

The following equations are utilized.

$$E(y_{t-1}\epsilon_t) = 0$$

$$E(y_{t-1}^2) = \text{Var}(y_{t-1}) = \gamma(0)$$

8. Asymmmptotic distribution of OLSE  $\hat{\phi}_1$ :

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$$

**Proof:**

$y_{t-1}\epsilon_t$ ,  $t = 1, 2, \dots, T$ , are distributed with mean zero and variance  $\frac{\sigma_\epsilon^4}{1 - \phi_1^2}$ .

From the central limit theorem,

$$\frac{(1/T) \sum_{t=1}^T y_{t-1} \epsilon_t}{\sqrt{\sigma_\epsilon^4 / (1 - \phi_1^2) / \sqrt{T}}} \longrightarrow N(0, 1)$$

Rewriting,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T y_{t-1} \epsilon_t \longrightarrow N\left(0, \frac{\sigma_\epsilon^4}{1 - \phi_1^2}\right).$$

Next,

$$\frac{1}{T} \sum_{t=1}^T y_{t-1}^2 \longrightarrow E(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2}$$

yields:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{(1/\sqrt{T}) \sum_{t=1}^T y_{t-1} \epsilon_t}{(1/T) \sum_{t=1}^T y_{t-1}^2} \longrightarrow N(0, 1 - \phi_1^2)$$

9. Some formulas:

(a) Central Limit Theorem

Random variables  $x_1, x_2, \dots, x_T$  are mutually independently distributed with mean  $\mu$  and variance  $\sigma^2$ .

Define  $\bar{x} = (1/T) \sum_{t=1}^T x_t$ .

Then,

$$\frac{\bar{x} - E(\bar{x})}{\sqrt{V(\bar{x})}} = \frac{\bar{x} - \mu}{\sigma / \sqrt{T}} \longrightarrow N(0, 1)$$

(b) Central Limit Theorem II

Random variables  $x_1, x_2, \dots, x_T$  are distributed with mean  $\mu$  and variance  $\sigma^2$ .

Define  $\bar{x} = (1/T) \sum_{t=1}^T x_t$ .

Then,

$$\frac{\bar{x} - E(\bar{x})}{\sqrt{V(\bar{x})}} \longrightarrow N(0, 1)$$

(c) Let  $x$  and  $y$  be random variables.

$y$  converges in distribution to a distribution, and  $x$  converges in probability to a fixed value.

Then,  $xy$  converges in distribution.

For example, consider:

$$y \longrightarrow N(\mu, \sigma^2), \quad x \longrightarrow c.$$

Then, we obtain:

$$xy \longrightarrow N(\mu, c^2\sigma^2)$$

10. **AR(1) +drift:**  $y_t = \mu + \phi_1 y_{t-1} + \epsilon_t$

Mean:

Using the lag operator,

$$\phi(L)y_t = \mu + \epsilon_t$$

where  $\phi(L) = 1 - \phi_1 L$ .

Multiply  $\phi(L)^{-1}$  on both sides. Then, when  $|\phi_1| < 1$ , we have:

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Taking the expectation on both sides,

$$\begin{aligned} E(y_t) &= \phi(L)^{-1}\mu + \phi(L)^{-1}E(\epsilon_t) \\ &= \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1} \end{aligned}$$

**Example: AR(2) Model:** Consider  $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$ .

1. The stationarity condition is: two solutions of  $x$  from  $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0$  are outside the unit circle.
2. Rewriting the AR(2) model,

$$(1 - \phi_1 L - \phi_2 L^2)y_t = \epsilon_t.$$

Let  $1/\alpha_1$  and  $1/\alpha_2$  be the solutions of  $\phi(x) = 0$ .

Then, the AR(2) model is written as:

$$(1 - \alpha_1 L)(1 - \alpha_2 L)y_t = \epsilon_t,$$

which is rewritten as:

$$\begin{aligned}y_t &= \frac{1}{(1 - \alpha_1 L)(1 - \alpha_2 L)} \epsilon_t \\ &= \left( \frac{\alpha_1 / (\alpha_1 - \alpha_2)}{1 - \alpha_1 L} + \frac{-\alpha_2 / (\alpha_1 - \alpha_2)}{1 - \alpha_2 L} \right) \epsilon_t\end{aligned}$$

### 3. Mean of AR(2) Model:

When  $y_t$  is stationary, i.e.,  $\alpha_1$  and  $\alpha_2$  are within the unit circle,

$$\mu = E(y_t) = E(\phi(L)\epsilon_t) = 0$$

#### 4. Autocovariance Function of AR(2) Model:

$$\begin{aligned}\gamma(\tau) &= E((y_t - \mu)(y_{t-\tau} - \mu)) = E(y_t y_{t-\tau}) \\ &= E((\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) y_{t-\tau}) \\ &= \phi_1 E(y_{t-1} y_{t-\tau}) + \phi_2 E(y_{t-2} y_{t-\tau}) + E(\epsilon_t y_{t-\tau}) \\ &= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}\end{aligned}$$

The initial condition is obtained by solving the following three equations:

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_\epsilon^2,$$

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1),$$

$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0).$$

Therefore, the initial conditions are given by:

$$\gamma(0) = \left( \frac{1 - \phi_2}{1 + \phi_2} \right) \frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2},$$

$$\gamma(1) = \frac{\phi_1}{1 - \phi_2} \gamma(0) = \left( \frac{\phi_1}{1 - \phi_2} \right) \left( \frac{1 - \phi_2}{1 + \phi_2} \right) \frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2}.$$

Given  $\gamma(0)$  and  $\gamma(1)$ , we obtain  $\gamma(\tau)$  as follows:

$$\gamma(\tau) = \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2), \quad \text{for } \tau = 2, 3, \dots$$

## 5. Another solution for $\gamma(0)$ :

From  $\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2$ ,

$$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1\rho(1) - \phi_2\rho(2)}$$

where

$$\rho(1) = \frac{\phi_1}{1 - \phi_2}, \quad \rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2 + (1 - \phi_2)\phi_2}{1 - \phi_2}.$$

## 6. Autocorrelation Function of AR(2) Model:

Given  $\rho(1)$  and  $\rho(2)$ ,

$$\rho(\tau) = \phi_1\rho(\tau - 1) + \phi_2\rho(\tau - 2), \quad \text{for } \tau = 3, 4, \dots,$$

## 7. Log-Likelihood Function — Innovation Form:

$$\log f(y_T, \dots, y_1) = \log f(y_2, y_1) + \sum_{t=3}^T \log f(y_t | y_{t-1}, \dots, y_1)$$

where

$$f(y_2, y_1) = \frac{1}{2\pi} \begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{vmatrix}^{-1/2} \exp\left(-\frac{1}{2}(y_1 \ y_2) \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right),$$
$$f(y_t | y_{t-1}, \dots, y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2\right).$$

Note as follows:

$$\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} = \gamma(0) \begin{pmatrix} 1 & \phi_1/(1 - \phi_2) \\ \phi_1/(1 - \phi_2) & 1 \end{pmatrix}.$$