8. **Lag Operator (ラグ作要素) :**

$$L^\tau y_t = y_{t-\tau}, \qquad \tau = 1, 2, \cdots$$

9. **Likelihood Function (尤度関数)** — Innovation Form :

The joint distribution of $y_1, y_2, \cdots, y_T$ is written as:

$$
\begin{aligned}
f(y_1, , y_2, \cdots, y_T) &= f(y_T|y_{T-1}, \cdots, y_1)f(y_{T-1}, \cdots, y_1) \\
&= f(y_T|y_{T-1}, \cdots, y_1)f(y_{T-1}|y_{T-2}, \cdots, y_1)f(y_{T-2}, \cdots, y_1) \\
&\ \ \vdots \\
&= f(y_T|y_{T-1}, \cdots, y_1)f(y_{T-1}|y_{T-2}, \cdots, y_1) \cdots f(y_2|y_1)f(y_1) \\
&= f(y_1) \prod_{t=2}^{T} f(y_t|y_{t-1}, \cdots, y_1).
\end{aligned}
$$

Therefore, the log-likelihood function is given by:

$$\log f(y_1, y_2, \cdots, y_T) = \log f(y_1) + \sum_{t=2}^{T} \log f(y_t|y_{t-1}, \cdots, y_1).$$

Under the normality assumption, $f(y_t|y_{t-1}, \cdots, y_1)$ is given by the normal distribution with conditional mean $E(y_t|y_{t-1}, \cdots, y_1)$ and conditional variance $Var(y_t|y_{t-1}, \cdots$

6.2 Time Series Models (時系列モデル)

Autoregressive Model (自己回帰モデル or AR モデル): AR(p)

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$

Moving Average Model (移動平均モデル or MA モデル): MA(q)

$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$

ARMA Model: ARMA(p, q)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

ARIMA Model: ARIMA(p, d, q)

$$\Delta y_t = y_t - y_{t-1} = (1 - L) y_t,$$
$$\Delta^2 y_t = \Delta y_t - \Delta y_{t-1} = (1 - L)^2 y_t,$$
$$\vdots$$
$$\Delta^d y_t = (1 - L)^d y_t.$$

$$\Delta^d y_t \sim \text{ARMA}(p, q) \iff y_t \sim \text{ARIMA}(p, d, q)$$

$$\Delta^d y_t = \phi_1 \Delta^d y_{t-1} + \phi_2 \Delta^d y_{t-2} + \cdots + \phi_p \Delta^d y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

SARIMA Model: SARIMA(p, d, q)

${}^s\Delta y_t = y_t - y_{t-s}$, $s = 4$ for quarterly data $s = 12$ for monthly data

$${}^s\Delta\Delta^d y_t \sim \mathrm{ARMA}(p, q) \iff y_t \sim \mathrm{SARIMA}(p, d, q)$$

$${}^s\Delta\Delta^d y_t = \phi_1 {}^s\Delta\Delta^d y_{t-1} + \phi_2 {}^s\Delta\Delta^d y_{t-2} + \cdots + \phi_p {}^s\Delta\Delta^d y_{t-p} + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$$

6.3 Autoregressive Model (自己回帰モデル or AR モデル)

1. **AR(p) Model :**

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

which is rewritten as:

$$\phi(L)y_t = \epsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p.$$

2. **Stationarity (定常性) :**

Suppose that all the p solutions of x from $\phi(x) = 0$ are real numbers

When the p solutions are greater than one in absolute value, y_t is stationary.

Suppose that the *p* solutions include imaginary numbers.

When the *p* solutions are outside unit circle, y_t is stationary.

3. **Partial Autocorrelation Coefficient (偏自己相関係数), $\phi_{k,k}$:**

   The partial autocorrelation coefficient between y_t and y_{t-k}, denoted by $\phi_{k,k}$, is a measure of strength of the relationship between y_t and y_{t-k}, after removing influence of $y_{t-1}, \cdots, y_{t-k+1}$.

   $$\phi_{1,1} = \rho(1)$$

   $$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{pmatrix}$$

Use Cramer's rule (クラメールの公式) to obtain $\phi_{k,k}$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Example: AR(1) Model: $\quad y_t = \phi_1 y_{t-1} + \epsilon_t$

1. The stationarity condition is: the solution of $\phi(x) = 1 - \phi_1 x = 0$, i.e., $x = 1/\phi_1$, is greater than one in absolute value, or equivalently, $|\phi_1| < 1$.

