

8. $\phi_{k,k}$ = **Partial Autocorrelation Coefficient of AR(2) Process:**

$$\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{pmatrix},$$

for $k = 1, 2, \dots$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Autocovariance Functions:

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1),$$

$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0),$$

$$\gamma(\tau) = \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2), \quad \text{for } \tau = 3, 4, \dots.$$

Autocorrelation Functions:

$$\rho(1) = \phi_1 + \phi_2\rho(1) = \frac{\phi_1}{1 - \phi_2},$$

$$\rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2,$$

$$\rho(\tau) = \phi_1\rho(\tau - 1) + \phi_2\rho(\tau - 2), \quad \text{for } \tau = 3, 4, \dots.$$

$$\phi_{1,1} = \rho(1) = \frac{\phi_1}{1 - \phi_2}$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \phi_2$$

$$\phi_{3,3} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}}$$

$$= \frac{(\rho(3) - \rho(1)\rho(2)) - \rho(1)^2(\rho(3) - \rho(1)) + \rho(2)\rho(1)(\rho(2) - 1)}{(1 - \rho(1)^2) - \rho(1)^2(1 - \rho(2)) + \rho(2)(\rho(1)^2 - \rho(2))} = 0.$$

9. **AR(2) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$

Mean:

Rewriting the AR(2)+drift model,

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$.

Under the stationarity assumption, we can rewrite the AR(2)+drift model as follows:

$$y_t = \phi(L)^{-1} \mu + \phi(L)^{-1} \epsilon_t.$$

Therefore,

$$E(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}E(\epsilon_t) = \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1 - \phi_2}$$

Example: AR(p) model: Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t$.

1. Variance of AR(p) Process:

Under the stationarity condition (i.e., the p solutions of x from $\phi(x) = 0$ are outside the unit circle),

$$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1 \rho(1) - \dots - \phi_p \rho(p)}.$$

Note that $\gamma(\tau) = \rho(\tau)\gamma(0)$.

Solve the following simultaneous equations for $\tau = 0, 1, \dots, p$:

$$\begin{aligned}\gamma(\tau) &= \mathbb{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathbb{E}(y_t y_{t-\tau}) \\ &= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \dots + \phi_p \gamma(\tau - p), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \dots + \phi_p \gamma(\tau - p) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}\end{aligned}$$

2. Estimation of AR(p) Model:

1. OLS:

$$\min_{\phi_1, \dots, \phi_p} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \dots - \phi_p y_{t-p})^2$$

2. MLE:

$$\min_{\phi_1, \dots, \phi_p} \log f(y_T, \dots, y_1)$$

where

$$\log f(y_T, \dots, y_1) = \log f(y_p, \dots, y_2, y_1) + \sum_{t=p+1}^T \log f(y_t | y_{t-1}, \dots, y_1),$$

$$f(y_p, \dots, y_2, y_1) = (2\pi)^{-p/2} |V|^{-1/2} \exp \left(-\frac{1}{2} (y_1 \ y_2 \ \dots \ y_p) V^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} \right)$$

$$V = \gamma(0) \begin{pmatrix} 1 & \rho(1) & \dots & \rho(p-2) & \rho(p-1) \\ \rho(1) & 1 & & \rho(p-3) & \rho(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(p-1) & \rho(p-2) & \dots & \rho(1) & 1 \end{pmatrix}$$

$$f(y_t | y_{t-1}, \dots, y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp \left(-\frac{1}{2\sigma_\epsilon^2} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \dots - \phi_p y_{t-p})^2 \right)$$

3. Yule-Walker (ユール・ウォーカー) Equation:

Multiply $y_{t-1}, y_{t-2}, \dots, y_{t-p}$ on both sides of $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t = y_t$, take expectations for each case, and divide by the sample variance $\hat{\gamma}(0)$.

$$\begin{pmatrix} 1 & \hat{\rho}(1) & \cdots & \hat{\rho}(p-2) & \hat{\rho}(p-1) \\ \hat{\rho}(1) & 1 & & \hat{\rho}(p-3) & \hat{\rho}(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \hat{\rho}(p-1) & \hat{\rho}(p-2) & \cdots & \hat{\rho}(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{pmatrix} = \begin{pmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(p) \end{pmatrix}$$

where

$$\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=\tau+1}^T (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu}), \quad \hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}.$$

3. **AR(p) + drift:** $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t$

Mean:

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$.

