15.3 Cointegration (共和分)

1. For a scalar y_t, when $(1-L)^d y_t$ is stationary, we write $y_t \sim I(d)$.

   When $\Delta y_t = y_t - y_{t-1}$ is stationary, we write $\Delta y_t \sim I(0)$ or $y_t \sim I(1)$.

2. **Definition of Cointegration:**

   Suppose that each series in a $g \times 1$ vector y_t is $I(1)$, i.e., each series has unit root, and that a linear combination of each series (i.e, $a'y_t$ for a nonzero vector a) is $I(0)$, i.e., stationary.

   Then, we say that y_t has a cointegration.

3. **Example:**

Suppose that $y_t = (y_{1,t},\ y_{2,t})'$ is the following vector autoregressive process:

$$y_{1,t} = \gamma y_{2,t} + \epsilon_{1,t},$$

$$y_{2,t} = y_{2,t-1} + \epsilon_{2,t}.$$

Then,

$$\Delta y_{1,t} = \gamma \epsilon_{2,t} + \epsilon_{1,t} - \epsilon_{1,t-1}, \quad \text{(MA(1) process)},$$

$$\Delta y_{2,t} = \epsilon_{2,t},$$

where both $y_{1,t}$ and $y_{2,t}$ are $I(1)$ processes.

The linear combination $y_{1,t} - \gamma y_{2,t}$ is $I(0)$.

In this case, we say that $y_t = (y_{1,t},\ y_{2,t})'$ is cointegrated with $a = (1,\ -\gamma)$.

$a = (1,\ -\gamma)$ is called the **cointegrating vector (共和分ベクトル)**, which is not unique. Therefore, the first element of a is set to be one.

4. Suppose that $y_t \sim I(1)$ and $x_t \sim I(1)$.

   For the regression model $y_t = x_t\beta + u_t$, OLS does not work well if we do not have the β which satisfies $u_t \sim I(0)$.

   $\Longrightarrow$ **Spurious regression (見せかけの回帰)**

5. Suppose that $y_t \sim I(1)$, y_t is a $g \times 1$ vector and $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$.

$y_{2,t}$ is a $k \times 1$ vector, where $k = g - 1$.

Consider the following regression model:

$$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t, \qquad t = 1, 2, \cdots, T.$$

OLSE is given by:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} \sum y_{1,t} \\ \sum y_{1,t} y_{2,t} \end{pmatrix}.$$

Next, consider testing the null hypothesis $H_0: \ R\gamma = d$, where R is a $G \times k$

matrix ($G \leq k$) and r is a $G \times 1$ vector. G denotes the number of the linear restrictions.

The F statistic, denoted by F, is given by:

$$F = \frac{1}{G}(R\hat{\gamma} - d)' \left(s^2 \begin{pmatrix} 0 & R \end{pmatrix} \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R' \end{pmatrix} \right)^{-1} (R\hat{\gamma} - d),$$

where

$$s^2 = \frac{1}{T-g} \sum_{t=1}^{T} (y_{1,t} - \hat{\alpha} - \hat{\gamma}' y_{2,t})^2.$$

When we have the γ such that $y_{1,t} - \gamma y_{2,t}$ is stationary, OLSE of γ, i.e., $\hat{\gamma}$, is not statistically equal to zero.

When the sample size T is large enough, H_0 is rejected by the F test.

6. Phillips, P.C.B. (1986) "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, Vol.33, pp.95 – 131.

Consider a $g \times 1$ vector y_t whose first difference is described by:

$$\Delta y_t = \Psi(L)\epsilon_t = \sum_{s=0}^{\infty} \Psi_s \epsilon_{t-s},$$

for ϵ_t an i.i.d. $g \times 1$ vector with mean zero , variance $\mathrm{E}(\epsilon_t \epsilon_t') = PP'$, and finite fourth moments and where $\{s\Psi_s\}_{s=0}^{\infty}$ is absolutely summable.

Let $k = g - 1$ and $\Lambda = \Psi(1)P$.

Partition y_t as $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$ and $\Lambda\Lambda'$ as $\Lambda\Lambda' = \begin{pmatrix} \Sigma_{11} & \Sigma_{21}' \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where $y_{1,t}$ and Σ_{11} are scalars, $y_{2,t}$ and Σ_{21} are $k \times 1$ vectors, and Σ_{22} is a $k \times k$ matrix.

Suppose that $\Lambda\Lambda'$ is nonsingular,and define $\sigma_1^2 = \Sigma_{11} - \Sigma_{21}'\Sigma_{22}^{-1}\Sigma_{21}$.

Let L_{22} denote the Cholesky factor of Σ_{22}^{-1}, i.e., L_{22} is the lower triangular matrix satisfying $\Sigma_{22}^{-1} = L_{22}L_{22}'$.

Then, (a) – (c) hold.

