Remark: The regression model with AR(1) error is:

$$y_t = x_t\beta + u_t, \qquad u_t = \rho u_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{ iid } N(0, \sigma_\epsilon^2).$$

$$\mathrm{V}(u) = \sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 & & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{n-3} \\ \rho^3 & \rho^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \rho \\ \rho^{n-1} & \rho^{n-2} & \cdots & \rho^2 & \rho & 1 \end{pmatrix} = \sigma^2\Omega, \qquad \text{where } \sigma^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}.$$

where $\mathrm{Cov}(u_i, u_j) = \mathrm{E}(u_i u_j) = \sigma^2\rho^{|i-j|}$, i.e., the ith row and jth column of Ω is $\rho^{|i-j|}$.

The regression model with AR(1) error is: $y = X\beta + u, \qquad u \sim N(0, \sigma^2\Omega)$.

There exists P which satisfies that $\Omega = PP'$, because Ω is a positive definite matrix.

Multiply P^{-1} on both sides from the left.

$$P^{-1}y = P^{-1}X\beta + P^{-1}u \qquad \Longrightarrow \qquad y^* = X^*\beta + u^* \text{ and } u^* \sim N(0, \sigma^2 I_n)$$

$$\Longrightarrow \qquad \text{Apply OLS.}$$

$$
y^* = \begin{pmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}\,y_1 \\ y_2 - \rho y_1 \\ \vdots \\ y_n - \rho y_{n-1} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & \cdots & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = P^{-1} y
$$

$$
X^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2}\,x_1 \\ x_2 - \rho x_1 \\ \vdots \\ x_n - \rho x_{n-1} \end{pmatrix} = P^{-1} X
$$

$\Longrightarrow$ Check $P^{-1}\Omega P^{-1\prime} = aI_n$, where a is constant.

9.6 MLE: Regression Model with Heteroscedastic Errors

In the case where the error term depends on the other exogenous variables, the regression model is written as follows:

$$y_i = x_i\beta + u_i, \qquad u_i \sim \text{ id } N(0, \sigma_i^2), \qquad \sigma_i^2 = (z_i\alpha)^2.$$

The joint distribution of $u_n, u_{n-1}, \cdots, u_1$, denoted by $f_u(\cdot;\cdot)$, is given by:

$$\begin{aligned}
\log f_u(u_n, u_{n-1}, \cdots, u_1; \sigma_1^2, \cdots, \sigma_n^2) &= \sum_{i=1}^{n} \log f_u(u_t; \sigma_i^2) \\
&= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\log(\sigma_i^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{u_i}{\sigma_i}\right)^2 \\
&= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\log(z_i\alpha)^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{u_i}{z_i\alpha}\right)^2
\end{aligned}$$

By the transformation of variables from $u_n, u_{n-1}, \cdots, u_1$ to $y_n, y_{n-1}, \cdots, y_1$, the log-likelihood function is:

$$
\begin{aligned}
L(\alpha, \beta; y_n, y_{n-1}, \cdots, y_1) &= \log f_y(y_n, y_{n-1}, \cdots, y_1; \alpha, \beta) \\
&= \log f_u(y_n - x_n\beta, y_{n-1} - x_{n-1}\beta, \cdots, y_1 - x_1\beta; \sigma_i^2) \left| \frac{\partial u}{\partial y'} \right| \\
&= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n} \log(z_i\alpha)^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i - x_i\beta}{z_i\alpha}\right)^2
\end{aligned}
$$

$\Longrightarrow$ Maximize the above log-likelihood function with respect to β and α.

10 Asymptotic Theory

1. **Definition: Convergence in Distribution (分布収束)**

   A series of random variables X_1, X_2, $\cdots$, X_n, $\cdots$ have distribution functions F_1, F_2, $\cdots$, respectively.

   If

   $$\lim_{n\to\infty} F_n = F,$$

   then we say that a series of random variables X_1, X_2, $\cdots$ converges to F in distribution.

2. **Consistency (一致性):**

   (a) **Definition: Convergence in Probability (確率収束)**

   Let $\{Z_n :\ n = 1, 2, \cdots\}$ be a series of random variables.

If the following holds,

$$\lim_{n\to\infty} P(|Z_n - \theta| < \epsilon) = 1,$$

for any positive ϵ, then we say that Z_n converges to θ in probability.

θ is called a **probability limit (確率極限)** of Z_n.

$$\text{plim}\, Z_n = \theta.$$

(b) Let $\hat{\theta}_n$ be an estimator of parameter θ.

If $\hat{\theta}_n$ converges to θ in probability, we say that $\hat{\theta}_n$ is a consistent estimator of θ.

3. A General Case of **Chebyshev's Inequality:**

For $g(X) \geq 0$,

$$P(g(X) \geq k) \leq \frac{\mathrm{E}(g(X))}{k},$$

where k is a positive constant.

4. **Example:** For a random variable X, set $g(X) = (X-\mu)'(X-\mu)$, $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \Sigma$.

   Then, we have the following inequality:

$$P((X-\mu)'(X-\mu) \geq k) \leq \frac{\mathrm{tr}(\Sigma)}{k}.$$

   Note as follows:

$$\begin{aligned}\mathrm{E}((X-\mu)'(X-\mu)) &= \mathrm{E}\Big(\mathrm{tr}((X-\mu)'(X-\mu))\Big) = \mathrm{E}\Big(\mathrm{tr}((X-\mu)(X-\mu)')\Big) \\ &= \mathrm{tr}\Big(\mathrm{E}((X-\mu)(X-\mu)')\Big) = \mathrm{tr}(\Sigma).\end{aligned}$$

5. **Example 1 (Univariate Case):**

Suppose that $X_i \sim (\mu, \sigma^2)$, $i = 1, 2, \cdots, n$.

Then, the sample average $\overline{X}$ is a consistent estimator of μ.

Proof:

Note that $g(\overline{X}) = (\overline{X} - \mu)^2$, $\epsilon^2 = k$, $\mathrm{E}(g(\overline{X})) = \mathrm{V}(\overline{X}) = \dfrac{\sigma^2}{n}$.

Use Chebyshev's inequality.

If $n \longrightarrow \infty$,

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \longrightarrow 0, \qquad \text{for any } \epsilon.$$

That is. for any ϵ,

$$\lim_{n\to\infty} P(|\overline{X} - \mu| < \epsilon) = 1.$$

$\Longrightarrow$ **Chebyshev's inequality**

6. **Example 2 (Multivariate Case):**

Suppose that $X_i \sim (\mu, \Sigma)$, $i = 1, 2, \cdots, n$.

Then, the sample average $\overline{X}$ is a consistent estimator of μ.

Proof:

Note that $g(\overline{X}) = (\overline{X} - \mu)'(\overline{X} - \mu)$, $\epsilon^2 = k$, $\mathrm{E}(g(\overline{X})) = \mathrm{V}(\overline{X}) = \dfrac{1}{n}\Sigma$.

Use Chebyshev's inequality.

If $n \longrightarrow \infty$,

$$P((\overline{X} - \mu)'(\overline{X} - \mu) \geq k) = P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\mathrm{tr}(\Sigma)}{n\epsilon^2} \longrightarrow 0, \quad \text{for any positive } \epsilon.$$

That is. for any positive ϵ, $\lim_{n\to\infty} P((\overline{X} - \mu)'(\overline{X} - \mu) < k) = 1$.

Note that $|\overline{X} - \mu| = \sqrt{(\overline{X} - \mu)'(\overline{X} - \mu)}$, which is the distance between X and μ.

$\Longrightarrow$ **Chebyshev's inequality**