$$y_t = \phi(L)^{-1} \mu + \phi(L)^{-1} \epsilon_t$$

Taking the expectation on both sides,

$$\begin{aligned} E(y_t) &= \phi(L)^{-1}\mu + \phi(L)^{-1}E(\epsilon_t) = \phi(1)^{-1}\mu \\ &= \frac{\mu}{1 - \phi_1 - \phi_2 - \cdots - \phi_p} \end{aligned}$$

4. **Partial Autocorrelation of AR(p) Process:**

$$\phi_{k,k} = 0 \text{ for } k = p + 1, p + 2, \dots$$

1.3 MA Model

MA (Moving Average) Model:

1. MA(q)

$$y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q},$$

which is rewritten as:

$$y_t = \theta(L)\epsilon_t,$$

where

$$\theta(L) = 1 + \theta_1L + \theta_2L^2 + \cdots + \theta_qL^q.$$

2. Invertibility (反転可能性):

The q solutions of x from $\theta(x) = 1 + \theta_1x + \theta_2x^2 + \cdots + \theta_q x^q = 0$ are outside the unit circle.

\implies MA(q) model is rewritten as AR(∞) model.

Example: MA(1) Model: $y_t = \epsilon_t + \theta_1\epsilon_{t-1}$

1. Mean of MA(1) Process:

$$E(y_t) = E(\epsilon_t + \theta_1\epsilon_{t-1}) = E(\epsilon_t) + \theta_1E(\epsilon_{t-1}) = 0$$

2. Autocovariance Function of MA(1) Process:

$$\begin{aligned}\gamma(0) &= E(y_t^2) = E(\epsilon_t + \theta_1 \epsilon_{t-1})^2 = E(\epsilon_t^2 + 2\theta_1 \epsilon_t \epsilon_{t-1} + \theta_1^2 \epsilon_{t-1}^2) \\ &= E(\epsilon_t^2) + 2\theta_1 E(\epsilon_t \epsilon_{t-1}) + \theta_1^2 E(\epsilon_{t-1}^2) = (1 + \theta_1^2) \sigma_\epsilon^2\end{aligned}$$

$$\gamma(1) = E(y_t y_{t-1}) = E((\epsilon_t + \theta_1 \epsilon_{t-1})(\epsilon_{t-1} + \theta_1 \epsilon_{t-2})) = \theta_1 \sigma_\epsilon^2$$

$$\gamma(2) = E(y_t y_{t-2}) = E((\epsilon_t + \theta_1 \epsilon_{t-1})(\epsilon_{t-2} + \theta_1 \epsilon_{t-3})) = 0$$

3. Autocorrelation Function of MA(1) Process:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \begin{cases} \frac{\theta_1}{1 + \theta_1^2}, & \text{for } \tau = 1, \\ 0, & \text{for } \tau = 2, 3, \dots \end{cases}$$

Let x be $\rho(1)$.

$$\frac{\theta_1}{1 + \theta_1^2} = x, \quad \text{i.e.,} \quad x\theta_1^2 - \theta_1 + x = 0.$$

θ_1 should be a real number.

$$1 - 4x^2 > 0, \quad \text{i.e.,} \quad -\frac{1}{2} \leq \rho(1) \leq \frac{1}{2}.$$

4. Invertibility Condition of MA(1) Process:

$$\begin{aligned}\epsilon_t &= -\theta_1\epsilon_{t-1} + y_t \\ &= (-\theta_1)^2\epsilon_{t-2} + y_t + (-\theta_1)y_{t-1} \\ &= (-\theta_1)^3\epsilon_{t-3} + y_t + (-\theta_1)y_{t-1} + (-\theta_1)^2y_{t-2} \\ &\quad \vdots \\ &= (-\theta_1)^s\epsilon_{t-s} + y_t + (-\theta_1)y_{t-1} + (-\theta_1)^2y_{t-2} + \cdots + (-\theta_1)^{t-s+1}y_{t-s+1}\end{aligned}$$

When $(-\theta_1)^s \epsilon_{t-s} \rightarrow 0$, the MA(1) model is written as the AR(∞) model, i.e.,

$$y_t = -(-\theta_1)y_{t-1} - (-\theta_1)^2 y_{t-2} - \dots - (-\theta_1)^{t-s+1} y_{t-s+1} - \dots + \epsilon_t$$

5. Likelihood Function of MA(1) Process:

The autocovariance functions are: $\gamma(0) = (1 + \theta_1^2)\sigma_\epsilon^2$, $\gamma(1) = \theta_1\sigma_\epsilon^2$, and $\gamma(\tau) = 0$ for $\tau = 2, 3, \dots$.

