Homework (Due: July 8, 2014 AM10:20)

Let y, X, β and u be a $n \times 1$ vector, a $n \times k$ matrix, a $k \times 1$ vector and a $n \times 1$ vector, respectively, where X is nonstochastic, u is an error term, and β is a parameter.

Consider the regression model: $y = X\beta + u$.

- $\mathrm{E}(u) = 0$ and $\mathrm{V}(u) = \sigma^2 I_n$ are assumed.

  (1) Derive an OLS estimator of β, which is denoted by $\widehat{\beta}$.

  (2) Obtain the mean and variance of $\widehat{\beta}$.

  (3) Prove that $\widehat{\beta}$ is a best linear unbiased estimator.

  (4) Show that $\widehat{\beta}$ is a consistent estimator of β. Which assumptions are used?

  (5) As n goes to infinity, what is a distribution of $\sqrt{n}(\widehat{\beta} - \beta)$?

  (6) Let R and r be a $G \times k$ matrix and a $G \times 1$ vector, where $G \leq k$ and $\mathrm{Rank}(R) = G$. Suppose that the linear restriction is given by $R\beta = r$ (G restrictions). Derive the restricted OLS estimator, denoted by $\widetilde{\beta}$.

  (7) Consider $\widehat{\sigma}^2 = \dfrac{1}{n-k}\widehat{u}'\widehat{u}$ as an estimator of σ^2. Prove that $\widehat{\sigma}^2$ is an unbiased estimator of σ^2.

- $u \sim N(0, \sigma^2 I_n)$ is assumed.

  (8) Derive distributions of $\widehat{\beta}$ and $\dfrac{(n-k)\widehat{\sigma}^2}{\sigma^2}$.

  (9) Show that $\widehat{\sigma}^2$ is a consistent estimator of σ^2. Assume that $u \sim N(0, \sigma^2 I_n)$.

  (10) Using the distributions of $\widehat{\beta}$ and $\dfrac{(n-k)\widehat{\sigma}^2}{\sigma^2}$ obtained in Question (8), explain how to test $H_0:\ R\beta = r$ against $H_1:\ R\beta \neq r$.

  (11) Using $\widehat{u} = y - X\widehat{\beta}$ and $\widetilde{u} = y - X\widetilde{\beta}$, show that the test statistic obtained in Question (10) is given by $\dfrac{(\widetilde{u}'\widetilde{u} - \widehat{u}'\widehat{u})/G}{\widehat{u}'\widehat{u}/(n-k)}$.

  (12) Obtain ML estimators of β and σ^2, which are denoted by $\overline{\beta}$ and $\overline{\sigma}^2$.

  (13) Derive a distribution of $\sqrt{n}\begin{pmatrix} \overline{\beta} - \beta \\ \overline{\sigma}^2 - \sigma^2 \end{pmatrix}$ when n goes to infinity.

- $u_t = \rho u_{t-1} + \epsilon_t$, $t = 1, 2, \cdots, n$, and $\epsilon \sim N(0, \sigma^2 I_n)$ are assumed.

(14) Obtain the unconditional distribution of u_t.

(15) Obtain the conditional distribution of u_t, given u_{t-1}, u_{t-2}, $\cdots$, u_1.

(16) Derive the likelihood function of β, σ^2 and ρ, given data y and X.

(17) Explain how to obtain the maximum likelihood estimators of β, σ^2 and ρ.