$$= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\log\left(\frac{1}{1-\phi_1^2}\right)$$
$$-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

Note as follows:

$$f(y_1) = \frac{1}{\sqrt{2\pi\sigma^2/(1-\phi_1^2)}}\exp\left(-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2\right)$$
$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right)$$

$$\frac{\partial \log f(y_T,\cdots,y_1)}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4/(1-\phi_1^2)}y_1^2 + \frac{1}{2\sigma^4}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2 = 0$$

$$\frac{\partial \log f(y_T,\cdots,y_1)}{\partial \phi_1} = -\frac{\phi_1}{1-\phi_1^2} + \frac{\phi_1}{\sigma^2}y_1^2 + \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})y_{t-1} = 0$$

The MLE of $\phi_1$ and $\sigma^2$ satisfies the above two equation.

$$\tilde{\sigma}^2 = \frac{1}{T}\left((1 - \tilde{\phi}_1^2)y_1^2 + \sum_{t=2}^{T}(y_t - \tilde{\phi}_1 y_{t-1})^2\right)$$

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} + \left(\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2 \tilde{\phi}_1}{1 - \tilde{\phi}_1^2}\right)/\sum_{t=2}^{T} y_{t-1}^2$$

(b) Ordinary Least Squares (OLS) Method

$$S(\phi_1) = \sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

is minimized with respect to $\phi_1$.

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{T} y_{t-1} y_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{\sum_{t=2}^{T} y_{t-1}\epsilon_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{(1/T)\sum_{t=2}^{T} y_{t-1}\epsilon_t}{(1/T)\sum_{t=2}^{T} y_{t-1}^2}$$

$$\longrightarrow \phi_1 + \frac{\mathrm{E}(y_{t-1}\epsilon_t)}{\mathrm{E}(y_{t-1}^2)} = \phi_1$$

OLSE of $\phi_1$ is a consistent estimator.

The following equations are utilized.

$$E(y_{t-1}\epsilon_t) = 0$$

$$E(y_{t-1}^2) = \text{Var}(y_{t-1}) = \gamma(0)$$

10. Asymptotic distribution of OLSE $\hat{\phi}_1$:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$$

**Proof:**

$y_{t-1}\epsilon_t$, $t = 1, 2, \cdots, T$, are distributed with mean zero and variance $\dfrac{\sigma_\epsilon^4}{1 - \phi_1^2}$.

From the central limit theorem,

$$\frac{(1/T) \sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sqrt{\sigma_\epsilon^4/(1 - \phi_1^2)}/\sqrt{T}} \longrightarrow N(0, 1)$$

Rewriting,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} y_{t-1} \epsilon_t \longrightarrow N(0, \frac{\sigma_\epsilon^4}{1 - \phi_1^2}).$$

Next,

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1}^2 \longrightarrow E(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2}$$

yields:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{(1/\sqrt{T}) \sum_{t=1}^{T} y_{t-1} \epsilon_t}{(1/T) \sum_{t=1}^{T} y_{t-1}^2} \longrightarrow N(0, 1 - \phi_1^2)$$

11. Some formulas:

(a) Central Limit Theorem

Random variables $x_1$, $x_2$, $\cdots$, $x_T$ are mutually independently distributed with mean $\mu$ and variance $\sigma^2$.

Define $\overline{x} = (1/T) \sum_{t=1}^{T} x_t$.

Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} = \frac{\overline{x} - \mu}{\sigma/\sqrt{T}} \longrightarrow N(0, 1)$$

(b) Central Limit Theorem II

Random variables $x_1, x_2, \cdots, x_T$ are distributed with mean $\mu$ and variance $\sigma^2$.

Define $\overline{x} = (1/T) \sum_{t=1}^{T} x_t$.

Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} \longrightarrow N(0, 1)$$

(c) Let $x$ and $y$ be random variables.

$y$ converges in distribution to a distribution, and $x$ converges in probability to a fixed value.

Then, $xy$ converges in distribution.

For example, consider:

$$y \longrightarrow N(\mu, \sigma^2), \qquad x \longrightarrow c.$$

Then, we obtain:

$$xy \longrightarrow N(c\mu, c^2\sigma^2)$$

12. **AR(1) +drift:** $\quad y_t = \mu + \phi_1 y_{t-1} + \epsilon_t$

    Mean:

    Using the lag operator,

    $$\phi(L)y_t = \mu + \epsilon_t$$

    where $\phi(L) = 1 - \phi_1 L$.

    Multiply $\phi(L)^{-1}$ on both sides. Then, when $|\phi_1| < 1$, we have:

    $$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Taking the expectation on both sides,

$$E(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}E(\epsilon_t)$$
$$= \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1}$$

**Example: AR(2) Model:** Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$.

1. The stationarity condition is: two solutions of $x$ from $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0$ are outside the unit circle.

