6.11 ARCH and GARCH Models

Autoregressive Conditional Heteroskedasticity (ARCH)

Generalized Autoregressive Conditional Heteroskedasticity (GARCH)

1. ARCH (p) Model

$$\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \cdots, \epsilon_1 \sim N(0, h_t),$$

   where,

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2.$$

   The unconditional variance of ϵ_t is:

$$\sigma_\epsilon^2 = \frac{\alpha_0}{1 - \alpha_1 - \alpha_2 - \cdots - \alpha_p}$$

2. GARCH (p, q) Model

$$\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \cdots, \epsilon_1 \sim N(0, h_t),$$

where

$$h_t = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \cdots + \alpha_p\epsilon_{t-p}^2 + \beta_1 h_{t-1} + \cdots + \beta_q h_{t-q}.$$

3. Application to OLS (Case of ARCH(1) Model):

$$y_t = x_t\beta + \epsilon_t, \qquad \epsilon_t|\epsilon_{t-1}, \epsilon_{t-2}, \cdots, \epsilon_1 \sim N(0, \alpha_0 + \alpha_1\epsilon_{t-1}^2).$$

The joint density of $\epsilon_1, \epsilon_2, \cdots, \epsilon_T$ is:

$$\begin{aligned} f(\epsilon_1, \cdots, \epsilon_T) &= f(\epsilon_1)\prod_{t=2}^{T} f(\epsilon_t|\epsilon_{t-1}, \cdots, \epsilon_1) \\ &= (2\pi)^{-1/2}\left(\frac{\alpha_0}{1-\alpha_1}\right)^{-1/2} \exp\left(-\frac{1}{2\alpha_0/(1-\alpha_1)}\epsilon_1^2\right) \\ &\quad \times (2\pi)^{-(T-1)/2}\prod_{t=2}^{T}(\alpha_0 + \alpha_1\epsilon_{t-1}^2)^{-1/2} \exp\left(-\frac{1}{2}\sum_{t=2}^{T}\frac{\epsilon_t^2}{\alpha_0 + \alpha_1\epsilon_{t-1}^2}\right). \end{aligned}$$

The log-likelihood function is:

$$
\begin{aligned}
&\log L(\beta, \alpha_0, \alpha_1; y_1, \cdots, y_T) \\
&\quad = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\frac{\alpha_0}{1-\alpha_1}\right) - \frac{1}{2\alpha_0/(1-\alpha_1)}(y_1 - x_1\beta)^2 \\
&\qquad -\frac{T-1}{2}\log(2\pi) - \frac{1}{2}\sum_{t=2}^{T}\log\left(\alpha_0 + \alpha_1(y_{t-1} - x_{t-1}\beta)^2\right) \\
&\qquad -\frac{1}{2}\sum_{t=2}^{T}\frac{(y_t - x_t\beta)^2}{\alpha_0 + \alpha_1(y_{t-1} - x_{t-1}\beta)^2}.
\end{aligned}
$$

Obtain α_0, α_1 and β such that the log-likelihood function is maximized.

$\alpha_0 > 0$ and $\alpha_1 > 0$ have to be satisfied.

These two conditions are explicitly included, when the model is modified to:

$\mathrm{E}(\epsilon_t^2|\epsilon_{t-1}, \epsilon_{t-2}, \cdots, \epsilon_1) = \alpha_0^2 + \alpha_1^2\epsilon_{t-1}^2$.

Testing the ARCH(1) Effect:

(a) Estimate $y_t = x_t\beta + u_t$ by OLS, and compute $\hat{\beta}$ and $\hat{u}_t = y_t - x_t\hat{\beta}$.

(b) Estimate $\hat{u}_t^2 = \alpha_0 + \alpha_1\hat{u}_{t-1}^2$ by OLS. If $\hat{\alpha}_1$ is significant, there is the ARCH(1) effect in the error term.

This test corresponds to LM test.

Example: GARCH(1,1) Model

```
. arch sdex l.sdex l2.sdex, arch(1) garch(1)

(setting optimization to BHHH)
Iteration 0:   log likelihood = -5089.3558
Iteration 1:   log likelihood = -5086.7468
.........
.........

Iteration 22:  log likelihood = -5064.9328  (backed up)
Iteration 23:  log likelihood = -5064.9328

ARCH family regression
```

```
Sample: 16 - 516                                   Number of obs   =       501
Distribution: Gaussian                             Wald chi2(2)    =    225.19
Log likelihood = -5064.933                         Prob > chi2     =    0.0000

------------------------------------------------------------------------------
             |                 OPG
        sdex |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
sdex         |
        sdex |
         L1. |  -.6357273   .0426939   -14.89   0.000    -.7194059   -.5520488
         L2. |   -.370862   .0466222    -7.95   0.000    -.4622398   -.2794842
             |
       _cons |  -55.28043   261.2057    -0.21   0.832    -567.2341    456.6733
-------------+----------------------------------------------------------------
ARCH         |
        arch |
         L1. |    .041632   .0123474     3.37   0.001     .0174317    .0658324
             |
       garch |
         L1. |   .9526041   .0148639    64.09   0.000     .9234715    .9817367
             |
       _cons |   312143.8   227564.3     1.37   0.170    -133873.9    758161.6
------------------------------------------------------------------------------
```

