(b) $\boldsymbol{H_0 : y_t = y_{t-1} + \epsilon_t}$ **and** $\boldsymbol{H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{|\phi_1| < 1}$

$$\begin{pmatrix} \hat{\alpha}_0 \\ \hat{\phi}_1 \end{pmatrix} = \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_t \\ \sum y_{t-1} y_t \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1} \epsilon_t \end{pmatrix}$$

In the true model, $\alpha_0 = 0$ and $\phi_1 = 1$.

$$\begin{pmatrix} \hat{\alpha}_0 \\ \hat{\phi}_1 - 1 \end{pmatrix} = \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1}\epsilon_t \end{pmatrix}$$

$$= \begin{pmatrix} O_p(T) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) \end{pmatrix}^{-1} \begin{pmatrix} O_p(T^{1/2}) \\ O_p(T) \end{pmatrix}$$

(*) For random variable x and constant k, $x = O_p(k)$ implies that x/k converges in distribution.

To change each element of the matrices to $O_p(1)$, we use the following matrix:

$$\Gamma = \begin{pmatrix} T^{1/2} & 0 \\ 0 & T \end{pmatrix}.$$

Multiplying the above matrix from the left, we obtain the following:

$$\Gamma\begin{pmatrix}\hat{\alpha}_0\\ \hat{\phi}_1-1\end{pmatrix}=\begin{pmatrix}T^{1/2}\hat{\alpha}_0\\ T(\hat{\phi}_1-1)\end{pmatrix}=\Gamma\begin{pmatrix}O_p(T) & O_p(T^{3/2})\\ O_p(T^{3/2}) & O_p(T^2)\end{pmatrix}^{-1}\Gamma\Gamma^{-1}\begin{pmatrix}O_p(T^{1/2})\\ O_p(T)\end{pmatrix}$$

$$=\left(\Gamma^{-1}\begin{pmatrix}O_p(T) & O_p(T^{3/2})\\ O_p(T^{3/2}) & O_p(T^2)\end{pmatrix}\Gamma^{-1}\right)^{-1}\Gamma^{-1}\begin{pmatrix}O_p(T^{1/2})\\ O_p(T)\end{pmatrix}$$

$$=\left(\Gamma^{-1}\begin{pmatrix}T & \sum y_{t-1}\\ \sum y_{t-1} & \sum y_{t-1}^2\end{pmatrix}\Gamma^{-1}\right)^{-1}\Gamma^{-1}\begin{pmatrix}\sum \epsilon_t\\ \sum y_{t-1}\epsilon_t\end{pmatrix}$$

$$=\begin{pmatrix}1 & T^{-3/2}\sum y_{t-1}\\ T^{-3/2}\sum y_{t-1} & T^{-2}\sum y_{t-1}^2\end{pmatrix}^{-1}\begin{pmatrix}T^{-1/2}\sum \epsilon_t\\ T^{-1}\sum y_{t-1}\epsilon_t\end{pmatrix}.$$

Each matrix converges in distribution as follows:

$$\begin{pmatrix} 1 & T^{-3/2}\sum y_{t-1} \\ T^{-3/2}\sum y_{t-1} & T^{-2}\sum y_{t-1}^2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r \\ \sigma_\epsilon \int_0^1 W(r)\mathrm{d}r & \sigma_\epsilon^2 \int_0^1 (W(r))^2\,\mathrm{d}r \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W(r)\mathrm{d}r \\ \int_0^1 W(r)\mathrm{d}r & \int_0^1 (W(r))^2\,\mathrm{d}r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix},$$

$$\begin{pmatrix} T^{-1/2}\sum \epsilon_t \\ T^{-1}\sum y_{t-1}\epsilon_t \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_\epsilon W(1) \\ \frac{1}{2}\sigma_\epsilon^2 \left((W(1))^2 - 1\right) \end{pmatrix} = \sigma_\epsilon \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} W(1) \\ \frac{1}{2}\left((W(1))^2 - 1\right) \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} T^{1/2}\hat{\alpha}_0 \\ T(\hat{\phi}_1 - 1) \end{pmatrix} \longrightarrow \left[\begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W(r)\mathrm{d}r \\ \int_0^1 W(r)\mathrm{d}r & \int_0^1 (W(r))^2\,\mathrm{d}r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \right]^{-1}$$

$$\times \sigma_\epsilon \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} W(1) \\ \frac{1}{2}\left((W(1))^2 - 1\right) \end{pmatrix}.$$