2. Rewriting the AR(2) model,

$$(1 - \phi_1 L - \phi_2 L^2)y_t = \epsilon_t.$$

Let $1/\alpha_1$ and $1/\alpha_2$ be the solutions of $\phi(x) = 0$.

Then, the AR(2) model is written as:

$$(1 - \alpha_1 L)(1 - \alpha_2 L)y_t = \epsilon_t,$$

which is rewritten as:

$$y_t = \frac{1}{(1 - \alpha_1 L)(1 - \alpha_2 L)}\epsilon_t$$
$$= \left(\frac{\alpha_1/(\alpha_1 - \alpha_2)}{1 - \alpha_1 L} + \frac{-\alpha_2/(\alpha_1 - \alpha_2)}{1 - \alpha_2 L}\right)\epsilon_t$$

3. **Mean of AR(2) Model:**

When $y_t$ is stationary, i.e., $\alpha_1$ and $\alpha_2$ are within the unit circle,

$$\mu = \mathrm{E}(y_t) = \mathrm{E}(\phi(L)\epsilon_t) = 0$$

4. **Autocovariance Function of AR(2) Model:**

$$\gamma(\tau) = \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau})$$

$$= \mathrm{E}\big((\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) y_{t-\tau}\big)$$

$$= \phi_1 \mathrm{E}(y_{t-1} y_{t-\tau}) + \phi_2 \mathrm{E}(y_{t-2} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau})$$

$$= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}$$

The initial condition is obtained by solving the following three equations:

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_\epsilon^2,$$

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1),$$

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0).$$

Therefore, the initial conditions are given by:

$$\gamma(0) = \left(\frac{1 - \phi_2}{1 + \phi_2}\right) \frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2},$$

$$\gamma(1) = \frac{\phi_1}{1 - \phi_2} \gamma(0) = \left(\frac{\phi_1}{1 - \phi_2}\right) \left(\frac{1 - \phi_2}{1 + \phi_2}\right) \frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2}.$$

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Given $\gamma(0)$ and $\gamma(1)$, we obtain $\gamma(\tau)$ as follows:

$$\gamma(\tau) = \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2), \qquad \text{for } \tau = 2, 3, \cdots.$$

5. **Another solution for $\gamma(0)$:**

From $\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_\epsilon^2$,

$$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1 \rho(1) - \phi_2 \rho(2)}$$

where

$$\rho(1) = \frac{\phi_1}{1 - \phi_2}, \qquad \rho(2) = \phi_1 \rho(1) + \phi_2 = \frac{\phi_1^2 + (1 - \phi_2)\phi_2}{1 - \phi_2}.$$

6. **Autocorrelation Function of AR(2) Model:**

Given $\rho(1)$ and $\rho(2)$,

$$\rho(\tau) = \phi_1 \rho(\tau - 1) + \phi_2 \rho(\tau - 2), \qquad \text{for } \tau = 3, 4, \cdots,$$

7. $\phi_{k,k}$ = **Partial Autocorrelation Coefficient of AR(2) Process:**

$$
\begin{pmatrix}
1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\
\rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\
\vdots & \vdots & & \vdots & \vdots \\
\rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1
\end{pmatrix}
\begin{pmatrix}
\phi_{k,1} \\
\phi_{k,2} \\
\vdots \\
\phi_{k,k-1} \\
\phi_{k,k}
\end{pmatrix}
=
\begin{pmatrix}
\rho(1) \\
\rho(2) \\
\vdots \\
\rho(k)
\end{pmatrix},
$$

for $k = 1, 2, \cdots$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Autocovariance Functions:

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1),$$

$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0),$$

$$\gamma(\tau) = \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2), \qquad \text{for } \tau = 3, 4, \cdots.$$

Autocorrelation Functions:

$$\rho(1) = \phi_1 + \phi_2\rho(1) = \frac{\phi_1}{1 - \phi_2},$$

$$\rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2,$$

$$\rho(\tau) = \phi_1\rho(\tau - 1) + \phi_2\rho(\tau - 2), \qquad \text{for } \tau = 3, 4, \cdots.$$

$$\phi_{1,1} = \rho(1) = \frac{\phi_1}{1 - \phi_2}$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \phi_2$$

$$\phi_{3,3} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}}$$

$$= \frac{(\rho(3) - \rho(1)\rho(2)) - \rho(1)^2(\rho(3) - \rho(1)) + \rho(2)\rho(1)(\rho(2) - 1)}{(1 - \rho(1)^2) - \rho(1)^2(1 - \rho(2)) + \rho(2)(\rho(1)^2 - \rho(2))} = 0.$$

8. **Log-Likelihood Function — Innovation Form:**

$$\log f(y_T, \cdots, y_1) = \log f(y_2, y_1) + \sum_{t=3}^{T} \log f(y_t | y_{t-1}, \cdots, y_1)$$

where

$$f(y_2, y_1) = \frac{1}{2\pi} \begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{vmatrix}^{-1/2} \exp\left(-\frac{1}{2}(y_1 \ y_2)\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right),$$