(b) Ordinary Least Squares (OLS) Method

$$S(\phi_1) = \sum_{t=2}^{T} (y_t - \phi_1 y_{t-1})^2$$

is minimized with respect to ϕ_1.

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{T} y_{t-1} y_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{\sum_{t=2}^{T} y_{t-1}\epsilon_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{(1/T)\sum_{t=2}^{T} y_{t-1}\epsilon_t}{(1/T)\sum_{t=2}^{T} y_{t-1}^2}$$

$$\longrightarrow \phi_1 + \frac{\mathrm{E}(y_{t-1}\epsilon_t)}{\mathrm{E}(y_{t-1}^2)} = \phi_1$$

OLSE of ϕ_1 is a consistent estimator.

The following equations are utilized.

$$\mathrm{E}(y_{t-1}\epsilon_t) = 0$$

$$\mathrm{E}(y_{t-1}^2) = \mathrm{Var}(y_{t-1}) = \gamma(0)$$

8. Asymptotic distribution of OLSE $\hat{\phi}_1$:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$$

Proof:

$y_{t-1}\epsilon_t$, $t = 1, 2, \cdots, T$, are distributed with mean zero and variance $\dfrac{\sigma_\epsilon^4}{1 - \phi_1^2}$.

From the central limit theorem,

$$\frac{(1/T)\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sqrt{\sigma_\epsilon^4/(1 - \phi_1^2)}/\sqrt{T}} \longrightarrow N(0, 1)$$

Rewriting,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0, \frac{\sigma_\epsilon^4}{1 - \phi_1^2}).$$

Next,

$$\frac{1}{T}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \mathrm{E}(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2}$$

yields:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{(1/\sqrt{T})\sum_{t=1}^{T} y_{t-1}\epsilon_t}{(1/T)\sum_{t=1}^{T} y_{t-1}^2} \longrightarrow N(0, 1-\phi_1^2)$$

9. Some formulas:

   (a) Central Limit Theorem

   Random variables x_1, x_2, $\cdots$, x_T are mutually independently distributed with mean μ and variance σ^2.

   Define $\overline{x} = (1/T)\sum_{t=1}^{T} x_t$.

   Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} = \frac{\overline{x} - \mu}{\sigma/\sqrt{T}} \longrightarrow N(0,1)$$

   (b) Central Limit Theorem II

   Random variables x_1, x_2, $\cdots$, x_T are distributed with mean μ and variance σ^2.

Define $\overline{x} = (1/T)\sum_{t=1}^{T} x_t$.

Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} \longrightarrow N(0, 1)$$

(c) Let x and y be random variables.

y converges in distribution to a distribution, and x converges in probability to a fixed value.

Then, xy converges in distribution.

For example, consider:

$$y \longrightarrow N(\mu, \sigma^2), \qquad x \longrightarrow c.$$

Then, we obtain:

$$xy \longrightarrow N(c\mu, c^2\sigma^2)$$

10. **AR(1) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \epsilon_t$

Mean:

Using the lag operator,

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L$.

Multiply $\phi(L)^{-1}$ on both sides. Then, when $|\phi_1| < 1$, we have:

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Taking the expectation on both sides,

$$\mathrm{E}(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}\mathrm{E}(\epsilon_t)$$
$$= \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1}$$