7 Vector Autoregressive (VAR) Model – Causality, Impulse Response Function and etc

Vector Autoregressive Process:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

where

$$y_t : k \times 1, \qquad \mu : k \times 1, \qquad \epsilon_t : k \times 1, \qquad \phi_i : k \times k.$$

Rewriting the above equation,

$$\phi(L) y_t = \mu + \epsilon_t,$$

where $\phi(L) = I_k - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$.

VAR(1) Model:

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \qquad \text{i.e.,} \qquad (I_k - \phi_1 L) y_t = \epsilon_t.$$

When y_t is stationary, we obtain:

$$\begin{aligned} y_t &= (I_k - \phi_1 L)^{-1} \epsilon_t \\ &= (I_k + \phi_1 L + \phi_1^2 L^2 + \phi_1^3 L^3 + \cdots) \epsilon_t \\ &= \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \cdots \end{aligned}$$

VAR(1)=VMA(∞)

VAR(2) Model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \qquad \text{i.e.,} \qquad (I_k - \phi_1 L - \phi_2 L^2) y_{t-1} = \epsilon_t.$$

When y_t is stationary, we obtain:

$$y_{t-1} = (I_k - \phi_1 L - \phi_2 L^2)^{-1} \epsilon_t$$

$$= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots$$

VAR(2)=VMA(∞)

VAR(p) Model:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

i.e.,

$$(I_k - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) y_{t-1} = \epsilon_t.$$

When y_t is stationary, we obtain:

$$y_t = (I_k - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1} \epsilon_t$$

$$= \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots$$

VAR(p)=VMA(∞)

7.1 Autocovariance Matrix and Autocorrelation Matrix

Let y_t be a $k \times 1$ vector.

Autocovariance Function Matrix:

$$\Gamma(\tau) = \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)'), \qquad \tau = 0, 1, 2, \cdots,$$

where $\mathrm{E}(y_t) = \mu$. $\Gamma(\tau)$ is a $k \times k$ matrix.

$$\Gamma(\tau) = \Gamma(-\tau)'$$

Autocorrelation Function Matrix:

$$\rho(\tau) = D^{-1/2}\Gamma(\tau)D^{-1/2},$$

where the (i, j)th element of D is given by $\gamma_{ii}(\tau) = \mathrm{V}(y_{it})$ for $i = j$ and zero otherwise.

$$\rho(\tau) = \rho(-\tau)'$$

7.2 Granger Cuasality Test (グレンジャー因果性テスト)

Consider a bivariate case.

Unrestricted Model (Sum of Squared Residuals, denoted by SSR_1):

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \phi_{11,1} & \phi_{12,1} \\ \phi_{21,1} & \phi_{22,1} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \phi_{11,p} & \phi_{12,p} \\ \phi_{21,p} & \phi_{22,p} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ y_{2,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

$H_0: \ \phi_{12,1} = \phi_{12,2} = \cdots = \phi_{12,p} = 0$

When H_0 is correct, we say there is no causality from y_2 to y_1.

$\Longrightarrow$ Granger Causality Test.

Restricted Model (Sum of Squared Residuals, denoted by SSR_0):

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \phi_{11,1} & 0 \\ \phi_{21,1} & \phi_{22,1} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \phi_{11,p} & 0 \\ \phi_{21,p} & \phi_{22,p} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ y_{2,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

Asymptotically, we have the following distribution:

$$F = \frac{(\mathrm{SSR}_0 - \mathrm{SSR}_1)/p}{\mathrm{SSR}_1/(T-2p-1)} \sim F(p, T-2p-1),$$

or

$$pF \sim \chi^2(p).$$

In general, we consider testing the Granger causality from y_j to y_i.

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t.$$

$y_t : k \times 1, \qquad \mu : k \times 1, \qquad \phi_p : k \times k, \qquad \epsilon_t : k \times 1.$

The null hypothesis is: $H_0 : \phi_{ij,1} = \phi_{ij,2} = \cdots = \phi_{ij,p} = 0.$

The alternative hypothesis is: $H_1 :$ not H_0.