Finally, $T(\hat{\phi}_1 - 1)$ converges to the following distribution:

$$T(\hat{\phi}_1 - 1) \longrightarrow \frac{\frac{1}{2}\left((W(1))^2 - 1\right) - W(1) \displaystyle\int_0^1 W(r)\mathrm{d}r}{\displaystyle\int_0^1 (W(r))^2\,\mathrm{d}r - \left(\int_0^1 W(r)\mathrm{d}r\right)^2}.$$

The t test statistic is:

$$t_T = \frac{\hat{\phi}_1 - 1}{\left(s_\phi^2\right)^{1/2}} = \frac{T(\hat{\phi}_1 - 1)}{\left(T^2 s_\phi^2\right)^{1/2}},$$

where

$$s_\phi^2 = s_T^2 \,(\,0 \quad 1\,)\begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$s_T^2 = \frac{1}{T-2}\sum_{t=1}^{T}(y_t - \hat{\alpha}_0 - \hat{\phi}_1 y_{t-1})^2.$$

The denominator $T^2 s_\phi^2$ converges in distribution as follows:

$$T^2 s_\phi^2 \longrightarrow \sigma_\epsilon^2 \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W(r)\mathrm{d}r \\ \int_0^1 W(r)\mathrm{d}r & \int_0^1 (W(r))^2 \,\mathrm{d}r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\epsilon \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\int_0^1 (W(r))^2 \,\mathrm{d}r - \left(\int_0^1 W(r)\mathrm{d}r \right)^2}$$

Thus, the t test statistic converges to the following distribution:

$$t_T \longrightarrow \frac{\frac{1}{2}\left((W(1))^2 - 1\right) - W(1) \int_0^1 W(r)\mathrm{d}r}{\left(\int_0^1 (W(r))^2 \,\mathrm{d}r - \left(\int_0^1 W(r)\mathrm{d}r\right)^2\right)^{1/2}}.$$

(c) $\boldsymbol{H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t}$ **and** $\boldsymbol{H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t}$ **for** $|\boldsymbol{\phi_1}| < 1$

The model is written as follows:

$$\begin{aligned} y_t &= y_0 + \alpha_0 t + (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t) \\ &= y_0 + \alpha_0 t + u_t, \end{aligned}$$

where $u_t = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_t$.

○ For $\sum_{t=1}^{T} y_{t-1}$,

$$\sum_{t=1}^{T} y_{t-1} = \sum_{t=1}^{T} y_0 + \sum_{t=1}^{T} \alpha_0(t-1) + \sum_{t=1}^{T} u_{t-1}$$
$$= O_p(T) + O_p(T^2) + O_p(T^{3/2}).$$

Therefore, we obtain:

$$T^{-2} \sum_{t=1}^{T} y_{t-1} \longrightarrow \frac{\alpha_0}{2}.$$

○ For $\sum_{t=1}^{T} y_{t-1}^2$,

$$\sum_{t=1}^{T} y_{t-1}^2 = \sum_{t=1}^{T} (y_0 + \alpha_0(t-1) + u_{t-1})^2$$

$$= \sum_{t=1}^{T} y_0^2 + \sum_{t=1}^{T} \alpha_0^2(t-1)^2 + \sum_{t=1}^{T} u_{t-1}^2 + \sum_{t=1}^{T} 2y_0\alpha_0(t-1) + \sum_{t=1}^{T} 2y_0 u_{t-1} + \sum_{t=1}^{T} 2\alpha_0(t-1)u_{t-1}$$

$$= O_p(T) + O_p(T^3) + O_p(T^2) + O_p(T^2) + O_p(T^{3/2}) + O_p(T^{5/2})$$

Therefore, we have:

$$T^{-3} \sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \frac{\alpha_0^2}{3}$$

○ For $\sum_{t=1}^{T} y_{t-1}\epsilon_t$,

$$\begin{aligned}\sum_{t=1}^{T} y_{t-1}\epsilon_t &= \sum_{t=1}^{T} \left(y_0 + \alpha_0(t-1) + u_{t-1}\right)\epsilon_t \\ &= \sum_{t=1}^{T} y_0\epsilon_t + \sum_{t=1}^{T} \alpha_0(t-1)\epsilon_t + \sum_{t=1}^{T} u_{t-1}\epsilon_t \\ &= O_p(T^{1/2}) + O_p(T^{3/2}) + O_p(T).\end{aligned}$$

