6. Summarizing the results up to now, $T(\hat{\phi}_1 - \phi_1)$, not $\sqrt{T}(\hat{\phi}_1 - \phi_1)$, has limiting distribution in the case of $\phi_1 = 1$.

$$T(\hat{\phi}_1 - \phi_1) = \frac{(1/T)\sum y_{t-1}\epsilon_t}{(1/T^2)\sum y_{t-1}^2} \longrightarrow \text{ a distribution.}$$

7. Basic Concepts of Random Walk Process:

   (a) Model: $y_t = y_{t-1} + \epsilon_t, \qquad y_0 = 0, \qquad \epsilon_t \sim N(0,1).$

   Then,

$$y_t = \epsilon_t + \epsilon_{t-1} + \cdots + \epsilon_1.$$

Therefore,

$$y_t \sim N(0, t).$$

$\Longrightarrow$ Nonstationary Process (i.e., variance depends on time t.)

Difference between y_s and y_t ($s > t$) is:

$$y_s - y_t = \epsilon_s + \epsilon_{s-1} + \cdots + \epsilon_{t+2} + \epsilon_{t+1}.$$

The distribution of $y_s - y_t$ is:

$$y_s - y_t \sim N(0, s - t).$$

(b) Rewrite as follows:

$$
\begin{aligned}
y_t &= y_{t-1} + \epsilon_t \\
&= y_{t-1} + e_{1,t} + e_{2,t} + \cdots + e_{N,t},
\end{aligned}
$$

where $\epsilon_t = e_{1,t} + e_{2,t} + \cdots + e_{N,t}$.

$e_{1,t}, e_{2,t}, \cdots, e_{N,t}$ are iid with $e_{i,t} \sim N(0, 1/N)$.

That is, suppose that there are N subperiods between time t and time $t + 1$.

The limit when $N \to \infty$ is a **continuous time (連続時間)** process known as **standard Brownian motion** or **Wiener process**.

The value of this process at time t is denoted by $W(t)$.

Definition:

Standard Brownian motion $W(t)$ denotes a continuous-time variable at time t and a stochastic function.

$W(t)$ for $t \in [0, 1]$ satisfies the following:

i. $W(0) = 0$

ii. For any time periods $0 \leq r_1 < r_2 < \cdots < r_k \leq 1$, $W(r_2) - W(r_1)$, $W(r_3) - W(r_2), \cdots, W(r_k) - W(r_{k-1})$ are independently multivariate normal with $W(s) - W(t) \sim N(0, s - t)$ for $s > t$.

iii. $W(t)$ is continuous in t with probability 1.

An example:

$$\sigma W(t) \sim N(0, \sigma^2 t),$$

which denotes the Brownian motion with variance σ^2.

Another example;

$$W(t)^2 \sim t \times \chi^2(1).$$

(c) Assume $\epsilon_t \sim$ iid $(0, \sigma_\epsilon^2)$. Define $X_T(r)$ for $r \in [0,1]$ as follows:

$$X_T(r) = \begin{cases} 0, & 0 \le r < \frac{1}{T} \\ \frac{\epsilon_1}{T}, & \frac{1}{T} \le r < \frac{2}{T} \\ \frac{\epsilon_1 + \epsilon_2}{T}, & \frac{2}{T} \le r < \frac{3}{T} \\ \quad \vdots & \quad \vdots \\ \frac{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_T}{T}, & r = 1 \end{cases}$$

Let $[Tr]$ be the largest integer which is less than or equal to $T \times r$.

$$X_T(r) \equiv \frac{1}{T} \sum_{t=1}^{[Tr]} \epsilon_t, \qquad \sqrt{T} X_T(r) \longrightarrow N(0, r\sigma_\epsilon^2).$$

Note that

$$\frac{1}{T} \sum_{t=1}^{[Tr]} \epsilon_t = \frac{[Tr]}{T} \frac{1}{[Tr]} \sum_{t=1}^{[Tr]} \epsilon_t,$$

$$\frac{[Tr]}{T} \longrightarrow r, \qquad \frac{1}{\sqrt{[Tr]}} \sum_{t=1}^{[Tr]} \epsilon_t \longrightarrow N(0, \sigma_\epsilon^2),$$

$$\sqrt{T} X_T(r) = \frac{[Tr]}{T} \sqrt{\frac{T}{[Tr]}} \frac{1}{\sqrt{[Tr]}} \sum_{t=1}^{[Tr]} \epsilon_t, \qquad \sqrt{\frac{T}{[Tr]}} \longrightarrow \frac{1}{\sqrt{r}}.$$

