000426576_ja.html`