Computational Procedure:

(1) Compute $\hat{S}^{(i)} = \hat{\Gamma}_0 + \sum_{i=1}^{q}\left(1 - \frac{i}{q+1}\right)(\hat{\Gamma}_i + \hat{\Gamma}_i')$, where $\hat{\Gamma}_\tau = \frac{1}{n}\sum_{i=\tau+1}^{n} h(\hat{\theta}; w_i)h(\hat{\theta}; w_{i-\tau})'$.

q is set by a researcher.

(2) Use the following iterative procedure:

$$\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - (\hat{D}^{(i)\prime}\hat{S}^{(i)-1}\hat{D}^{(i)})^{-1}\hat{D}^{(i)\prime}\hat{S}^{(i)-1}g(\hat{\theta}^{(i)}; W).$$

(3) Repeat (1) and (2) until $\hat{\theta}^{(i+1)}$ is equal to $\hat{\theta}^{(i)}$.

In (2), remember that when S is given we take the following iterative procedure:

$$\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - (\hat{D}^{(i)\prime}S^{-1}\hat{D}^{(i)})^{-1}\hat{D}^{(i)\prime}S^{-1}g(\hat{\theta}^{(i)}; W),$$

where $\hat{D}^{(i)} = \frac{\partial g(\hat{\theta}^{(i)}; W)}{\partial \theta'}$. S is replaced by $\hat{S}^{(i)}$.

● If the assumption $\mathrm{E}\big(h(\theta; w)\big) = 0$ is violated, the GMM estimator $\hat{\theta}$ is no longer consistent.

Therefore, we need to check if $\mathrm{E}\big(h(\theta; w)\big) = 0$.

From Assumption 2, note as follows:

$$J = \Big(\sqrt{n} g(\hat{\theta}; W) \Big)' \hat{S}^{-1} \Big(\sqrt{n} g(\hat{\theta}; W) \Big) \longrightarrow \chi^2(r-k),$$

which is called Hansen's J test.

Because of r equations and k parameters, the degree of freedom is given by $r - k$.

If J is small enough, we can judge that the specified model is correct.

Testing Hypothesis:

Remember that the GMM estimator $\hat{\theta}$ has the following asymptotic distribution:

$$\sqrt{n}(\hat{\theta} - \theta) \longrightarrow N\Big(0, (D'S^{-1}D)^{-1}\Big).$$

Consider testing the following null and alternative hypotheses:

- The null hypothesis: $H_0 : R(\theta) = 0$,
- The alternative hypothesis: $H_1 : R(\theta) \neq 0$,

where $R(\theta)$ is a $p \times k$ vector function for $p \leq k$.

p denotes the number of restrictions.

$R(\theta)$ is linearized as: $R(\hat{\theta}) = R(\theta) + R_{\overline{\theta}}(\hat{\theta} - \theta)$, where $R_{\overline{\theta}} = \dfrac{\partial R(\overline{\theta})}{\partial \theta'}$, which is a $p \times k$ matrix.

Note that $\overline{\theta}$ is bewteen $\hat{\theta}$ and θ. If $\hat{\theta} \longrightarrow \theta$, then $\overline{\theta} \longrightarrow \theta$ and $R_{\overline{\theta}} \longrightarrow R_\theta$.

Under the null hypothesis $R(\theta) = 0$, we have $R(\hat{\theta}) = R_{\overline{\theta}}(\hat{\theta} - \theta)$, which implies that the distribution of $R(\hat{\theta})$ is equivalent to that of $R_{\overline{\theta}}(\hat{\theta} - \theta)$.

The distribution of $\sqrt{n}R(\hat{\theta})$ is given by:

$$\sqrt{n}R(\hat{\theta}) = \sqrt{n}R_{\overline{\theta}}(\hat{\theta} - \theta) \longrightarrow N\Big(0, R_\theta(D'S^{-1}D)^{-1}R'_\theta\Big).$$

Therefore, under the null hypothesis, we have the following distribution:

$$nR(\hat{\theta})\Big(R_\theta(D'S^{-1}D)^{-1}R'_\theta\Big)^{-1}R(\hat{\theta})' \longrightarrow \chi^2(p).$$

Practically, replacing θ by $\hat{\theta}$ in R_θ, D and S, we use the following test statistic:

$$nR(\hat{\theta})\Big(R_{\hat{\theta}}(\hat{D}'\hat{S}^{-1}\hat{D})^{-1}R'_{\hat{\theta}}\Big)^{-1}R(\hat{\theta})' \longrightarrow \chi^2(p).$$

$\Longrightarrow$ Wald type test

Examples of $h(\theta; w)$:

1. **OLS:**

   Regression Model: $y_i = x_i\beta + \epsilon_i, \quad \mathrm{E}(x_i'\epsilon_i) = 0$

   $h(\theta; w_i)$ is taken as:

$$h(\theta; w_i) = x_i'(y_i - x_i\beta).$$

2. **IV (Instrumental Variable, 操作変数法):**

   Regression Model: $y_i = x_i\beta + \epsilon_i, \quad \mathrm{E}(x_i'\epsilon_i) \neq 0, \quad \mathrm{E}(z_i'\epsilon_i) = 0$

   $h(\theta; w_i)$ is taken as:

$$h(\theta; w_i) = z_i'(y_i - x_i\beta),$$

   where z_i is a vector of instrumental variables.

When z_i is a $1 \times k$ vector, the GMM of β is equivalent to the instrumental variable (IV) estimator.

When z_i is a $1 \times r$ vector for $r > k$, the GMM of β is equivalent to the two-stage least squares (2SLS) estimator.

3. **NLS (Nonlinear Least Squares, 非線形最小二乗法)**:

Regression Model: $\quad f(y_i, x_i, \beta) = \epsilon_i, \quad \mathrm{E}(x_i'\epsilon_i) \neq 0, \quad \mathrm{E}(z_i'\epsilon_i) = 0$

$h(\theta; w_i)$ is taken as:

$$h(\theta; w_i) = z_i' f(y_i, x_i, \beta)$$

where z_i is a vector of instrumental variables.