(a) OLSEs of α and γ in the regression model $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$, denoted by $\hat{\alpha}_T$ and $\hat{\gamma}_T$, are characterized by:

$$\begin{pmatrix} T^{-1/2}\hat{\alpha}_T \\ \hat{\gamma}_T - \Sigma_{22}^{-1}\Sigma_{21} \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_1 h_1 \\ \sigma_1 L_{22} h_2 \end{pmatrix},$$

where

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 & \int_0^1 W_2(r)' \mathrm{d}r \\ \int_0^1 W_2(r)\mathrm{d}r & \int_0^1 W_2(r)W_2(r)'\mathrm{d}r \end{pmatrix}^{-1} \begin{pmatrix} \int_0^1 W_1(r)\mathrm{d}r \\ \int_0^1 W_2(r)W_1(r)\mathrm{d}r \end{pmatrix},$$

where $W_1(r)$ and $W_2(r)$ denote scalar and g-dimensional standard Brownian motions, and $W_1(r)$ is independent of $W_2(r)$.

(b) The sum of squared residuals, denoted by $\mathrm{RSS}_T = \sum_{t=1}^{T} \hat{u}_t^2$, satisfies

$$T^{-2}\mathrm{RSS}_T \longrightarrow \sigma_1^2 H,$$

where

$$H = \int_0^1 (W_1(r))^2 \mathrm{d}r - \left(\begin{pmatrix} \int_0^1 W_1(r)\mathrm{d}r \\ \int_0^1 W_2(r)W_1(r)\mathrm{d}r \end{pmatrix}' \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right)^{-1}.$$

(c) The F test satisfies:

$$
\begin{aligned}
T^{-1}F \longrightarrow\; & \frac{1}{G}(\sigma_1 R^* h_2 - d^*)' \\
& \times\left(\sigma_1^2 H \begin{pmatrix} 0 & R^* \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W_2(r)'\mathrm{d}r \\ \int_0^1 W_2(r)\mathrm{d}r & \int_0^1 W_2(r)W_2^*(r)'\mathrm{d}r \end{pmatrix}^{-1} \begin{pmatrix} 0 & R^* \end{pmatrix}'\right)^{-1} \\
& \times(\sigma_1 R^* h_2 - d^*),
\end{aligned}
$$

where $R^* = RL_{22}$ and $d^* = d - R\Sigma_{22}^{-1}\Sigma_{21}$.

(a) indicates that OLSE $\hat{\gamma}_T$ is not consistent.

(b) indicates that $s^2 = \dfrac{1}{T-g}\sum_{t=1}^{T}\hat{u}_t^2$ diverges.

(c) indicates that F diverges.

$\Longrightarrow$ **Spurious regression (見せかけの回帰)**

7. **Resolution for Spurious Regression:**

Suppose that $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ is a spurious regression.

(1) Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + \phi y_{1,t-1} + \delta y_{2,t-1} + u_t$.

Then, $\hat{\gamma}_T$ is $\sqrt{T}$-consistent, and the t test statistic goes to the standard normal distribution under $H_0: \ \gamma = 0$.

(2) Estimate $\Delta y_{1,t} = \alpha + \gamma' \Delta y_{2,t} + u_t$. Then, $\hat{\alpha}_T$ and $\hat{\beta}_T$ are $\sqrt{T}$-consistent, and the t test and F test make sense.

(3) Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ by the Cochrane-Orcutt method, assuming that u_t is the first-order serially correlated error.

Usually, choose (2).

However, there are two exceptions.

(i) The true value of ϕ in (1) above is not one, i.e., less than one.

(ii) $y_{1,t}$ and $y_{2,t}$ are the cointegrated processes.

In these two cases, taking the first difference leads to the misspecified regression.

8. **Cointegrating Vector:**

Suppose that each element of y_t is $I(1)$ and that $a'y_t$ is $I(0)$.

a is called a **cointegrating vector (共和分ベクトル)**, which is not unique.

Set $z_t = a'y_t$, where z_t is scalar, and a and y_t are $g \times 1$ vectors.

For $z_t \sim I(0)$ (i.e., stationary),

$$T^{-1} \sum_{t=1}^{T} z_t^2 = T^{-1} \sum_{t=1}^{T} (a' y_t)^2 \longrightarrow \mathrm{E}(z_t^2).$$

For $z_t \sim I(1)$ (i.e., nonstationary, i.e., a is not a cointegrating vector),

$$T^{-2} \sum_{t=1}^{T} (a' y_t)^2 \longrightarrow \lambda^2 \int_0^1 (W(r))^2 \,\mathrm{d}r,$$

where $W(r)$ denotes a standard Brownian motion and λ^2 indicates variance of $(1 - L)z_t$.

If a is not a cointegrating vector, $T^{-1} \sum_{t=1}^{T} z_t^2$ diverges.

$\Longrightarrow$ We can obtain a consistent estimate of a cointegrating vector by minimizing $\sum_{t=1}^{T} z_t^2$ with respect to a, where a normalization condition on a has to be imposed.

The estimator of the a including the normalization condition is super-consistent (T-consistent).

Stock, J.H. (1987) "Asymptotic Properties of Least Squares Estimators of Cointegrating Vectors," *Econometrica*, Vol.55, pp.1035 – 1056.

