7. **Some Formulas:**

Let X_n and Y_n be the random variables which satisfy $\text{plim}\, X_n = c$ and $\text{plim}\, Y_n = d$. Then,

(a) $\text{plim}\,(X_n + Y_n) = c + d$

(b) $\text{plim}\, X_n Y_n = cd$

(c) $\text{plim}\, X_n/Y_n = c/d$ for $d \neq 0$

(d) $\text{plim}\, g(X_n) = g(c)$ for a function $g(\cdot)$

$\Longrightarrow$ **Slutsky's Theorem (スルツキー定理)**

8. **Central Limit Theorem (中心極限定理)**

Univariate Case: $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \sigma^2)$.

Then,

$$\frac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \longrightarrow N(0, 1),$$

which implies

$$\sqrt{n}(\overline{X} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \sigma^2).$$

Multivariate Case: $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \Sigma)$.

Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \Sigma)$$

9. **Central Limit Theorem (Generalization)**

$X_1,\ X_2,\ \cdots,\ X_n$ are mutually independently and identically distributed as $X_i \sim (\mu,\ \Sigma_i)$.

Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \longrightarrow N(0, \Sigma),$$

where

$$\Sigma = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \Sigma_i \right).$$

10. **Definition:** Let $\hat{\theta}_n$ be a consistent estimator of θ.

Suppose that $\sqrt{n}(\hat{\theta}_n - \theta)$ converges to $N(0, \Sigma)$ in distribution.

Then, we say that $\hat{\theta}_n$ has an **asymptotic distribution (漸近分布)**: $N(\theta, \Sigma/n)$.

11. $X_1, X_2, \cdots, X_n$ are random variables with density function $f(x;\theta)$.

Let $\hat{\theta}_n$ be a maximum likelihood estimator of θ.

Then, under some **regularity conditions**. $\hat{\theta}_n$ is a consistent estimator of θ and the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is given by: $N\left(0, \lim\left(\dfrac{I(\theta)}{n}\right)^{-1}\right)$.

12. **Regularity Conditions:**

(a) The domain of X_i does not depend on θ.

(b) There exists at least third-order derivative of $f(x;\theta)$ with respect to θ, and their derivatives are finite.

13. Thus, MLE is

(i) consistent,

(ii) asymptotically normal, and

(iii) asymptotically efficient.

11 Consistency and Asymptotic Normality of OLSE

Regression model: $y = X\beta + u, \qquad u \sim (0, \sigma^2 I_n).$

Consistency:

1. Let $\hat{\beta}_n = (X'X)^{-1}X'y$ be the OLS with sample size n.

   Consistency: As n is large, $\hat{\beta}_n$ converges to β.

2. Assume the stationarity assumption for X, i.e.,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

   Then, we have the following result:

$$\frac{1}{n}X'u \longrightarrow 0.$$

Proof:

According to Chebyshev's inequality, for $g(Z) \geq 0$,

$$P(g(Z) \geq k) \leq \frac{\mathrm{E}(g(Z))}{k},$$

where k is a positive constant.

Set $g(Z) = Z'Z$, and $Z = \frac{1}{n}X'u$.

Apply Chebyshev's inequality.

$$\begin{aligned}\mathrm{E}\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u\Big) &= \frac{1}{n^2}\mathrm{E}\Big(u'XX'u\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(u'XX'u)\Big) = \frac{1}{n^2}\mathrm{E}\Big(\mathrm{tr}(XX'uu')\Big) \\ &= \frac{1}{n^2}\mathrm{tr}\Big(XX'\mathrm{E}(uu')\Big) = \frac{\sigma^2}{n^2}\mathrm{tr}(XX') = \frac{\sigma^2}{n^2}\mathrm{tr}(X'X) = \frac{\sigma^2}{n}\mathrm{tr}(\frac{1}{n}X'X).\end{aligned}$$

Therefore,

$$P\Big((\frac{1}{n}X'u)'\frac{1}{n}X'u \geq k\Big) \leq \frac{\sigma^2}{nk}\mathrm{tr}(\frac{1}{n}X'X) \longrightarrow 0 \times \mathrm{tr}(M_{xx}) = 0.$$

Note that from the assumption,

$$\frac{1}{n}X'X \longrightarrow M_{xx}.$$

Therefore, we have:

$$(\frac{1}{n}X'u)'\frac{1}{n}X'u \longrightarrow 0,$$

which implies:

$$\frac{1}{n}X'u \longrightarrow 0,$$

because $(\frac{1}{n}X'u)'\frac{1}{n}X'u$ indicates a quadratic form.

3. Note that $\frac{1}{n}X'X \longrightarrow M_{xx}$ results in $(\frac{1}{n}X'X)^{-1} \longrightarrow M_{xx}^{-1}$.

$\Longrightarrow$ Slutsky's Theorem

(*) **Slutsky's Theorem** $g(\hat{\theta}) \longrightarrow g(\theta)$, when $\hat{\theta} \longrightarrow \theta$.

4. OLS is given by:

$$\hat{\beta}_n = \beta + (X'X)^{-1}X'u = \beta + (\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'u).$$

Therefore,

$$\hat{\beta}_n \longrightarrow \beta + M_{xx}^{-1} \times 0 = \beta$$

Thus, OLSE is a consitent estimator.