Therefore, we have:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0, \frac{\alpha_0^2}{3}\sigma^2\epsilon).$$

Therefore, OLSE is:

$$\begin{pmatrix} \hat{\alpha}_0 - \alpha_0 \\ \hat{\phi}_1 - 1 \end{pmatrix} = \begin{pmatrix} T & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \epsilon_t \\ \sum y_{t-1}\epsilon_t \end{pmatrix}$$
$$= \begin{pmatrix} O_p(T) & O_p(T^2) \\ O_p(T^2) & O_p(T^3) \end{pmatrix}^{-1} \begin{pmatrix} O_p(T^{1/2}) \\ O_p(T^{3/2}) \end{pmatrix}.$$

Set:

$$\Gamma = \begin{pmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{pmatrix}.$$

Multiplying Γ from the left,

$$\begin{pmatrix} T^{1/2}(\hat{\alpha}_0 - \alpha_0) \\ T^{3/2}(\hat{\phi}_1 - 1) \end{pmatrix} \longrightarrow N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma_\epsilon^2 \begin{pmatrix} 1 & \dfrac{\alpha_0}{2} \\ \dfrac{\alpha_0}{2} & \dfrac{\alpha_0^2}{3} \end{pmatrix} \right).$$

(d) $\boldsymbol{H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t}$ **and**

$\boldsymbol{H_1 : y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{|\phi_1| < 1}$

(abbr.)

9. The distributions of the t statistic: $\dfrac{\hat{\phi}_1 - 1}{s_\phi}$

t Distribution

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.49	−2.06	−1.71	−1.32	1.32	1.71	2.06	2.49
50	−2.40	−2.01	−1.68	−1.30	1.30	1.68	2.01	2.40
100	−2.36	−1.98	−1.66	−1.29	1.29	1.66	1.98	2.36
250	−2.34	−1.97	−1.65	−1.28	1.28	1.65	1.97	2.34
500	−2.33	−1.96	−1.65	−1.28	1.28	1.65	1.96	2.33
∞	−2.33	−1.96	−1.64	−1.28	1.28	1.64	1.96	2.33

(a) $\boldsymbol{H_0: \ y_t = y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1: \ y_t = \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$ **or** $\boldsymbol{-1 < \phi_1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00

(b) $\boldsymbol{H_0: \ y_t = y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1: \ y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$ **or** $\boldsymbol{-1 < \phi_1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−3.75	−3.33	−3.00	−2.63	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60

(d) $\boldsymbol{H_0 : \; y_t = \alpha_0 + y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1 : \; y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$ **or** $\boldsymbol{-1 < \phi_1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

3.2 Serially Correlated Errors

Consider the case where the error term is serially correlated.

3.2.1 Augmented Dickey-Fuller (ADF) Test

Consider the following AR(p) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \qquad \epsilon_t \sim \text{iid}(0, \sigma_\epsilon^2),$$

which is rewritten as:

$$\phi(L) y_t = \epsilon_t.$$

When the above model has a unit root, we have $\phi(1) = 0$, i.e., $\phi_1 + \phi_2 + \cdots + \phi_p = 1$.

The above AR(p) model is written as:

$$y_t = \rho y_{t-1} + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + +\delta_{p-1} \Delta y_{t-p+1} + \epsilon_t,$$

where $\rho = \phi_1 + \phi_2 + \cdots + \phi_p$ and $\delta_j = -(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p)$.

The null and alternative hypotheses are:

$$H_0 : \rho = 1 \text{ (Unit root)},$$

$$H_1 : \rho < 1 \text{ (Stationary)}.$$

Use the t test, where we have the same asymptotic distributions.

We can utilize the same tables as before.

Choose p by AIC or SBIC.

Use $N(0,1)$ to test $H_0:\ \delta_j = 0$ against $H_1:\ \delta_j \neq 0$ for $j = 1, 2, \cdots, p-1$.

Reference

Kurozumi (2008) "Economic Time Series Analysis and Unit Root Tests: Development and Perspective," *Japan Statistical Society*, Vol.38, Series J, No.1, pp.39 – 57.

Download the above paper from:

http://ci.nii.ac.jp/vol_issue/nels/AA11989749/ISS0000426576_ja.html