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2. Rewriting the AR(1) model,

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\
&= \phi_1^3 y_{t-3} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} \\
&\quad \vdots \\
&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1}.
\end{aligned}
$$

As s is large, ϕ_1^s approaches zero. $\implies$ Stationarity condition

3. For stationarity, $y_t = \phi_1 y_{t-1} + \epsilon_t$ is rewritten as:

$$
y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots
$$

MA representation of AR model.

(MA will be discussed later.)

4. Mean of AR(1) process, μ

$$\mu = \mathrm{E}(y_t) = \mathrm{E}(\epsilon_t + \phi_1\epsilon_{t-1} + \phi_1^2\epsilon_{t-2} + \cdots)$$

$$= \mathrm{E}(\epsilon_t) + \phi_1\mathrm{E}(\epsilon_{t-1}) + \phi_1^2\mathrm{E}(\epsilon_{t-2}) + \cdots = 0$$

5. Autocovariance and autocorrelation functions of the AR(1) process:

   Rewriting the AR(1) process, we have:

   $$y_t = \phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1\epsilon_{t-1} + \cdots + \phi_1^{\tau-1}\epsilon_{t-\tau+1}.$$

   Therefore, the autocovariance function of AR(1) process is:

   $$\begin{aligned}
   \gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\
   &= \mathrm{E}\Big((\phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1\epsilon_{t-1} + \cdots + \phi_1^{\tau-1}\epsilon_{t-\tau+1})y_{t-\tau}\Big) \\
   &= \phi_1^\tau \mathrm{E}(y_{t-\tau} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau}) + \phi_1\mathrm{E}(\epsilon_{t-1} y_{t-\tau}) + \cdots + \phi_1^{\tau-1}\mathrm{E}(\epsilon_{t-\tau+1} y_{t-\tau}) \\
   &= \phi_1^\tau \gamma(0).
   \end{aligned}$$

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The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi_1^\tau.$$

Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation:

$$E(y_t y_{t-\tau}) = \phi_1 E(y_{t-1} y_{t-\tau}) + E(\epsilon_t y_{t-\tau})$$

$$\gamma(\tau) = \begin{cases} \phi_1 \gamma(\tau - 1), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by:

$$\gamma(0) = \phi_1 \gamma(1) + \sigma^2 = \phi_1^2 \gamma(0) + \sigma^2.$$

Note that $\gamma(1) = \phi_1 \gamma(0)$.

Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1 - \phi_1^2}$$

6. Partial autocorrelation function of AR(1) process:

$$\phi_{1,1} = \rho(1) = \phi_1$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = 0$$

7. Estimation of AR(1) model:

   (a) Likelihood function

$$\log f(y_T, \cdots, y_1) = \log f(y_1) + \sum_{t=1}^{T} \log f(y_t | y_{t-1}, \cdots, y_1)$$

$$
\begin{aligned}
&= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\frac{\sigma^2}{1-\phi_1^2}\right) - \frac{1}{\sigma^2/(1-\phi_1^2)}y_1^2 \\
&\quad -\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\log\left(\frac{1}{1-\phi_1^2}\right) \\
&\quad -\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2
\end{aligned}
$$

Note as follows:

$$
f(y_1) = \frac{1}{\sqrt{2\pi\sigma^2/(1-\phi_1^2)}}\exp\left(-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2\right)
$$

$$
f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right)
$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4/(1-\phi_1^2)}y_1^2 + \frac{1}{2\sigma^4}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2 = 0$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \phi_1} = -\frac{\phi_1}{1-\phi_1^2} + \frac{\phi_1}{\sigma^2}y_1^2 + \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})y_{t-1} = 0$$

The MLE of ϕ_1 and σ^2 satisfies the above two equation.

$$\tilde{\sigma}^2 = \frac{1}{T}\left((1-\tilde{\phi}_1^2)y_1^2 + \sum_{t=2}^{T}(y_t - \tilde{\phi}_1 y_{t-1})^2\right)$$

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} + \left(\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2 \tilde{\phi}_1}{1-\tilde{\phi}_1^2}\right)\Big/\sum_{t=2}^{T} y_{t-1}^2$$