Moreover, the second term is derived from:

$$\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \mathrm{E}(\epsilon_t^2) = \sigma_\epsilon^2.$$

Therefore,

$$\frac{1}{\sigma_\epsilon^2 T}\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}\left(\frac{y_T}{\sigma_\epsilon\sqrt{T}}\right)^2 - \frac{1}{2\sigma_\epsilon^2}\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \frac{1}{2}(\chi^2(1)-1).$$

(b) Next, consider $\sum y_{t-1}^2$.

$$\mathrm{E}\left(\sum_{t=1}^{T} y_{t-1}^2\right) = \sum_{t=1}^{T}\mathrm{E}(y_{t-1}^2) = \sum_{t=1}^{T}\sigma_\epsilon^2(t-1) = \sigma_\epsilon^2\frac{T(T-1)}{2}.$$

Thus, we obtain the following result:

$$\frac{1}{T^2}\mathrm{E}\left(\sum_{t=1}^{T} y_{t-1}^2\right) \longrightarrow \text{a fixed value.}$$

Therefore,

$$\frac{1}{T^2}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \text{ a distribution.}$$

6. Summarizing the results up to now, $T(\hat{\phi}_1 - \phi_1)$, not $\sqrt{T}(\hat{\phi}_1 - \phi_1)$, has limiting distribution in the case of $\phi_1 = 1$.

$$T(\hat{\phi}_1 - \phi_1) = \frac{(1/T)\sum y_{t-1}\epsilon_t}{(1/T^2)\sum y_{t-1}^2} \longrightarrow \text{ a distribution.}$$

We say that $\hat{\phi}_1$ is **super-consistent (超一致性)** or ***T*-consistent**.

Remember that when $|\phi_1| < 1$ we have $\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$, and in this case we say that $\hat{\phi}_1$ is $\boldsymbol{\sqrt{T}}$**-consistent**.

Conventional $t$ test statistic is given by:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi},$$

where

$$s_\phi = \left( s^2 \Big/ \sum_{t=1}^{T} y_{t-1}^2 \right)^{1/2} \quad \text{and} \quad s^2 = \frac{1}{T-1} \sum_{t=1}^{T} (y_t - \hat{\phi}_1 y_{t-1})^2.$$

7. The distributions of the $t$ statistic: $\dfrac{\hat{\phi}_1 - 1}{s_\phi}$

Remember that the $t$ distribution is:

**$t$ Distribution**

| $T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|---|---|---|---|---|---|---|---|---|
| 25 | −2.49 | −2.06 | −1.71 | −1.32 | 1.32 | 1.71 | 2.06 | 2.49 |
| 50 | −2.40 | −2.01 | −1.68 | −1.30 | 1.30 | 1.68 | 2.01 | 2.40 |
| 100 | −2.36 | −1.98 | −1.66 | −1.29 | 1.29 | 1.66 | 1.98 | 2.36 |
| 250 | −2.34 | −1.97 | −1.65 | −1.28 | 1.28 | 1.65 | 1.97 | 2.34 |
| 500 | −2.33 | −1.96 | −1.65 | −1.28 | 1.28 | 1.65 | 1.96 | 2.33 |
| $\infty$ | −2.33 | −1.96 | −1.64 | −1.28 | 1.28 | 1.64 | 1.96 | 2.33 |

**(a)** $\boldsymbol{H_0 : \; y_t = y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1 : \; y_t = \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$ **or** $\boldsymbol{-1 < \phi_1}$

| $T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|---|---|---|---|---|---|---|---|---|
| 25 | −2.66 | −2.26 | −1.95 | −1.60 | 0.92 | 1.33 | 1.70 | 2.16 |
| 50 | −2.62 | −2.25 | −1.95 | −1.61 | 0.91 | 1.31 | 1.66 | 2.08 |
| 100 | −2.60 | −2.24 | −1.95 | −1.61 | 0.90 | 1.29 | 1.64 | 2.03 |
| 250 | −2.58 | −2.23 | −1.95 | −1.62 | 0.89 | 1.29 | 1.63 | 2.01 |
| 500 | −2.58 | −2.23 | −1.95 | −1.62 | 0.89 | 1.28 | 1.62 | 2.00 |
| $\infty$ | −2.58 | −2.23 | −1.95 | −1.62 | 0.89 | 1.28 | 1.62 | 2.00 |