Therefore, we obtain:

$$\sqrt{T} X_T(r) \longrightarrow N(0, r\sigma_\epsilon^2).$$

Moreover, we have the following results:

$$\frac{\sqrt{T}\,(X_T(r_2) - X_T(r_1))}{\sigma_\epsilon} \longrightarrow N(0, r_2 - r_1),$$

$$\frac{\sqrt{T} X_T(r)}{\sigma_\epsilon} \longrightarrow W(r)$$

For example, consider:

$$X_T(1) = \frac{1}{T}\sum_{t=1}^{T}\epsilon_t.$$

Then,

$$\frac{\sqrt{T}X_T(1)}{\sigma_\epsilon} = \frac{1}{\sigma_\epsilon\sqrt{T}}\sum_{t=1}^{T}\epsilon_t \longrightarrow W(1) = N(0,1).$$

(d) Consider $y_t = y_{t-1} + \epsilon_t,\ y_0 = 0$ and $\epsilon_t \sim N(0, \sigma_\epsilon^2)$.

$X_T(r)$ is defined as follows:

$$X_T(r) = \begin{cases} 0, & 0 \le r < \dfrac{1}{T}, \\ \dfrac{y_1}{T}, & \dfrac{1}{T} \le r < \dfrac{2}{T}, \\ \dfrac{y_2}{T}, & \dfrac{2}{T} \le r < \dfrac{3}{T}, \\ \quad\vdots & \quad\vdots \\ \dfrac{y_{T-1}}{T}, & \dfrac{T-1}{T} \le r < 1, \\ \dfrac{y_T}{T}, & r = 1. \end{cases}$$

Define $S_T(r)$ as follows:

$$S_T(r) = \begin{cases} 0, & 0 \le r < \frac{1}{T}, \\ \frac{y_1^2}{T}, & \frac{1}{T} \le r < \frac{2}{T}, \\ \frac{y_2^2}{T}, & \frac{2}{T} \le r < \frac{3}{T}, \\ \vdots & \vdots \\ \frac{y_{T-1}^2}{T}, & \frac{T-1}{T} \le r < 1, \\ \frac{y_T^2}{T}, & r = 1. \end{cases}$$

To obtain $\int_0^1 X_T(r)\mathrm{d}r$ and $\int_0^1 S_T(r)\mathrm{d}r$, we compute a sum of rectangulars as follows:

$$\begin{aligned}\int_0^1 X_T(r)\mathrm{d}r &\approx \frac{y_1}{T}\left(\frac{2}{T}-\frac{1}{T}\right)+\frac{y_2}{T}\left(\frac{3}{T}-\frac{2}{T}\right)+\cdots+\frac{y_{T-1}}{T}\left(1-\frac{T-1}{T}\right)\\ &= \frac{y_1}{T^2}+\frac{y_2}{T^2}+\cdots+\frac{y_{T-1}}{T^2}=\frac{1}{T^2}\sum_{t=1}^{T} y_t,\end{aligned}$$

$$\begin{aligned}\int_0^1 S_T(r)\mathrm{d}r &\approx \frac{y_1^2}{T}\left(\frac{2}{T}-\frac{1}{T}\right)+\frac{y_2^2}{T}\left(\frac{3}{T}-\frac{2}{T}\right)+\cdots+\frac{y_{T-1}^2}{T}\left(1-\frac{T-1}{T}\right)\\ &= \frac{y_1^2}{T^2}+\frac{y_2^2}{T^2}+\cdots+\frac{y_{T-1}^2}{T^2}=\frac{1}{T^2}\sum_{t=1}^{T} y_t^2.\end{aligned}$$

We have already known that $\sqrt{T}X_T(r) \longrightarrow \sigma_\epsilon W(r)$.

Therefore,

$$\int_0^1 \sqrt{T}X_T(r)\mathrm{d}r \longrightarrow \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r.$$

That is,

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T} y_t \longrightarrow \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r.$$

From $S_T(r) \equiv \left(\sqrt{T} X_T(r)\right)^2$,

$$S_T(r) \longrightarrow \sigma_\epsilon^2 \left(W(r)\right)^2,$$

which is called the continuous mapping theorem.

(*) **Continuous Mapping Theorem (連続写像定理):**

if $x_T \longrightarrow x$ (convergence in distribution) and $g(\cdot)$ is a continuous function, then $g(x_T) \longrightarrow g(x)$ (convergence in distribution).

