$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2\right).$$

Note as follows:

$$\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} = \gamma(0)\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} = \gamma(0)\begin{pmatrix} 1 & \phi_1/(1-\phi_2) \\ \phi_1/(1-\phi_2) & 1 \end{pmatrix}.$$

9. **AR(2) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$

Mean:

Rewriting the AR(2)+drift model,

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$.

Under the stationarity assumption, we can rewrite the AR(2)+drift model as follows:

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Therefore,

$$\mathrm{E}(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}\mathrm{E}(\epsilon_t) = \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1 - \phi_2}$$

Example: AR(p) model: Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t$.

1. **Variance of AR(p) Process:**

   Under the stationarity condition (i.e., the p solutions of x from $\phi(x) = 0$ are outside the unit circle),

   $$\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1\rho(1) - \cdots - \phi_p\rho(p)}.$$

   Note that $\gamma(\tau) = \rho(\tau)\gamma(0)$.

   Solve the following simultaneous equations for $\tau = 0, 1, \cdots, p$:

   $$\gamma(\tau) = \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau})$$

$$
= \begin{cases} \phi_1\gamma(\tau-1) + \phi_2\gamma(\tau-2) + \cdots + \phi_p\gamma(\tau-p), & \text{for } \tau \neq 0, \\ \phi_1\gamma(\tau-1) + \phi_2\gamma(\tau-2) + \cdots + \phi_p\gamma(\tau-p) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}
$$

2. **Estimation of AR(p) Model:**

   1. OLS:

$$
\min_{\phi_1,\cdots,\phi_p} \sum_{t=p+1}^{T} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \ \cdots \ - \phi_p y_{t-p})^2
$$

   2. MLE:

$$
\max_{\phi_1,\cdots,\phi_p} \quad \log f(y_T, \cdots, y_1)
$$

where

$$
\log f(y_T, \cdots, y_1) = \log f(y_p, \cdots, y_2, y_1) + \sum_{t=p+1}^{T} \log f(y_t | y_{t-1}, \cdots, y_1),
$$

$$f(y_p, \cdots, y_2, y_1) = (2\pi)^{-p/2}|V|^{-1/2} \exp\left(-\frac{1}{2}(y_1\ y_2\ \cdots\ y_p)V^{-1}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}\right)$$

$$V = \gamma(0)\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(p-2) & \rho(p-1) \\ \rho(1) & 1 & & \rho(p-3) & \rho(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(1) & 1 \end{pmatrix}$$

$$f(y_t|y_{t-1}, \cdots, y_1) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\Big(-\frac{1}{2\sigma_\epsilon^2}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p})^2\Big)$$

3. Yule=Walker (ユール・ウォーカー) Equation:

Multiply $y_{t-1}, y_{t-2}, \cdots, y_{t-p}$ on both sides of $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} +$

$\epsilon_t = y_t$, take expectations for each case, and divide by the sample variance $\hat{\gamma}(0)$.

$$\begin{pmatrix} 1 & \hat{\rho}(1) & \cdots & \hat{\rho}(p-2) & \hat{\rho}(p-1) \\ \hat{\rho}(1) & 1 & & \hat{\rho}(p-3) & \hat{\rho}(p-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \hat{\rho}(p-1) & \hat{\rho}(p-2) & \cdots & \hat{\rho}(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{pmatrix} = \begin{pmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(p) \end{pmatrix}$$

where

$$\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=\tau+1}^{T} (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu}), \qquad \hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}.$$

3. **AR(p) +drift:** $y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots \phi_p y_{t-p} + \epsilon_t$

Mean:

$$\phi(L) y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$.

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t$$

Taking the expectation on both sides,

$$\mathrm{E}(y_t) = \phi(L)^{-1}\mu + \phi(L)^{-1}\mathrm{E}(\epsilon_t) = \phi(1)^{-1}\mu$$
$$= \frac{\mu}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

4. **Partial Autocorrelation of AR(p) Process:**

$\phi_{k,k} = 0$ for $k = p + 1, p + 2, \cdots$.