**(b)** $H_0: \; y_t = y_{t-1} + \epsilon_t$

$H_1: \; y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t$ **for** $\phi_1 < 1$ **or** $-1 < \phi_1$

| $T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|---|---|---|---|---|---|---|---|---|
| 25 | −3.75 | −3.33 | −3.00 | −2.63 | −0.37 | 0.00 | 0.34 | 0.72 |
| 50 | −3.58 | −3.22 | −2.93 | −2.60 | −0.40 | −0.03 | 0.29 | 0.66 |
| 100 | −3.51 | −3.17 | −2.89 | −2.58 | −0.42 | −0.05 | 0.26 | 0.63 |
| 250 | −3.46 | −3.14 | −2.88 | −2.57 | −0.42 | −0.06 | 0.24 | 0.62 |
| 500 | −3.44 | −3.13 | −2.87 | −2.57 | −0.43 | −0.07 | 0.24 | 0.61 |
| $\infty$ | −3.43 | −3.12 | −2.86 | −2.57 | −0.44 | −0.07 | 0.23 | 0.60 |

**(d)** $H_0: \; y_t = \alpha_0 + y_{t-1} + \epsilon_t$

$H_1: \; y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t$ **for** $\phi_1 < 1$ **or** $-1 < \phi_1$

| $T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
|---|---|---|---|---|---|---|---|---|
| 25 | −4.38 | −3.95 | −3.60 | −3.24 | −1.14 | −0.80 | −0.50 | −0.15 |
| 50 | −4.15 | −3.80 | −3.50 | −3.18 | −1.19 | −0.87 | −0.58 | −0.24 |
| 100 | −4.04 | −3.73 | −3.45 | −3.15 | −1.22 | −0.90 | −0.62 | −0.28 |
| 250 | −3.99 | −3.69 | −3.43 | −3.13 | −1.23 | −0.92 | −0.64 | −0.31 |
| 500 | −3.98 | −3.68 | −3.42 | −3.13 | −1.24 | −0.93 | −0.65 | −0.32 |
| $\infty$ | −3.96 | −3.66 | −3.41 | −3.12 | −1.25 | −0.94 | −0.66 | −0.33 |

## 14.2 Serially Correlated Errors

Consider the case where the error term is serially correlated.

### 14.2.1 Augmented Dickey-Fuller (ADF) Test

Consider the following AR($p$) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \qquad \epsilon_t \sim \mathrm{iid}(0, \sigma_\epsilon^2),$$

which is rewritten as:

$$\phi(L) y_t = \epsilon_t.$$

When the above model has a unit root, we have $\phi(1) = 0$, i.e., $\phi_1 + \phi_2 + \cdots + \phi_p = 1$.

The above AR($p$) model is written as:

$$y_t = \rho y_{t-1} + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + +\delta_{p-1} \Delta y_{t-p+1} + \epsilon_t,$$

where $\rho = \phi_1 + \phi_2 + \cdots + \phi_p$ and $\delta_j = -(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p)$.

The null and alternative hypotheses are:

$$H_0:\ \rho = 1 \text{ (Unit root)}, \qquad H_1:\ \rho < 1 \text{ (Stationary)}.$$

Use the $t$ test, where we have the same asymptotic distributions as before.

Choose $p$ by AIC or SBIC.

Use $N(0,1)$ to test $H_0:\ \delta_j = 0$ against $H_1:\ \delta_j \neq 0$ for $j = 1, 2, \cdots, p-1$.

**Reference**

Kurozumi (2008) "Economic Time Series Analysis and Unit Root Tests: Development and Perspective," *Japan Statistical Society*, Vol.38, Series J, No.1, pp.39 – 57.

