- However, β_{MM} is inconsistent when $\mathrm{E}(x'u) \neq 0$, i.e.,

$$\beta_{MM} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u = \beta + \Big(\frac{1}{n}X'X\Big)^{-1}\Big(\frac{1}{n}X'u\Big) \not\longrightarrow \beta.$$

Note as follows:

$$\frac{1}{n}X'u = \frac{1}{n}\sum_{i=1}^{n} x_i'u_i \;\longrightarrow\; \mathrm{E}(x'u) \neq 0.$$

In order to obtain a consistent estimator of β, we find the instrumental variable z which satisfies $\mathrm{E}(z'u) = 0$.

Let z_i be the ith realization of z, where z_i is a $1 \times k$ vector.

Then, we have the following:

$$\frac{1}{n}Z'u = \frac{1}{n}\sum_{i=1}^{n} z_i'u_i \;\longrightarrow\; \mathrm{E}(z'u) = 0.$$

The β which satisfies $\displaystyle\frac{1}{n}\sum_{i=1}^{n} z_i'u_i = 0$ is denoted by β_{IV}, i.e., $\displaystyle\frac{1}{n}\sum_{i=1}^{n} z_i'(y_i - x_i\beta_{IV}) = 0$.

Thus, β_{IV} is obtained as:

$$\beta_{IV} = \Big(\frac{1}{n}\sum_{i=1}^{n} z_i' x_i\Big)^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} z_i' y_i\Big) = (Z'X)^{-1}Z'y.$$

Note that $Z'X$ is a $k \times k$ square matrix, where we assume that the inverse matrix of $Z'X$ exists.

Assume that as n goes to infinity there exist the following moment matrices:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} z_i' x_i &= \frac{1}{n}Z'X \longrightarrow M_{zx},\\
\frac{1}{n}\sum_{i=1}^{n} z_i' z_i &= \frac{1}{n}Z'Z \longrightarrow M_{zz},\\
\frac{1}{n}\sum_{i=1}^{n} z_i' u_i &= \frac{1}{n}Z'u \longrightarrow 0.
\end{aligned}$$

As n goes to infinity, β_{IV} is rewritten as:

$$\begin{aligned}\beta_{IV} &= (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u \\ &= \beta + (\frac{1}{n}Z'X)^{-1}(\frac{1}{n}Z'u) \longrightarrow \beta + M_{zx} \times 0 = \beta,\end{aligned}$$

Thus, β_{IV} is a consistent estimator of β.

- We consider the asymptotic distribution of β_{IV}.

By the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz})$$

Note that $\mathrm{V}(\frac{1}{\sqrt{n}}Z'u) = \frac{1}{n}\mathrm{V}(Z'u) = \frac{1}{n}\mathrm{E}(Z'uu'Z) = \frac{1}{n}\mathrm{E}\Big(\mathrm{E}(Z'uu'Z|Z)\Big) = \frac{1}{n}\mathrm{E}\Big(Z'\mathrm{E}(uu'|Z)Z\Big) =$
$\frac{1}{n}\mathrm{E}(\sigma^2 Z'Z) = \mathrm{E}(\sigma^2\frac{1}{n}Z'Z) \longrightarrow \mathrm{E}(\sigma^2 M_{zz}) = \sigma^2 M_{zz}$.

We obtain the following asymmptotic distribution:

$$\sqrt{n}(\beta_{IV} - \beta) = (\frac{1}{n}Z'X)^{-1}(\frac{1}{\sqrt{n}}Z'u) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M_{zx}^{-1\prime})$$

Practically, for large n we use the following distribution:

$$\beta_{IV} \sim N\Big(\beta, s^2 (Z'X)^{-1} Z'Z (Z'X)^{-1\prime}\Big),$$

where $s^2 = \dfrac{1}{n-k}(y - X\beta_{IV})'(y - X\beta_{IV})$.

● In the case where z_i is a $1 \times r$ vector for $r > k$, $Z'X$ is a $r \times k$ matrix, which is not a square matrix. $\Longrightarrow$ **Generalized Method of Moments (GMM, 一般化積率法)**

4.2 Generalized Method of Moments (GMM, 一般化積率法)

Consider the following regression model:

$$Z'y = Z'X\beta + Z'u,$$

where Z, y, X, β and u are $n \times r$, $n \times 1$, $n \times k$, $k \times 1$ and $n \times 1$ matrices or vectors.

Note that $r \geq k$.

$y^* = Z'y$, $X^* = Z'X$ and $u^* = Z'u$ denote $r \times 1$, $r \times k$ and $r \times 1$ matrices or vectors, where $r \geq k$.