The joint distribution of y_1, y_2, \dots, y_T is:

$$f(y_1, y_2, \dots, y_T) = \frac{1}{(2\pi)^{T/2}} |V|^{-1/2} \exp\left(-\frac{1}{2} Y' V^{-1} Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \quad V = \sigma_\epsilon^2 \begin{pmatrix} 1 + \theta_1^2 & \theta_1 & 0 & \cdots & 0 \\ \theta_1 & 1 + \theta_1^2 & \theta_1 & \ddots & \vdots \\ 0 & \theta_1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 + \theta_1^2 & \theta_1 \\ 0 & \cdots & 0 & \theta_1 & 1 + \theta_1^2 \end{pmatrix}.$$

6. **MA(1) +drift:** $y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1}$

Mean of MA(1) Process:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1 L$.

Taking the expectation,

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu.$$

Example: MA(2) Model: $y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$

1. Autocovariance Function of MA(2) Process:

$$\gamma(\tau) = \begin{cases} (1 + \theta_1^2 + \theta_2^2)\sigma_\epsilon^2, & \text{for } \tau = 0, \\ (\theta_1 + \theta_1\theta_2)\sigma_\epsilon^2, & \text{for } \tau = 1, \\ \theta_2\sigma_\epsilon^2, & \text{for } \tau = 2, \\ 0, & \text{otherwise.} \end{cases}$$

2. let $-1/\beta_1$ and $-1/\beta_2$ be two solutions of x from $\theta(x) = 0$.

For invertibility condition, both β_1 and β_2 should be less than one in absolute value.

Then, the MA(2) model is represented as:

$$\begin{aligned}y_t &= \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} \\ &= (1 + \theta_1L + \theta_2L^2)\epsilon_t \\ &= (1 + \beta_1L)(1 + \beta_2L)\epsilon_t\end{aligned}$$

AR(∞) representation of the MA(2) model is given by:

$$\begin{aligned}\epsilon_t &= \frac{1}{(1 + \beta_1L)(1 + \beta_2L)}y_t \\ &= \left(\frac{\beta_1/(\beta_1 - \beta_2)}{1 + \beta_1L} + \frac{-\beta_2/(\beta_1 - \beta_2)}{1 + \beta_2L} \right) y_t\end{aligned}$$

3. Likelihood Function:

$$f(y_1, y_2, \dots, y_T) = \frac{1}{(2\pi)^{T/2}} |V|^{-1/2} \exp\left(-\frac{1}{2} Y' V^{-1} Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \quad V = \sigma_\epsilon^2 \begin{pmatrix} 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 & \theta_2 & & 0 \\ \theta_1 + \theta_1\theta_2 & 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 & \ddots & \\ \theta_2 & \theta_1 + \theta_1\theta_2 & \ddots & \ddots & \theta_2 \\ & \ddots & \ddots & 1 + \theta_1^2 + \theta_2^2 & \theta_1 + \theta_1\theta_2 \\ 0 & & \theta_2 & \theta_1 + \theta_1\theta_2 & 1 + \theta_1^2 + \theta_2^2 \end{pmatrix}$$

4. **MA(2) +drift:** $y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1L + \theta_2L^2$.

Therefore,

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu$$

Example: MA(q) モデル: $y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$

1. Mean of MA(q) Process:

$$E(y_t) = E(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}) = 0$$

2. Autocovariance Function of MA(q) Process:

$$\gamma(\tau) = \begin{cases} \sigma_\epsilon^2(\theta_0\theta_\tau + \theta_1\theta_{\tau+1} + \cdots + \theta_{q-\tau}\theta_q) = \sigma_\epsilon^2 \sum_{i=0}^{q-\tau} \theta_i\theta_{\tau+i}, & \tau = 1, 2, \cdots, q, \\ 0, & \tau = q + 1, q + 2, \cdots, \end{cases}$$

where $\theta_0 = 1$.

3. MA(q) process is stationary.

4. **MA(q) +drift:** $y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \dots + \theta_q\epsilon_{t-q}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1L + \theta_2L^2 + \dots + \theta_qL^q$.

Therefore, we have:

$$E(y_t) = \mu + \theta(L)E(\epsilon_t) = \mu.$$