SSR_0 = Sum of Squared Residuals under H_0

SSR_1 = Sum of Squared Residuals under H_1

Under H_0, the asymptotic distribution is given by:

$$F = \frac{(\text{SSR}_0 - \text{SSR}_1)/p}{\text{SSR}_1/(T - kp - 1)} \sim F(p, T - kp - 1),$$

or

$$pF \sim \chi^2(p).$$

Example:

Data:　1994 年第一四半期～2014 年第一四半期

gdp = GDP (実質，10 億円，季調済，内閣府 HP から取得)

def = GDP デフレータ (季調済，内閣府 HP から取得)

r = 貸出約定平均金利 (%，新規，総合・国内銀行，日銀 HP から取得)

m = 通貨流通高（平均発行高，億円，季調済，日銀 HP から取得)

```
. gen time=_n

. tsset time
        time variable:  time, 1 to 81
                delta:  1 unit

. gen lgdp=log(gdp)

. gen lm=log(m/(def/10))

. varsoc d.lgdp d.r d.lm

   Selection-order criteria
   Sample:  6 - 81                                          Number of obs      =        76
  +-----------------------------------------------------------------------------+
  |lag |    LL      LR      df    p      FPE        AIC       HQIC       SBIC    |
```

```
  |----+--------------------------------------------------------------------------|
  |  0 |  541.22                                1.4e-10  -14.1637  -14.1269  -14.0717  |
  |  1 |  571.181  59.923*    9  0.000  8.2e-11* -14.7153* -14.5682* -14.3473* |
  |  2 |  575.715  9.0675     9  0.431  9.2e-11  -14.5978  -14.3404  -13.9537  |
  |  3 |   579.55  7.6704     9  0.568  1.1e-10  -14.4619  -14.0942  -13.5418  |
  |  4 |  583.767  8.4328     9  0.491  1.2e-10   -14.336   -13.858  -13.1399  |
  +--------------------------------------------------------------------------+
   Endogenous:  D.lgdp D.r D.lm
    Exogenous:  _cons

. var d.lgdp d.r d.lm, lags(1)

Vector autoregression

Sample:  3 - 81                                    No. of obs      =        79
Log likelihood =  592.2334                         AIC             = -14.68945
FPE            =  8.38e-11                         HQIC            = -14.54526
Det(Sigma_ml)  =  6.18e-11                         SBIC            = -14.32954

Equation           Parms      RMSE     R-sq      chi2     P>chi2
----------------------------------------------------------------
D_lgdp               4     .010717   0.0422   3.480972   0.3232
D_r                  4     .087186   0.2553    27.0782   0.0000
D_lm                 4     .009434   0.2903   32.30929   0.0000
----------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
D_lgdp       |
        lgdp |
         LD. |   .2031129   .1119361     1.81   0.070    -.0162778    .4225037
             |
```

r						
LD.	.0045431	.0120151	0.38	0.705	-.0190061	.0280922
lm						
LD.	.0152162	.1086739	0.14	0.889	-.1977807	.228213
_cons	.0019504	.0019124	1.02	0.308	-.0017978	.0056986
D_r						
lgdp						
LD.	.4341641	.9106374	0.48	0.634	-1.350652	2.218981
r						
LD.	.5085677	.0977469	5.20	0.000	.3169874	.7001481
lm						
LD.	.1845222	.8840978	0.21	0.835	-1.548278	1.917322
_cons	-.0202984	.0155578	-1.30	0.192	-.0507912	.0101943
D_lm						
lgdp						
LD.	-.1972406	.098541	-2.00	0.045	-.3903774	-.0041037
r						
LD.	-.029395	.0105773	-2.78	0.005	-.0501261	-.0086639
lm						
LD.	.4472679	.0956691	4.68	0.000	.2597599	.634776
_cons	.0071036	.0016835	4.22	0.000	.0038039	.0104033

```
. vargranger
```

Granger causality Wald tests

Equation	Excluded	chi2	df	Prob > chi2
D_lgdp	D.r	.14297	1	0.705
D_lgdp	D.lm	.0196	1	0.889
D_lgdp	ALL	.15705	2	0.924
D_r	D.lgdp	.22731	1	0.634
D_r	D.lm	.04356	1	0.835
D_r	ALL	.3039	2	0.859
D_lm	D.lgdp	4.0064	1	0.045
D_lm	D.r	7.7232	1	0.005
D_lm	ALL	10.798	2	0.005