2.3 Properties of Least Squares Estimator

Equation (10) is rewritten as:

$$\hat{\beta}_2 = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n (x_i - \overline{x})y_i}{\sum_{i=1}^n (x_i - \overline{x})^2} - \frac{\overline{y}\sum_{i=1}^n (x_i - \overline{x})}{\sum_{i=1}^n (x_i - \overline{x})^2}$$
$$= \sum_{i=1}^n \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2} y_i = \sum_{i=1}^n \omega_i y_i. \qquad (12)$$

In the third equality, $\sum_{i=1}^n (x_i - \overline{x}) = 0$ is utilized because of $\overline{x} = \frac{1}{n}\sum_{i=1}^n x_i$.

In the fourth equality, ω_i is defined as: $\omega_i = \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2}$.

ω_i is nonstochastic because x_i is assumed to be nonstochastic.

ω_i has the following properties:

$$\sum_{i=1}^n \omega_i = \sum_{i=1}^n \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n (x_i - \overline{x})}{\sum_{i=1}^n (x_i - \overline{x})^2} = 0, \qquad (13)$$

$$\sum_{i=1}^{n} \omega_i x_i = \sum_{i=1}^{n} \omega_i (x_i - \overline{x}) = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = 1, \tag{14}$$

$$\sum_{i=1}^{n} \omega_i^2 = \sum_{i=1}^{n} \left(\frac{x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2} \right)^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\left(\sum_{i=1}^{n}(x_i - \overline{x})^2\right)^2} = \frac{1}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \tag{15}$$

The first equality of (14) comes from (13).

From now on, we focus only on $\hat{\beta}_2$, because usually β_2 is more important than β_1 in the regression model (4).

In order to obtain the properties of the least squares estimator $\hat{\beta}_2$, we rewrite (12) as:

$$\begin{aligned} \hat{\beta}_2 &= \sum_{i=1}^{n} \omega_i y_i = \sum_{i=1}^{n} \omega_i (\beta_1 + \beta_2 x_i + u_i) \\ &= \beta_1 \sum_{i=1}^{n} \omega_i + \beta_2 \sum_{i=1}^{n} \omega_i x_i + \sum_{i=1}^{n} \omega_i u_i = \beta_2 + \sum_{i=1}^{n} \omega_i u_i. \end{aligned} \tag{16}$$

In the fourth equality of (16), (13) and (14) are utilized.

[Review] Random Variables:

Let X_1, X_2, $\cdots$, X_n be n random variavles, which are mutually independently and identically distributed.

mutually independent $\Longrightarrow$ $f(x_i, x_j) = f_i(x_i) f_j(x_j)$ for $i \neq j$.

$f(x_i, x_j)$ denotes a joint distribution of X_i and X_j.

$f_i(x)$ indicates a marginal distribution of X_i.

identical $\Longrightarrow$ $f_i(x) = f_j(x)$ for $i \neq j$.

[End of Review]

[Review] Mean and Variance:

Let X and Y be random variables (continuous type), which are independently distributed.

Definition and Formulas:

- $\mathrm{E}(g(X)) = \int g(x) f(x) \mathrm{d}x$ for a function $g(\cdot)$ and a density function $f(\cdot)$.
- $\mathrm{V}(X) = \mathrm{E}((X-\mu)^2) = \int (x-\mu)^2 f(x) \mathrm{d}x$ for $\mu = \mathrm{E}(X)$.
- $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and $\mathrm{V}(aX + b) = a^2 \mathrm{V}(X)$.
- $\mathrm{E}(X \pm Y) = \mathrm{E}(X) \pm \mathrm{E}(Y)$ and $\mathrm{V}(X \pm Y) = \mathrm{V}(X) + \mathrm{V}(Y)$.

[End of Review]

Mean and Variance of $\hat{\beta}_2$: $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed with mean zero and variance σ^2, but they are not necessarily normal.

Remember that we do not need normality assumption to obtain mean and variance but the normality assumption is required to test a hypothesis.

From (16), the expectation of $\hat{\beta}_2$ is derived as follows:

$$\mathrm{E}(\hat{\beta}_2) = \mathrm{E}(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = \beta_2 + \mathrm{E}(\sum_{i=1}^{n} \omega_i u_i) = \beta_2 + \sum_{i=1}^{n} \omega_i \mathrm{E}(u_i) = \beta_2. \qquad (17)$$

It is shown from (17) that the ordinary least squares estimator $\hat{\beta}_2$ is an **unbiased estimator (不偏推定量)** of β_2.

From (16), the variance of $\hat{\beta}_2$ is computed as:

$$\mathrm{V}(\hat{\beta}_2) = \mathrm{V}(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = \mathrm{V}(\sum_{i=1}^{n} \omega_i u_i) = \sum_{i=1}^{n} \mathrm{V}(\omega_i u_i) = \sum_{i=1}^{n} \omega_i^2 \mathrm{V}(u_i)$$
$$= \sigma^2 \sum_{i=1}^{n} \omega_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \qquad (18)$$

The third equality holds because $u_1, u_2, \cdots, u_n$ are mutually independent.

The last equality comes from (15).