Rewrite as follows:

$$y^* = X^*\beta + u^*,$$

$\Longrightarrow$ r is taken as sample size.

Mean and variance of u^* are given by:

$$\mathrm{E}(u^*) = 0 \quad \text{and} \quad \mathrm{V}(u^*) = \mathrm{E}(u^* u^{*\prime}) = \sigma^2 Z'Z = \sigma^2 \Omega.$$

Using GLS, GMM is obtained as:

$$\beta_{GMM} = (X^{*\prime}\Omega^{-1}X^*)^{-1}X^{*\prime}\Omega^{-1}y^* = \Big(X'Z(Z'Z)^{-1}Z'X\Big)^{-1}X'Z(Z'Z)^{-1}Z'y.$$

β_{GMM} is rewritten as:

$$\begin{aligned}\beta_{GMM} &= \Big(X'Z(Z'Z)^{-1}Z'X\Big)^{-1}X'Z(Z'Z)^{-1}Z'y \\ &= \Big(X'Z(Z'Z)^{-1}Z'X\Big)^{-1}X'Z(Z'Z)^{-1}Z'(X\beta + u) \\ &= \beta + \Big(X'Z(Z'Z)^{-1}Z'X\Big)^{-1}X'Z(Z'Z)^{-1}Z'u.\end{aligned}$$

Assume:

$$\frac{1}{n}X'Z \longrightarrow M_{xz}, \qquad \frac{1}{n}Z'Z \longrightarrow M_{zz}, \qquad \frac{1}{n}Z'u \longrightarrow 0.$$

Then, β_{GMM} is a consistent estimator of β, which is shown as follows:

$$\beta_{GMM} = \beta + \Big((\frac{1}{n}X'Z)(\frac{1}{n}Z'Z)^{-1}(\frac{1}{n}Z'X)\Big)^{-1}(\frac{1}{n}X'Z)(\frac{1}{n}Z'Z)^{-1}(\frac{1}{n}Z'u)$$
$$\longrightarrow \ \beta + (M_{xz}M_{zz}^{-1}M_{xz}')^{-1}M_{xz}M_{zz}^{-1} \times 0 = \beta.$$

● We derive the asymptotic distribution of β_{GMM}.

From the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \ \longrightarrow \ N(0, \sigma^2 M_{zz}).$$

Accordingly, β_{GMM} is distributed as:

$$\sqrt{n}(\beta_{GMM} - \beta) = \Big((\frac{1}{n}X'Z)(\frac{1}{n}Z'Z)^{-1}(\frac{1}{n}Z'X)\Big)^{-1}(\frac{1}{n}X'Z)(\frac{1}{n}Z'Z)^{-1}(\frac{1}{\sqrt{n}}Z'u)$$
$$\longrightarrow \ N\Big(0, \sigma^2(M_x z M_{zz}^{-1}M_{xz}')^{-1}\Big).$$

Practically, for large n we use the following distribution:

$$\beta_{GMM} \sim N\Big(\beta, s^2(X'Z(Z'Z)^{-1}Z'X)^{-1}\Big),$$

where $s^2 = \dfrac{1}{n-k}(y - X\beta_{GMM})'(y - X\beta_{GMM})$.

- The above GMM is equivalent to 2SLS.

X: $n \times k$, Z: $n \times r$, $r > k$.

Assume:

$$\frac{1}{n}X'u = \frac{1}{n}\sum_{i=1}^{n} x_i' u_i \longrightarrow \mathrm{E}(x'u) \neq 0,$$

$$\frac{1}{n}Z'u = \frac{1}{n}\sum_{i=1}^{n} z_i' u_i \longrightarrow \mathrm{E}(z'u) = 0.$$

Regress X on Z, i.e., $X = Z\Gamma + V$ by OLS, where Γ is a $r \times k$ unknown parameter matrix and V is an error term, Denote the predicted value of X by $\hat{X} = Z\hat{\Gamma} = Z(Z'Z)^{-1}Z'X$, where $\hat{\Gamma} = (Z'Z)^{-1}Z'X$.

Note that 2SLS is equivalent to IV in the case of $Z = \hat{X}$, where this Z is different from the above Z.

This Z is a $n \times k$ matrix, while the above Z is a $n \times r$ matrix.

When Z is a $n \times k$ instrumental variable, the IV estimator is given by:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

Z is replaced by $\hat{X}$. Then,

$$\beta_{2SLS} = (\hat{X}'X)^{-1}\hat{X}'y = \left(X'Z(Z'Z)^{-1}Z'X\right)^{-1}X'Z(Z'Z)^{-1}Z'y = \beta_{GMM}.$$

GMM is interpreted as the GLS applied to MM.