Proposition:

Let $y_{1,t}$ be a scalar, $y_{2,t}$ be a $k \times 1$ vector, and $(y_{1,t}, y'_{2,t})'$ be a $g \times 1$ vector, where $g = k + 1$.

Consider the following model:

$$y_{1,t} = \alpha + \gamma' y_{2,t} + u_{1,t}$$

$$\Delta y_{2,t} = u_{2,t}$$

$$\begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \Psi(L)\epsilon_t$$

ϵ_t is a $g \times 1$ i.i.d. vector with $\mathrm{E}(\epsilon_t) = 0$ and $\mathrm{E}(\epsilon_t \epsilon_t') = PP'$.

OLSE is given by:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} T & \sum y_{2,t}' \\ \sum y_{2,t} & \sum y_{2,t} y_{2,t}' \end{pmatrix}^{-1} \begin{pmatrix} \sum y_{1,t} \\ \sum y_{1,t} y_{2,t} \end{pmatrix}.$$

Define λ_1, which is a $g \times 1$ vector, and Λ_2, which is a $k \times g$ matrix, as follows:

$$\Psi(1)\,P = \begin{pmatrix} \lambda_1{}' \\ \Lambda_2 \end{pmatrix}.$$

Then, we have the following results:

$$\begin{pmatrix} T^{1/2}(\hat{\alpha}-\alpha) \\ T(\hat{\gamma}-\gamma) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \left(\Lambda_2 \int W(r)\mathrm{d}r\right)' \\ \Lambda_2 \int W(r)\mathrm{d}r & \Lambda_2 \left(\int (W(r))\,(W(r))'\,\mathrm{d}r\right)\Lambda_2{}' \end{pmatrix}^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \lambda_1{}'W(1) \\ \Lambda_2 \left(\int W(r)\,(\mathrm{d}W(r))'\right)\lambda_1 + \sum_{\tau=0}^{\infty} \mathrm{E}(u_{2,t}u_{1,t+\tau}) \end{pmatrix}.$$

$W(r)$ denotes a g-dimensional standard Brownian motion.

1) OLSE of the cointegrating vector is consistent even though u_t is serially correlated.

2) The consistency of OLSE implies that $T^{-1}\sum \hat{u}_t^2 \longrightarrow \sigma^2$.

3) Because $T^{-1}\sum(y_{1,t} - \bar{y}_1)^2$ goes to infinity, a coefficient of determination, R^2, goes to one.

15.4 Testing Cointegration

15.4.1 Engle-Granger Test

$y_t \sim I(1)$

$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$

- $u_t \sim I(0) \implies$ Cointegration
- $u_t \sim I(1) \implies$ Spurious Regression

Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ by OLS, and obtain $\hat{u}_t$.

Estimate $\hat{u}_t = \rho \hat{u}_{t-1} + \delta_1 \Delta \hat{u}_{t-1} + \delta_2 \Delta \hat{u}_{t-2} + \cdots + \delta_{p-1} \Delta \hat{u}_{t-p+1} + e_t$ by OLS.

ADF Test:

- $H_0:\ \rho = 1$ (Sprious Regression)
- $H_1:\ \rho < 1$ (Cointegration)

$\Longrightarrow$ **Engle-Granger Test**

For example, see Engle and Granger (1987), Phillips and Ouliaris (1990) and Hansen (1992).

Asymmptotic Distribution of Residual-Based ADF Test for Cointegration

# of Refressors,	(a) Regressors have no drift				(b) Some regressors have drift			
excluding constant	1%	2.5%	5%	10%	1%	2.5%	5%	10%
1	−3.96	−3.64	−3.37	−3.07	−3.96	−3.67	−3.41	−3.13
2	−4.31	−4.02	−3.77	−3.45	−4.36	−4.07	−3.80	−3.52
3	−4.73	−4.37	−4.11	−3.83	−4.65	−4.39	−4.16	−3.84
4	−5.07	−4.71	−4.45	−4.16	−5.04	−4.77	−4.49	−4.20
5	−5.28	−4.98	−4.71	−4.43	−5.36	−5.02	−4.74	−4.46

J.D. Hamilton (1994), *Time Series Analysis*, p.766.

The Other Topics

- Generalized Method of Moments (一般化積率法，GMM)
- System of Equations (Seemingly Unrelated Regression (SUR), Simultaneous Equation (連立方程式), and etc.)
- Panel Data (パネル・データ)
- Discrete Dependent Variable, and Limited Dependent Variable
- Bayesian Estimation (ベイズ推定)
- Semiparametric and Nonparametric Regressions and Tests (セミパラメトリック，ノンパラメトリック推定・検定)
- $\cdots$

Exam — Jan. 29, 2014 (AM8:50-10:20), and # 509

- 60 - 70% from two homeworks (2 つの宿題から 60 - 70%)
- 30 - 40% of new questions (30 - 40% の新しい問題)
- Questions are written in English, and answers should be in English or Japanese.

(出題は英語，解答は英語または日本語)

- With no carrying in (持ち込みなし)