Download the above paper from:

`http://ci.nii.ac.jp/vol_issue/nels/AA11989749/ISS0000426576_ja.html`

## 14.3 Cointegration (共和分)

1. For a scalar $y_t$, when $(1-L)^d y_t$ is stationary, we write $y_t \sim I(d)$.

   When $\Delta y_t = y_t - y_{t-1}$ is stationary, we write $\Delta y_t \sim I(0)$ or $y_t \sim I(1)$.

2. **Definition of Cointegration:**

   Suppose that each series in a $g \times 1$ vector $y_t$ is $I(1)$, i.e., each series has unit root, and that a linear combination of each series (i.e, $a' y_t$ for a nonzero vector $a$) is $I(0)$, i.e., stationary.

   Then, we say that $y_t$ has a cointegration.

3. **Example:**

   Suppose that $y_t = (y_{1,t},\ y_{2,t})'$ is the following vector autoregressive process:

$$y_{1,t} = \gamma y_{2,t} + \epsilon_{1,t},$$

$$y_{2,t} = y_{2,t-1} + \epsilon_{2,t}.$$

Then,

$$\Delta y_{1,t} = \gamma\epsilon_{2,t} + \epsilon_{1,t} - \epsilon_{1,t-1}, \quad \text{(MA(1) process)},$$

$$\Delta y_{2,t} = \epsilon_{2,t},$$

where both $y_{1,t}$ and $y_{2,t}$ are $I(1)$ processes.

The linear combination $y_{1,t} - \gamma y_{2,t}$ is $I(0)$.

In this case, we say that $y_t = (y_{1,t},\ y_{2,t})'$ is cointegrated with $a = (1,\ -\gamma)$.

$a = (1,\ -\gamma)$ is called the **cointegrating vector (共和分ベクトル)**, which is not unique. Therefore, the first element of $a$ is set to be one.

4. Suppose that $y_t \sim I(1)$ and $x_t \sim I(1)$.

For the regression model $y_t = x_t\beta + u_t$, OLS does not work well if we do not have the $\beta$ which satisfies $u_t \sim I(0)$.

$\Longrightarrow$ **Spurious regression (見せかけの回帰)**

(a) OLSE $\hat{\gamma}_T$ is not consistent.

(b) $s^2 = \dfrac{1}{T-g}\displaystyle\sum_{t=1}^{T}\hat{u}_t^2$ diverges.

(c) The $t$ test statistic diverges.

5. **Resolution for Spurious Regression:**

Suppose that $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ is a spurious regression.

(1) Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + \phi y_{1,t-1} + \delta y_{2,t-1} + u_t$.

Then, $\hat{\gamma}_T$ is $\sqrt{T}$-consistent, and the $t$ test statistic goes to the standard normal distribution under $H_0 : \ \gamma = 0$.

(2) Estimate $\Delta y_{1,t} = \alpha + \gamma' \Delta y_{2,t} + u_t$. Then, $\hat{\alpha}_T$ and $\hat{\beta}_T$ are $\sqrt{T}$-consistent, and the $t$ test and $F$ test make sense.

(3) Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ by the Cochrane-Orcutt method, assuming that $u_t$ is the first-order serially correlated error.

Usually, choose (2).

However, there are two exceptions.

(i) The true value of $\phi$ in (1) above is not one, i.e., less than one.

(ii) $y_{1,t}$ and $y_{2,t}$ are the cointegrated processes.

In these two cases, taking the first difference leads to the misspecified regression.

6. **Cointegrating Vector:**

Suppose that each element of $y_t$ is $I(1)$ and that $a'y_t$ is $I(0)$.

$a$ is called a **cointegrating vector (共和分ベクトル)**, which is not unique.

Set $z_t = a'y_t$, where $z_t$ is scalar, and $a$ and $y_t$ are $g \times 1$ vectors.