Thus, $\mathrm{E}(\hat{\beta}_2)$ and $\mathrm{V}(\hat{\beta}_2)$ are given by (17) and (18).

Gauss-Markov Theorem (ガウス・マルコフ定理): $\hat{\beta}_2$ has minimum variance within a class of the linear unbiased estimators.

$\longrightarrow$ **best linear unbiased estimator (BLUE, 最良線型不偏推定量)**

(Proof is omitted.)

Distribution of $\hat{\beta}_2$: We discuss the small sample properties of $\hat{\beta}_2$.

In order to obtain the distribution of $\hat{\beta}_2$ in small sample, the distribution of the error term has to be assumed.

Therefore, the extra assumption is that $u_i \sim N(0, \sigma^2)$.

Writing (16), again, $\hat{\beta}_2$ is represented as:

$$\hat{\beta}_2 = \beta_2 + \sum_{i=1}^{n} \omega_i u_i.$$

First, we obtain the distribution of the second term in the above equation.

It is well known that sum of normal random variables results in a normal distribution.

Therefore, $\sum_{i=1}^{n} \omega_i u_i$ is distributed as:

$$\sum_{i=1}^{n} \omega_i u_i \sim N(0,\ \sigma^2 \sum_{i=1}^{n} \omega_i^2).$$

Therefore, $\hat{\beta}_2$ is distributed as:

$$\hat{\beta}_2 = \beta_2 + \sum_{i=1}^{n} \omega_i u_i \sim N(\beta_2,\ \sigma^2 \sum_{i=1}^{n} \omega_i^2),$$

or equivalently,

$$\frac{\hat{\beta}_2 - \beta_2}{\sigma \sqrt{\sum_{i=1}^{n} \omega_i^2}} = \frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} \sim N(0, 1),$$

for any n.

Moreover, replacing σ^2 by its estimator $s^2 = \dfrac{1}{n-2} \displaystyle\sum_{i=1}^{n}(y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2$, it is known that we have:

$$\frac{\hat{\beta}_2 - \beta_2}{s / \sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} \sim t(n-2),$$

where $t(n-2)$ denotes t distribution with $n-2$ degrees of freedom.

Thus, under normality assumption on the error term u_i, the $t(n-2)$ distribution is used for the confidence interval and the testing hypothesis in small sample.

Or, taking the square on both sides,

$$\left(\frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}}\right)^2 \sim F(1, n-2).$$

[Review] Confidence Interval (信頼区間，区間推定)):

Suppose that $X_1, X_2, \cdots, X_n$ are mutually independently, identically and normally distributed with mean μ and variance σ^2.

Then, we can obtain: $\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$, where $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

That is,

$$P\Big(-t_{\alpha/2}(n-1) < \frac{\overline{X}-\mu}{S/\sqrt{n}} < t_{\alpha/2}(n-1)\Big) = 1-\alpha$$

i.e.,

$$P\Big(\overline{X} - t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}} < \mu < \overline{X} + t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}}\Big) = 1-\alpha.$$

Note that $t_{\alpha/2}(n-1)$ is obtained from the t distribution table, given α and $n-1$.

Then, replacing $\overline{X}$ by $\overline{x}$, we obtain the $100(1-\alpha)\%$ confidence interval of μ as follows:

$$(\overline{x} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}},\ \overline{x} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}).$$

[End of Review]

In the case of OLS,

$$P\left(-t_{\alpha/2}(n-2) < \frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} < t_{\alpha/2}(n-2)\right) = 1 - \alpha,$$

where $t_{\alpha/2}(n-2)$ denotes $100 \times \alpha/2\%$ point from the $t(n-2)$ distribution.

Rewriting,

$$P\left(\hat{\beta}_2 - t_{\alpha/2}(n-2)\frac{s}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} < \beta_2 < \hat{\beta}_2 + t_{\alpha/2}(n-2)\frac{s}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}}\right) = 1 - \alpha.$$

Replacing $\hat{\beta}_2$ and s^2 by observed data, the $100(1-\alpha)\%$ confidence interval of β_2 is given by:

$$\left(\hat{\beta}_2 - t_{\alpha/2}(n-2)\frac{s}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}},\ \hat{\beta}_2 + t_{\alpha/2}(n-2)\frac{s}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}}\right).$$

[Review] Testing the Hypothesis (仮説検定):

Suppose that $X_1, X_2, \cdots, X_n$ are mutually independently, identically and normally distributed with mean μ and variance σ^2.

Then, we obtain: $\dfrac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$, where $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$, which is known as the unbiased estimator of σ^2.

- The null hypothesis $H_0: \ \mu = \mu_0$, where μ_0 is a fixed number.
- The alternative hypothesis $H_1: \ \mu \neq \mu_0$

Under the null hypothesis, we have the disribution: $\dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$.

Replacing $\overline{X}$ and S^2 by $\overline{x}$ and s^2, compare $\dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$ and $t(n-1)$.

H_0 is rejected when $\left|\dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}\right| > t_{\alpha/2}(n-1)$.

$t_{\alpha/2}(n-1)$ is obtained from the significance level α and the degrees of freedom $n-1$.

[End of Review]