Asymptotic Normality:

1. Asymptotic Normality of OLSE

$$\sqrt{n}(\hat{\beta}_n - \beta) \longrightarrow N(0.\sigma^2 M_{xx}^{-1}), \quad \text{when } n \longrightarrow \infty.$$

2. **Central Limit Theorem:** Greenberg and Webster (1983)

   $Z_1, Z_2, \cdots, Z_n$ are mutually indelendently distributed with mean μ and variance Σ_i.

   Then, we have the following result:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_i-\mu) \longrightarrow N(0,\Sigma),$$

   where

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\Sigma_i\right).$$

   The distribution of Z_i is not assumed.

3. Define $Z_i = x_i' u_i$. Then, $\Sigma_i = \mathrm{Var}(Z_i) = \sigma^2 x_i' x_i$.

4. Σ is defined as:

$$\Sigma = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma^2 x_i' x_i\right) = \sigma^2 \lim_{n\to\infty}\left(\frac{1}{n}X'X\right) = \sigma^2 M_{xx},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

5. Applying Central Limit Theorem (Greenberg and Webster (1983), we obtain the following:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i' u_i = \frac{1}{\sqrt{n}}X'u \longrightarrow N(0, \sigma^2 M_{xx}).$$

On the other hand, from $\hat{\beta}_n = \beta + (X'X)^{-1}X'u$, we can rewrite as:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u.$$

$$
\begin{aligned}
\mathrm{Var}\Big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\Big) &= \mathrm{E}\Big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\Big(\big(\frac{1}{n}X'X\big)^{-1}\frac{1}{\sqrt{n}}X'u\Big)'\Big) \\
&= \big(\frac{1}{n}X'X\big)^{-1}\Big(\frac{1}{n}X'\mathrm{E}(uu')X\Big)\big(\frac{1}{n}X'X\big)^{-1} \\
&= \sigma^2\big(\frac{1}{n}X'X\big)^{-1} \longrightarrow \sigma^2 M_{xx}^{-1}.
\end{aligned}
$$

Therefore,

$$
\sqrt{n}(\hat{\beta}-\beta) \longrightarrow N(0, \sigma^2 M_{xx}^{-1})
$$

$\Longrightarrow$ Asymptotic normality (漸近的正規性) of OLSE

The distribution of u_i is not assumed.

12 Instrumental Variable (操作変数法)

12.1 Measurement Error (測定誤差)

Errors in Variables

1. True regression model:

$$y = \tilde{X}\beta + u$$

2. Observed variable:

$$X = \tilde{X} + V$$

   V: is called the **measurement error (測定誤差 or 観測誤差)**.

3. For the elements which do not include measurement errors in X, the corresponding elements in V are zeros.

4. Regression using observed variable:

$$y = X\beta + (u - V\beta)$$

OLS of β is:

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'(u - V\beta)$$

5. Assumptions:

   (a) The measurement error in X is uncorrelated with $\tilde{X}$ in the limit. i.e.,

$$\text{plim}\Big(\frac{1}{n}\tilde{X}'V\Big) = 0.$$

   Therefore, we obtain the following:

$$\text{plim}\Big(\frac{1}{n}X'X\Big) = \text{plim}\Big(\frac{1}{n}\tilde{X}'\tilde{X}\Big) + \text{plim}\Big(\frac{1}{n}V'V\Big) = \Sigma + \Omega$$

(b) u is not correlated with V.

u is not correlated with $\tilde{X}$.

That is,

$$\text{plim}\Big(\frac{1}{n}V'u\Big) = 0, \qquad \text{plim}\Big(\frac{1}{n}\tilde{X}'u\Big) = 0.$$

6. OLSE of β is:

$$\hat{\beta} = \beta + (X'X)^{-1}X'(u - V\beta) = \beta + (X'X)^{-1}(\tilde{X} + V)'(u - V\beta).$$

Therefore, we obtain the following:

$$\text{plim}\,\hat{\beta} = \beta - (\Sigma + \Omega)^{-1}\Omega\beta$$

7. **Example: The Case of Two Variables:**

The regression model is given by:

$$y_t = \alpha + \beta\tilde{x}_t + u_t, \qquad x_t = \tilde{x}_t + v_t.$$

Under the above model,

$$\Sigma = \text{plim}\Big(\frac{1}{n}\tilde{X}'\tilde{X}\Big) = \text{plim}\begin{pmatrix} 1 & \frac{1}{n}\sum\tilde{x}_i \\ \frac{1}{n}\sum\tilde{x}_i & \frac{1}{n}\sum\tilde{x}_i^2 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2+\sigma^2 \end{pmatrix},$$

where μ and σ^2 represent the mean and variance of $\tilde{x}_i$.

$$\Omega = \text{plim}\Big(\frac{1}{n}V'V\Big) = \text{plim}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{n}\sum v_i^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\text{plim}\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \left(\begin{pmatrix} 1 & \mu \\ \mu & \mu^2+\sigma^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \frac{1}{\sigma^2+\sigma_v^2}\begin{pmatrix} -\mu\sigma_v^2\beta \\ \sigma_v^2\beta \end{pmatrix}
\end{aligned}$$

Now we focus on β.

$\hat{\beta}$ is not consistent. because of:

$$\text{plim}(\hat{\beta}) = \beta - \frac{\sigma_v^2 \beta}{\sigma^2 + \sigma_v^2} = \frac{\beta}{1 + \sigma_v^2/\sigma^2} < \beta$$