5.3 MA Model

MA (Moving Average，移動平均) Model:

1. MA(q)

$$y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q},$$

which is rewritten as:

$$y_t = \theta(L)\epsilon_t,$$

where

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q.$$

2. **Invertibility (反転可能性):**

   The q solutions of x from $\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q = 0$ の q are outside the unit circle.

   $\Longrightarrow$ MA(q) model is rewritten as AR(∞) model.

Example: MA(1) Model: $y_t = \epsilon_t + \theta_1 \epsilon_{t-1}$

1. **Mean of MA(1) Process:**

$$\mathrm{E}(y_t) = \mathrm{E}(\epsilon_t + \theta_1 \epsilon_{t-1}) = \mathrm{E}(\epsilon_t) + \theta_1 \mathrm{E}(\epsilon_{t-1}) = 0$$

2. **Autocovariance Function of MA(1) Process:**

$$\begin{aligned}\gamma(0) &= \mathrm{E}(y_t^2) = \mathrm{E}(\epsilon_t + \theta_1 \epsilon_{t-1})^2 = \mathrm{E}(\epsilon_t^2 + 2\theta_1 \epsilon_t \epsilon_{t-1} + \theta_1^2 \epsilon_{t-1}^2) \\ &= \mathrm{E}(\epsilon_t^2) + 2\theta_1 \mathrm{E}(\epsilon_t \epsilon_{t-1}) + \theta_1^2 \mathrm{E}(\epsilon_{t-1}^2) = (1 + \theta_1^2)\sigma_\epsilon^2\end{aligned}$$

$$\gamma(1) = \mathrm{E}(y_t y_{t-1}) = \mathrm{E}((\epsilon_t + \theta_1\epsilon_{t-1})(\epsilon_{t-1} + \theta_1\epsilon_{t-2})) = \theta_1\sigma_\epsilon^2$$

$$\gamma(2) = \mathrm{E}(y_t y_{t-2}) = \mathrm{E}((\epsilon_t + \theta_1\epsilon_{t-1})(\epsilon_{t-2} + \theta_1\epsilon_{t-3})) = 0$$

3. **Autocorrelation Function of MA(1) Process:**

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \begin{cases} \dfrac{\theta_1}{1+\theta_1^2}, & \text{for } \tau = 1, \\ 0, & \text{for } \tau = 2, 3, \cdots. \end{cases}$$

Let x be $\rho(1)$.

$$\frac{\theta_1}{1+\theta_1^2} = x, \qquad \text{i.e.,} \qquad x\theta_1^2 - \theta + x = 0.$$

θ_1 should be a real number.

$$1 - 4x^2 > 0, \qquad \text{i.e.,} \qquad -\frac{1}{2} \le \rho(1) \le \frac{1}{2}.$$

4. **Invertibility Condition of MA(1) Process:**

$$
\begin{aligned}
\epsilon_t &= -\theta_1 \epsilon_{t-1} + y_t \\
&= (-\theta_1)^2 \epsilon_{t-2} + y_t + (-\theta_1) y_{t-1} \\
&= (-\theta_1)^3 \epsilon_{t-3} + y_t + (-\theta_1) y_{t-1} + (-\theta_1)^2 y_{t-2} \\
&\qquad \vdots \\
&= (-\theta_1)^s \epsilon_{t-s} + y_t + (-\theta_1) y_{t-1} + (-\theta_1)^2 y_{t-2} + \cdots + (-\theta_1)^{t-s+1} y_{t-s+1}
\end{aligned}
$$

When $(-\theta_1)^s \epsilon_{t-s} \longrightarrow 0$, the MA(1) model is written as the AR(∞) model, i.e.,

$$
y_t = -(-\theta_1) y_{t-1} - (-\theta_1)^2 y_{t-2} - \cdots - (-\theta_1)^{t-s+1} y_{t-s+1} - \cdots + \epsilon_t
$$

That is, $|\theta_1| < 1$ represents the invertibility condition.