Threfore, we have the follwoing result:

$$\frac{1}{T^2}\sum_{t=1}^{T} y_t^2 \longrightarrow \int_0^1 S_T(r)\mathrm{d}r = \sigma_\epsilon^2 \int_0^1 (W(r))^2\,\mathrm{d}r.$$

(e) Decompose $T^{-3/2}\sum_{t=1}^{T} y_{t-1}$ as follows:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} = T^{-3/2}(\epsilon_1 + (\epsilon_1+\epsilon_2) + (\epsilon_1+\epsilon_2+\epsilon_3) + \cdots$$
$$+(\epsilon_1+\epsilon_2+\cdots+\epsilon_{T-1}))$$

$$= T^{-3/2}((T-1)\epsilon_1 + (T-2)\epsilon_2 + (T-3)\epsilon_3 + \cdots$$

$$+2\epsilon_{T-2} + \epsilon_{T-1})$$

$$= T^{-3/2}\sum_{t=1}^{T}(T-t)\epsilon_t = T^{-1/2}\sum_{t=1}^{T}\epsilon_t - T^{-3/2}\sum_{t=1}^{T}t\epsilon_t$$

We utilize the following fact:

$$\begin{pmatrix} T^{-1/2}\sum_{t=1}^{T}\epsilon_t \\ T^{-3/2}\sum_{t=1}^{T}t\,\epsilon_t \end{pmatrix} \longrightarrow N\left(\begin{pmatrix}0\\0\end{pmatrix}, \sigma_\epsilon^2\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{pmatrix}\right)$$

ϵ_t is stationary. $\Longrightarrow$ Apply CLT to $(1/T)\sum_{t=1}^{T}\epsilon_t$.

$t\epsilon_t/T$ is stationary. $\Longrightarrow$ Apply CLT to $(1/T)\sum_{t=1}^{T} t\epsilon_t/T$.

Using a matrix form, we ca rewrite as follows:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} = (1 \quad -1)\begin{pmatrix} T^{-1/2}\displaystyle\sum_{t=1}^{T}\epsilon_t \\ T^{-3/2}\displaystyle\sum_{t=1}^{T} t\,\epsilon_t \end{pmatrix}.$$

Then, the variance of $T^{-3/2}\sum_{t=1}^{T} y_{t-1}$ is given by:

$$\mathrm{V}\Big(T^{-3/2}\sum_{t=1}^{T} y_{t-1}\Big) = \sigma_\epsilon^2(1 \quad -1)\begin{pmatrix} 1 & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\sigma_\epsilon^2}{3}.$$

Therefore, $T^{-3/2}\sum_{t=1}^{T} y_{t-1} \sim N(0, \sigma_\epsilon^2/3)$.

We have already known:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} \longrightarrow \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r,$$

$$\frac{1}{T}\sum_{t=1}^{T} \epsilon_t \longrightarrow \sigma_\epsilon W(1).$$

That is, the following relationship holds:

$$\sigma_\epsilon \int_0^1 W(r)\mathrm{d}r \approx T^{-3/2}\sum_{t=1}^{T} y_{t-1} = T^{-1/2}\sum_{t=1}^{T}\epsilon_t - T^{-3/2}\sum_{t=1}^{T} t\epsilon_t$$

$$\approx \sigma_\epsilon W(1) - T^{-3/2}\sum_{t=1}^{T} t\epsilon_t$$

Therefore, we obatain the following result:

$$T^{-3/2}\sum_{t=1}^{T} t\epsilon_t \longrightarrow \sigma_\epsilon W(1) - \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r = N(0, \frac{\sigma_\epsilon^2}{3}).$$

(f) **Some Formulas:** Model: $y_t = y_{t-1} + \epsilon_t$.

i. $T^{-1/2}\sum_{t=1}^{T} \epsilon_t \longrightarrow \sigma_\epsilon W(1) = N(0, \sigma_\epsilon^2)$

ii. $T^{-1}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow \frac{1}{2}\sigma_\epsilon^2\Big((W(1))^2 - 1\Big) = \frac{1}{2}\sigma_\epsilon^2\Big(\chi^2(1) - 1\Big)$

Note that we obtain $(W(1))^2 \sim \chi^2(1)$ from $W(1) = N(0, 1)$.

iii. $T^{-3/2}\sum_{t=1}^{T} t\,\epsilon_t \longrightarrow \sigma_\epsilon W(1) - \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r = N(0, \frac{\sigma_\epsilon^2}{3})$

iv. $T^{-3/2}\sum_{t=1}^{T} y_{t-1} \longrightarrow \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r = N(0, \frac{\sigma_\epsilon^2}{3})$

v. $T^{-2}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \sigma_\epsilon^2 \int_0^1 (W(r))^2\,\mathrm{d}r$

vi. $T^{-5/2}\sum_{t=1}^{T} t\, y_{t-1} \longrightarrow \sigma_\epsilon \int_0^1 r\, W(r)\mathrm{d}r$

vii. $T^{-3}\sum_{t=1}^{T} t\, y_{t-1}^2 \longrightarrow \sigma_\epsilon^2 \int_0^1 r\ (W(r))^2\,\mathrm{d}r$

viii. $T^{-(\nu+1)}\sum_{t=1}^{T} t^\nu \longrightarrow \frac{1}{\nu+1}$ for $\nu = 0, 1, \cdots$.