For $z_t \sim I(0)$ (i.e., stationary),

$$T^{-1}\sum_{t=1}^{T} z_t^2 = T^{-1}\sum_{t=1}^{T}(a'y_t)^2 \longrightarrow \mathrm{E}(z_t^2).$$

For $z_t \sim I(1)$ (i.e., nonstationary, i.e., $a$ is not a cointegrating vector),

$$T^{-2}\sum_{t=1}^{T} z_t^2 = T^{-2}\sum_{t=1}^{T}(a'y_t)^2 \longrightarrow \text{Distribution}.$$

If $a$ is not a cointegrating vector, $T^{-1}\sum_{t=1}^{T} z_t^2$ diverges.

$\Longrightarrow$ We can obtain a consistent estimate of a cointegrating vector by minimizing $\sum_{t=1}^{T} z_t^2$ with respect to $a$, where a normalization condition on $a$ has to be imposed.

The estimator of the $a$ including the normalization condition is super-consistent ($T$-consistent).

Stock, J.H. (1987) "Asymptotic Properties of Least Squares Estimators of Cointegrating Vectors," *Econometrica*, Vol.55, pp.1035 – 1056.

## 14.4 Testing Cointegration

### 14.4.1 Engle-Granger Test

$y_t \sim I(1)$

$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$

- $u_t \sim I(0) \implies$ Cointegration
- $u_t \sim I(1) \implies$ Spurious Regression

Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ by OLS, and obtain $\hat{u}_t$.

Estimate $\hat{u}_t = \rho\hat{u}_{t-1} + \delta_1\Delta\hat{u}_{t-1} + \delta_2\Delta\hat{u}_{t-2} + \cdots + \delta_{p-1}\Delta\hat{u}_{t-p+1} + e_t$ by OLS.

ADF Test:

- $H_0: \ \rho = 1$ (Sprious Regression)
- $H_1: \ \rho < 1$ (Cointegration)

$\Longrightarrow$ **Engle-Granger Test**

For example, see Engle and Granger (1987), Phillips and Ouliaris (1990) and Hansen (1992).

**Asymmptotic Distribution of Residual-Based ADF Test for Cointegration**

| # of Refressors, | (a) Regressors have no drift | | | | (b) Some regressors have drift | | | |
|---|---|---|---|---|---|---|---|---|
| excluding constant | 1% | 2.5% | 5% | 10% | 1% | 2.5% | 5% | 10% |
| 1 | −3.96 | −3.64 | −3.37 | −3.07 | −3.96 | −3.67 | −3.41 | −3.13 |
| 2 | −4.31 | −4.02 | −3.77 | −3.45 | −4.36 | −4.07 | −3.80 | −3.52 |
| 3 | −4.73 | −4.37 | −4.11 | −3.83 | −4.65 | −4.39 | −4.16 | −3.84 |
| 4 | −5.07 | −4.71 | −4.45 | −4.16 | −5.04 | −4.77 | −4.49 | −4.20 |
| 5 | −5.28 | −4.98 | −4.71 | −4.43 | −5.36 | −5.02 | −4.74 | −4.46 |

J.D. Hamilton (1994), *Time Series Analysis*, p.766.

## The Other Topics

- Generalized Method of Moments (一般化積率法，GMM)
- System of Equations (Seemingly Unrelated Regression (SUR), Simultaneous Equation (連立方程式), and etc.)
- Panel Data (パネル・データ)
- Discrete Dependent Variable, and Limited Dependent Variable
- Bayesian Estimation (ベイズ推定)
- Semiparametric and Nonparametric Regressions and Tests (セミパラメトリック, ノンパラメトリック推定・検定)
- $\cdots$

# Exam — Aug. 5, 2014 (AM8:50-10:20)

- 60 - 70% from two homeworks including optional an additional questions (2 つの宿題から 60 - 70%)
- 30 - 40% of new questions (30 - 40% の新しい問題)
- Questions are written in English, and answers should be in English or Japanese. (出題は英語，解答は英語または日本語)
- With no carrying in (持ち込みなし)