8. Asymptotic Distribution of AR(1) Model:

(a) **True Model: $y_t = y_{t-1} + \epsilon_t$ and Estimated Model: $y_t = \phi_1 y_{t-1} + \epsilon_t$**

OLSE of ϕ_1, denoted by $\hat{\phi}_1$, is given by:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^{T} y_{t-1} y_t}{\sum_{t=1}^{T} y_{t-1}^2} = \phi_1 + \frac{\sum_{t=1}^{T} y_{t-1} \epsilon_t}{\sum_{t=1}^{T} y_{t-1}^2}$$

Using $\phi_1 = 1$ and some formulas shown above, we obtain:

$$T(\hat{\phi}_1 - 1) = \frac{T^{-1} \sum_{t=1}^{T} y_{t-1} u_t}{T^{-2} \sum_{t=1}^{T} y_{t-1}^2} \longrightarrow \frac{\frac{1}{2}\Big((W(1))^2 - 1\Big)}{\int_0^1 (W(r))^2 \, \mathrm{d}r}$$

Remember that

$$T^{-1}\sum_{t=1}^{T} y_{t-1}u_t \;\longrightarrow\; \frac{1}{2}\sigma_\epsilon^2\Big((W(1))^2 - 1\Big)$$

and

$$T^{-2}\sum_{t=1}^{T} y_{t-1}^2 \;\longrightarrow\; \sigma_\epsilon^2\int_0^1 (W(r))^2\,\mathrm{d}r,$$

where $(W(1))^2 = \chi^2(1)$.

We say that $\hat{\phi}_1$ is **super-consistent (超一致性)** or ***T*-consistent**.

Remember that when $|\phi_1| < 1$ we have $\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$, and in this case we say that $\hat{\phi}_1$ is $\boldsymbol{\sqrt{T}}$**-consistent**.

Conventional t test statistic is given by:

$$t_T = \frac{\hat{\phi}_1 - 1}{s_\phi},$$

where

$$s_\phi = \left(s_T^2 \Big/ \sum_{t=1}^{T} y_{t-1}^2 \right)^{1/2} \quad \text{and} \quad s_T^2 = \frac{1}{T-1} \sum_{t=1}^{T} (y_t - \hat{\phi}_1 y_{t-1})^2.$$

Next, consider t statistic.

The t test statistic, denoted by t_T, is represented as follows:

$$t_T = \frac{\hat{\phi}_1 - 1}{s_\phi} = \frac{T(\hat{\phi}_1 - 1)}{T s_\phi}$$

The denominator is:

$$\begin{aligned} T s_\phi &= \left(s_T^2 \Big/ \frac{1}{T^2} \sum_{t=1}^{T} y_{t-1}^2 \right)^{1/2} \\ &\longrightarrow \left(\sigma_\epsilon^2 \Big/ \left(\sigma_\epsilon^2 \int_0^1 (W(r))^2 \, \mathrm{d}r \right) \right)^{1/2} = \left(\int_0^1 (W(r))^2 \, \mathrm{d}r \right)^{-1/2}, \end{aligned}$$

where $s^2 \longrightarrow \sigma_\epsilon^2$ is utilized.

Therefore, we have the following asymptotic distribution:

$$
\begin{aligned}
t_T = \frac{\hat{\phi}_1 - 1}{s_\phi} \longrightarrow & \frac{\frac{1}{2}\left((W(1))^2 - 1\right)}{\displaystyle\int_0^1 (W(r))^2 \, \mathrm{d}r} \Bigg/ \left(\int_0^1 (W(r))^2 \, \mathrm{d}r\right)^{-1/2} \\
= & \frac{\frac{1}{2}\left((W(1))^2 - 1\right)}{\left(\displaystyle\int_0^1 (W(r))^2 \, \mathrm{d}r\right)^{1/2}}.
\end{aligned}
$$

Therefore, the distribution of the t_T statistic shown above is different from the t distribution.